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PRECISE TAIL BEHAVIOR OF SOME RANDOM DIRICHLET SERIES WITH SUMMANDS HAVING AN INFINITE SECOND MOMENT

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Abstract. We point out a precise first-order tail behavior of a random Dirichlet series $\sum_{k\geq 1} k^{-\alpha}\eta_k$. Here, $\eta_1,\ \eta_2,\dots$ are independent copies of a random variable η with zero mean, which is almost surely bounded from above and satisfies $\mathbb{P}\{-\eta > x\} \sim \mathrm{const}\, x^{-\beta}$ as $x \to \infty$ for some $\beta \in (1,2)$, and $\alpha \in (1/\beta, 1]$. Our results complement those obtained recently by Iksanov and Wachtel (2024) under the assumptions that η has zero mean, is bounded from above, yet the second moment of η is finite. We make quantitatively precise the intuitive expectation that the tail behavior of $\sum_{k\geq 1} k^{-\alpha}\eta_k$ is mainly driven by the boundedness of η, the influence of the left distribution tail of η being small.

1. Introduction and main results

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathbb{N}$ is the set of positive integer numbers. The random series

$$D(z) := \sum_{n\geq 1} \frac{X_n}{n^z},$$

is called random Dirichlet series.

A better understanding of the structure of random Dirichlet series may lead to a better understanding of the behaviour of nonrandom Dirichlet series. In particular, this can be useful for the analysis of the Riemann zeta function, which is of major importance in contemporary cryptography and data security. Random Dirichlet series also admit other applications in Number Theory.

Let $\eta_1,\ \eta_2,\dots$ be independent copies of a random variable η. There are numerous publications which investigate random Dirichlet series. Some of them are devoted to laws of the iterated logarithm (LIL) and functional limit theorems (FLT). Indeed, FLTs and LILs for a random Dirichlet series $\sum_{k\geq 2} \frac{(\log k)^\alpha}{k^{1/2+s}}\eta_k$ were formulated and proved in the recent past, see [1] for a LIL in the case $\alpha = 0$ and $\mathbb{P}\{\eta = \pm 1\} = 1/2$, [4] for an FLT and a LIL and [6] for a functional LIL in the case $\alpha > -1/2$, and [9] for a LIL in the case $\alpha = -1/2$. It was found in [8] that the FLT and LIL for the series $\sum_{k\geq 2} \frac{(\log k)^{-1/2}}{k^{1/2+s}}\eta_k$ and $\sum_{p\in\mathcal{P}} \frac{\eta_p}{p^{1/2+s}}$, where $\mathcal{P}$ is the set of prime numbers, are identical. In contrast, much less attention has been paid to a precise tail behaviour of random Dirichlet series. In [10], which gave impetus to the present work, a precise tail behaviour for $\sum_{k\geq 1} k^{-\alpha}\eta_k$ was obtained under the assumptions that η is upper bounded, has zero mean and finite second moment.

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Throughout the paper our standing assumptions are that $\eta \le b$ almost surely (a.s.) for some $b > 0$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1, 2)$. Plainly, $\mathbb{E}[\eta^2] = \infty$.

Put $S(\alpha) := \sum_{k\ge1} k^{-\alpha}\eta_k$ for appropriate $\alpha \in \mathbb{R}$. Our purpose is to determine a precise first-order behavior of $\mathbb{P}\{S(\alpha) > x\}$ as $x \to \infty$, whenever $|S(\alpha)| < \infty$ a.s. First, we have to find the range of α, for which the series is well-defined.

Proposition 1.1. The series $\sum_{k\ge1} k^{-\alpha}\eta_k$ converges a.s. if, and only if, $\alpha\beta > 1$.

Since η is bounded from above, we infer

$$S(\alpha) \le b\sum_{k\ge1} k^{-\alpha}.$$

Thus, if $\alpha > 1$, the variable $S(\alpha)$ is a.s. upper bounded and does not have the right tail in the sense that $\mathbb{P}\{S(\alpha) > x\} = 0$ for $x > b\sum_{k\ge1} k^{-\alpha}$. Summarizing, we only need to consider the case $\alpha \in (1/\beta, 1]$.

To present our main results, we have to introduce some additional notation. Put $\psi(t) := \log \mathbb{E}[\mathrm{e}^{t\eta}]$ for $t \in \mathbb{R}$. Under the standing assumptions, the function ψ is infinitely differentiable, strictly convex and strictly increasing on $[0, \infty)$. Consequently, ψ' is positive and strictly increasing on $(0, \infty)$, and ψ'' is positive on the same interval. Recall that, for $\rho \in (0, 1]$, the Euler–Mascheroni constant γ_ρ is defined by

$$\gamma_\rho := \lim_{n\to\infty} \Big(\sum_{k=1}^{n} k^{-\rho} - (1-\rho)^{-1} n^{1-\rho} \Big) \tag{1.1}$$

if $\rho \in (0, 1)$, and

$$\gamma_1 := \lim_{n\to\infty} \Big(\sum_{k=1}^{n} k^{-1} - \log n \Big) \tag{1.2}$$

if $\rho = 1$.

Here are our main results. Theorems 1.2 and 1.3 only differ by the assumptions regarding the behavior of the distribution of η in a left vicinity of b.

Theorem 1.2. *Assume that $\eta \le b$ a.s. for some $b > 0$, $\mathbb{P}\{\eta = b\} = \theta \in (0, 1)$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1, 2)$.*
If $\alpha = 1$, then, as $x \to \infty$,

$$\mathbb{P}\{S(1) > x\} \sim \frac{1}{(2\pi\theta b)^{1/2}} \exp\Big(-\frac{x-q}{2b} - b\mathrm{e}^{(x-q)/b} \Big),$$

where

$$q := b\gamma_1 + \int_0^1 x^{-1}\psi'(x)\mathrm{d}x + \int_1^\infty x^{-1}(\psi'(x) - b)\mathrm{d}x \in \mathbb{R}$$

If $\alpha \in (1/\beta, 1)$, then, as $x \to \infty$,

$$\mathbb{P}\{S(\alpha) > x\} \sim \frac{1}{(2\pi\theta)^{1/2}} \left(\frac{\alpha(\sigma^2_{\alpha,\beta})^\alpha}{1-\alpha} \right)^{1/(2(1-\alpha))} \frac{1}{x^{1/(2(1-\alpha))}} \times \exp\left(-\left(\frac{1-\alpha}{(\alpha\sigma^2_{\alpha,\beta})^\alpha} \right)^{1/(1-\alpha)} (x - b\gamma_\alpha)^{1/(1-\alpha)} \right),$$

where

$$\sigma^2_{\alpha,\beta} := \alpha^{-1}\int_0^\infty x^{1-1/\alpha}\psi''(x)\mathrm{d}x = (1-\alpha)\alpha^{-3}\int_0^\infty x^{-1-1/\alpha}\psi(x)\mathrm{d}x \in (0, \infty).$$

Theorem 1.3. *Assume that $\eta \le b$ a.s. for some $b > 0$, $\mathbb{P}\{b - \eta \le x\} \sim \lambda x^r$ as $x \to 0+$ for some positive λ and r, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1, 2)$.*
If $\alpha = 1$, then, as $x \to \infty$,

$$\mathbb{P}\{S(1) > x\} \sim \left(\frac{(2\pi)^{r-1}}{\lambda\Gamma(r+1)b}\right)^{1/2} \exp\left(\frac{r-1}{2b}(x-q) - b\mathrm{e}^{(x-q)/b}\right)$$

with q as defined in Theorem 1.2 and Γ being the Euler gamma-function.
If $\alpha \in (1/\beta, 1)$, then, as $x \to \infty$,

$$\begin{aligned}\mathbb{P}\{S(\alpha) > x\} \sim \left(\frac{(2\pi)^{r\alpha-1}}{\lambda\Gamma(r+1)}\right)^{1/2} &\left(\frac{(1-\alpha)^{r\alpha-1}}{\alpha^{r\alpha-1}(\sigma^2_{\alpha,\beta})^{(r-1)\alpha}}\right)^{1/(2(1-\alpha))} \\ &\times \exp\left(\frac{r}{2}\left(\left(\frac{\alpha\sigma^2_{\alpha,\beta}}{1-\alpha}\right)^{\alpha/(1-\alpha)} - 1\right)\right) x^{(r\alpha-1)/(2(1-\alpha))} \\ &\times \exp\left(-\left(\frac{1-\alpha}{(\alpha\sigma^2_{\alpha,\beta})^{\alpha}}\right)^{1/(1-\alpha)} (x - b\gamma_\alpha)^{1/(1-\alpha)}\right)\end{aligned}$$

with $\sigma^2_{\alpha,\beta}$ as defined in Theorem 1.2.

One may expect that the asymptotic behavior of $\mathbb{P}\{S(\alpha) > x\}$ as $x \to \infty$ is mainly driven by the boundedness assumption and the behavior of the distribution of η in a left vicinity of b, whereas the contribution of the left distribution tail of η is significantly smaller. Making the words 'significantly smaller' quantitatively precise has been the original motivation behind the present article. Under the same assumptions as in Theorems 1.2 and 1.3, with the exception that the assumption $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ is replaced by $\mathbb{E}[\eta^2] < \infty$ the asymptotic behavior of $\mathbb{P}\{S(\alpha) > x\}$ was obtained in Theorems 1.1 and 1.2 of [10]. Comparing our results with those in [10] we conclude that the asymptotic forms of $\mathbb{P}\{S(\alpha) > x\}$ as $x \to \infty$ are identical, with different constants q and $\sigma_{\alpha,\beta}$ (σ_α in [10]). The influence of the left distribution tail of η is hidden in these constants, which gives the quantitative meaning to the words 'significantly smaller'.

In answer to a question put to us by one of the referees, we note that the statements of Theorems 1.2 and 1.3 do not change, if the assumption $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ is replaced by $\mathbb{P}\{-\eta > x\} \sim x^{-\beta}\ell(x)$ for some ℓ slowly varying at ∞. Under the modified assumption, the constant c has to be replaced with $\ell(t)$ in the asymptotics given in Lemma 4.3. Further, we have checked that all the arguments appealing to Lemma 4.3 still work in the presence of a slowly varying factor.

The proof of Proposition 1.1 is given in Section 3. The proofs of Theorems 1.2 and 1.3 which follow closely the argument worked out in [10] are presented in Sections 4 and 5.

2. Plan of the proof

We begin by performing an exponential change of measure, a standard technique in large deviation theory. Specifically, for each $t > 0$, define a new probability measure $\mathbb{P}^{(t)}$ on the σ-algebra $\sigma(\eta_1, \eta_2, \ldots)$ by

$$\mathbb{E}^{(t)}[g(\eta_1, \ldots, \eta_k)] = \frac{\mathbb{E}[\mathrm{e}^{tS(\alpha)} g(\eta_1, \ldots, \eta_k)]}{\mathbb{E}[\mathrm{e}^{tS(\alpha)}]}, \tag{2.1}$$

where $\mathbb{E}^{(t)}$ denotes expectation with respect to $\mathbb{P}^{(t)}$. This identity holds for all $k \in \mathbb{N}$ and every bounded Borel function $g : \mathbb{R}^k \to \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be any bounded Borel function. Setting $g(y_1, \dots, y_k) = f\Big(\sum_{j=1}^{k} j^{-\alpha} y_j\Big)$ for $(y_1, \dots, y_k) \in \mathbb{R}^k$ and letting $k \to \infty$, we obtain

$$\mathbb{E}^{(t)}[f(S(\alpha))] = \frac{\mathbb{E}[\mathrm{e}^{tS(\alpha)} f(S(\alpha))]}{\mathbb{E}[\mathrm{e}^{tS(\alpha)}]}. \tag{2.2}$$

Equality (2.2) remains valid for any (not necessarily bounded) Borel function $f : \mathbb{R} \to \mathbb{R}$, provided that at least one side of (2.2) is well defined, possibly infinite.

A key property of the exponential change of measure (2.1) is that the random variables $\eta_1, \eta_2, \dots$ remain independent under $\mathbb{P}^{(t)}$. In other words, the change of measure affects only the marginal distributions of the individual random variables.

Recall that $\psi(t) = \log \mathbb{E}[\mathrm{e}^{t\eta}]$ for $t \in \mathbb{R}$. Then $\mathbb{E}[\mathrm{e}^{tS(\alpha)}] = \exp(\sum_{k\geq 1} \psi(t/k^\alpha))$ for $t \in \mathbb{R}$. Applying (2.2) with $f(y) = y$, we have

$$\mathbb{E}^{(t)}[S(\alpha)] = \frac{\mathbb{E}[\mathrm{e}^{tS(\alpha)} S(\alpha)]}{\mathbb{E}[\mathrm{e}^{tS(\alpha)}]} = \sum_{k\geq 1} k^{-\alpha} \psi'(t/k^\alpha).$$

Fix any $x > 0$ and set $f(y) = \mathrm{e}^{-ty} 1_{(x,\infty)}(y)$. Then (2.2) gives

$$\begin{aligned}\mathbb{P}\{S(\alpha) > x\} &= \mathbb{E}[\mathrm{e}^{tS(\alpha)}]\mathbb{E}^{(t)}[\mathrm{e}^{-tS(\alpha)}\, \mathbb{1}_{\{S(\alpha)>x\}}] \\ &= \mathrm{e}^{-tx+\sum_{k\geq 1}\psi(t/k^\alpha)}\mathbb{E}^{(t)}[\mathrm{e}^{-t(S(\alpha)-x)}\, \mathbb{1}_{\{S(\alpha)-x>0\}}].\end{aligned}$$

Under $\mathbb{P}^{(t)}$, define $S_0^{(t)}(\alpha) = S(\alpha) - \mathbb{E}^{(t)}[S(\alpha)]$. The function $t \mapsto \sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha)$ is continuous, strictly increasing on $[0, \infty)$, and vanishes at $t = 0$. Hence, for each $x \geq 0$, the equation

$$\sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha) = x \tag{2.3}$$

has a unique solution $t = t(x)$.

We will study the asymptotic behaviour of $\mathbb{P}\{S(\alpha) > x\}$ as $x \to \infty$ using the representation

$$\begin{aligned}\mathbb{P}\{S(\alpha) > x\} = \exp\left\{\sum_{k\geq 1}\big(\psi(t(x)/k^\alpha) - (t(x)/k^\alpha)\psi'(t(x)/k^\alpha)\big)\right\} \\ \times\, \mathbb{E}^{(t(x))}\Big[\mathrm{e}^{-t(x)S_0^{(t)}(\alpha)}\, \mathbb{1}_{\{S_0^{(t)}(\alpha)>0\}}\Big].\end{aligned} \tag{2.4}$$

Our analysis of the terms appearing on the right-hand side of (2.4) proceeds through the following four steps.

Step 1. Obtain an asymptotic expansion of $\mathbb{E}^{(t)}[S(\alpha)] = \sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha)$ as $t \to \infty$, with a precision sufficient for future needs. It turns out that an expansion which is accurate up to the term $o(1/t)$ is adequate for our purposes. This expansion is then used to derive the asymptotic behaviour of the solution $t(x)$ to equation (2.3). The details of this step are presented in Propositions 4.4 and 4.5.

Step 2. Derive an asymptotic expansion of $\sum_{k\geq 1}\big(\psi(t/k^\alpha) - (t/k^\alpha)\psi'(t/k^\alpha)\big)$ as $t \to \infty$, up to the term $o(1)$. The $o(1)$ accuracy is required to obtain precise asymptotics for the exponential component in (2.4). This step is carried out in Propositions 4.6 and 4.7.

Step 3. Determine the first-order asymptotic behaviour of $\mathbb{E}^{(t)}\Big[\mathrm{e}^{-tS_0^{(t)}(\alpha)}\, \mathbb{1}_{\{S_0^{(t)}(\alpha)>0\}}\Big]$ as $t \to \infty$ (this is done in Proposition 4.8). To accomplish this, a local central limit theorem (CLT) for $S_0^{(t)}(\alpha)$ under $P^{(t)}$ is established in Theorem 4.10. Recall that CLT-type results are commonly employed in the study of large deviation probabilities.

Step 4. Finally, substitute $t = t(x)$ into the asymptotic expansions obtained in Steps 2 and 3. This yields the desired results, stated in Theorems 1.2 and 1.3.

The plan just outlined is analogous to the plan worked out in [10]. As far as a realization of the plan is concerned, we note that Lemmas 4.3 and 4.11 are new. Also, modifications of the proof given in [10] are needed at all passages affected by the asymptotic behavior at 0 of ψ and its derivatives.

3. Proof of Proposition 1.1

By Kolmogorov's three series theorem, see, for instance, p. 63 in [5], the random Dirichlet series $S(a)$ converges a.s. if, and only if, the series

$$\sum_{k\geq 1} \mathbb{E}[k^{-\alpha}\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}}], \qquad \sum_{k\geq 1} \mathrm{Var}[k^{-\alpha}\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}}]$$

$$\sum_{k\geq 1} \mathbb{P}\{|k^{-\alpha}\eta_k| > \varepsilon\}$$

converge for all/some $\varepsilon > 0$.

The last series converges if, and only if, $\alpha\beta > 1$. This follows from

$$\mathbb{P}\{|k^{-\alpha}\eta_k| > \varepsilon\} = \mathbb{P}\{|\eta| > \varepsilon k^{\alpha}) \ \sim \ \mathbb{P}(-\eta > \varepsilon k^{\alpha}) \ \sim \ c\varepsilon^{-\beta}k^{-\alpha\beta}, \quad k \to \infty.$$

The other two series also converge if, and only if, $\alpha\beta > 1$. Indeed,

$$-\mathbb{E}[k^{-\alpha}\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}}] = k^{-\alpha}\int_{(\varepsilon k^{\alpha}, \infty)} u \mathrm{d}\mathbb{P}\{-\eta \leq u\} \ \sim \ c\beta(\beta-1)^{-1}\varepsilon^{1-\beta}k^{-\alpha\beta}, \quad k \to \infty,$$

where the first equality holds for $k \in \mathbb{N}$ satisfying $\varepsilon k^{\alpha} > b$, and the asymptotic relation is justified by Karamata's theorem (Theorem 1.6.5 in [3]).

Invoking another form of Karamata's theorem (Theorem 1.6.4 in [3]) we infer

$$\mathbb{E}[k^{-2\alpha}(\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}})^2] = k^{-2\alpha}\int_{[0,\, \varepsilon k^{\alpha}]} u^2 \mathrm{d}\mathbb{P}\{|\eta| \leq u\} \sim c\beta(2-\beta)^{-1}\varepsilon^{2-\beta}k^{-\alpha\beta}, \quad k \to \infty,$$

whence

$$\mathrm{Var}[k^{-\alpha}\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}}] = \mathbb{E}[k^{-2\alpha}(\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}})^2] - \left(\mathbb{E}[k^{-\alpha}\eta_k \, \mathbb{1}_{\{|k^{-\alpha}\eta_k|\leq \varepsilon\}}]\right)^2 \sim \ c\beta(2-\beta)^{-1}\varepsilon^{2-\beta}k^{-\alpha\beta}, \quad k \to \infty.$$

4. Auxiliary tools

The following three lemmas provide the asymptotic behaviour of $\psi(t)$ and its derivatives as $t \to \infty$ and $t \to 0+$.

Lemma 4.1. *Assume, that $\eta \leq b$ a.s. with $\mathbb{P}\{\eta = b\} = \theta \in (0,1)$. Then, as $t \to \infty$,*
(a) $\psi(t) = bt + \log\theta + o(1)$;
(b) $t\psi'(t) = bt + o(1)$ and $L'(t) = o(1/t)$, where $L(t) := \log \mathbb{E}[\mathrm{e}^{-t(b-\eta)}]$ for $t \geq 0$;
(c) $\lim_{t\to\infty} t^2\psi''(t) = \lim_{t\to\infty} t^3|\psi'''(t)| = 0$.

Lemma 4.2. *Assume, that $\eta \leq b$ a.s. and $\mathbb{P}\{b - \eta \leq x\} \sim \lambda x^r$ as $x \to 0+$ for positive λ and r. Then, as $t \to \infty$,*
(a) $\psi(t) = bt - r\log t + \log(\lambda\Gamma(r+1)) + o(1)$;
(b) $L'(t) = -rt^{-1} + o(1/t)$ and $\psi'(t) = b - rt^{-1} + o(1/t)$;
(c) $\psi''(t) = L''(t) = rt^{-2} + o(1/t^2)$ and $\psi'''(t) = L'''(t) = -2rt^{-3} + o(1/t^3)$.

The proofs of Lemmas 4.1 and 4.2 can be found on p. 2713-2714 in [10].

Lemma 4.3. *Assume that $\eta \le b$ a.s. for some $b > 0$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1, 2)$. Then, as $t \to 0+$,*

$$\psi(t) \sim \frac{c\Gamma(2-\beta)t^{\beta}}{\beta-1}, \qquad \psi'(t) \sim \frac{c\beta\Gamma(2-\beta)t^{\beta-1}}{\beta-1}, \qquad \psi''(t) \sim c\beta\Gamma(2-\beta)t^{\beta-2}$$

and

$$\psi'''(t) \sim -c\beta\Gamma(3-\beta)t^{\beta-3}.$$

Proof. Put $\eta_b := b - \eta$ and note that $\eta_b \ge 0$ a.s. and $\mathbb{E}[\eta_b] = b$. To investigate the asymptotic behavior of ψ we use a representation

$$\psi(t) = \log(\mathrm{e}^{bt}\mathbb{E}[\mathrm{e}^{-t\eta_b}]) = bt + \log \mathbb{E}[\mathrm{e}^{-t\eta_b}], \quad t \in \mathbb{R}.$$

Since $\mathbb{P}\{\eta_b > x\} \sim cx^{-\beta}$ as $x \to \infty$, an application of Theorem 8.1.6 in [3] yields

$$1 - \mathbb{E}[\mathrm{e}^{-t\eta_b}] - bt \sim c\frac{\Gamma(2-\beta)}{\beta-1}t^{\beta}, \quad t \to 0+$$

and thereupon

$$\psi(t) = bt - 1 + \mathbb{E}[\mathrm{e}^{-t\eta_b}] + O((1 - \mathbb{E}[\mathrm{e}^{-t\eta_b}])^2) \ \sim \ c\frac{\Gamma(2-\beta)}{\beta-1}t^{\beta}, \quad t \to 0+$$

because

$$1 - \mathbb{E}[\mathrm{e}^{-t\eta_b}] \ \sim \ bt, \quad t \to 0+. \tag{4.1}$$

By the same theorem,

$$b - \mathbb{E}[\eta_b \mathrm{e}^{-t\eta_b}] \ \sim \ \frac{c\beta\Gamma(2-\beta)}{\beta-1}t^{\beta-1}, \quad t \to 0+,$$

whence

$$\psi'(t) = b - \frac{\mathbb{E}[\eta_b \mathrm{e}^{-t\eta_b}]}{\mathbb{E}[\mathrm{e}^{-t\eta_b}]} = \frac{b - \mathbb{E}[\eta_b \mathrm{e}^{-t\eta_b}] - b(1 - \mathbb{E}[\mathrm{e}^{-t\eta_b}])}{\mathbb{E}[\mathrm{e}^{-t\eta_b}]} \ \sim \ \frac{c\beta\Gamma(2-\beta)}{\beta-1}t^{\beta-1}, \quad t \to 0+.$$

having utilized (4.1).

Further,

$$\psi''(t) = \frac{\mathbb{E}[\eta_b^2 \mathrm{e}^{-t\eta_b}]}{\mathbb{E}[\mathrm{e}^{-t\eta_b}]} - \Big(\frac{\mathbb{E}[\eta_b \mathrm{e}^{-t\eta_b}]}{\mathbb{E}[\mathrm{e}^{-t\eta_b}]}\Big)^2. \tag{4.2}$$

Since

$$b - \mathbb{E}[\eta_b \mathrm{e}^{-t\eta_b}] = \int_0^t \mathbb{E}[\eta_b^2 \mathrm{e}^{-y\eta_b}]\mathrm{d}y \sim \frac{c\beta\Gamma(2-\beta)}{\beta-1}t^{\beta-1}, \quad t \to 0+,$$

and $y \to \mathbb{E}[\eta_b^2 \mathrm{e}^{-y\eta_b}]$ is a nonincreasing function we infer $\mathbb{E}[\eta_b^2 \mathrm{e}^{-t\eta_b}] \sim c\beta\Gamma(2-\beta)t^{\beta-2}, t \to 0+$ by a variant of the monotone density theorem (Theorem 1.7.2(b) in [3]). Using (4.2) we conclude that

$$\psi''(t) \sim c\beta\Gamma(2-\beta)t^{\beta-2}, \quad t \to 0+.$$

Finally, $\mathbb{E}[\eta_b^2 \mathrm{e}^{-t\eta_b}] = \int_t^\infty \mathbb{E}[\eta_b^3 \mathrm{e}^{-y\eta_b}]\mathrm{d}y \sim c\beta\Gamma(2-\beta)t^{\beta-2}$ as $t \to 0+$ entails

$$-\psi'''(t) \sim \mathbb{E}[\eta_b^3 \mathrm{e}^{-t\eta_b}] \ \sim \ c\beta\Gamma(3-\beta)t^{\beta-3}, \quad t \to 0+$$

by another version of the monotone density theorem. □

From now on, we follow the steps outlined in Section 2. Step 1 is realized by the following propositions.

Proposition 4.4. Assume that $\eta \le b$ a.s. for some $b > 0$ with $\mathbb{P}\{\eta = b\} = \theta \in (0,1)$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1,2)$. If $\alpha = 1$, then

$$\sum_{k\ge1} k^{-1}\psi'(t/k) = b\log t + q + o(1/t), \quad t \to \infty, \tag{4.3}$$

where $q = b\gamma_1 + \int_0^1 x^{-1}\psi'(x)\mathrm{d}x + \int_1^\infty x^{-1}(\psi'(x)-b)\mathrm{d}x \in \mathbb{R}$. For each $x \ge 0$, the equation $\sum_{k\ge1} k^{-1}\psi'(t/k) = x$ has a unique solution $t = t(x)$ satisfying

$$t(x) = \exp((x-q)/b) + o(1), \quad x \to \infty. \tag{4.4}$$

If $\alpha \in (1/\beta, 1)$, then

$$\sum_{k\ge1} k^{-\alpha}\psi'(t/k^\alpha) = r_{\alpha,\beta}t^{-1+1/\alpha} + b\gamma_\alpha + o(1/t), \quad t \to \infty, \tag{4.5}$$

where $r_{\alpha,\beta} := \alpha\sigma_{\alpha,\beta}^2/(1-\alpha) \in (0,\infty)$. For each $x \ge 0$, the equation $\sum_{k\ge1} k^{-\alpha}\psi'(t/k^\alpha) = x$ has a unique solution $t = t(x)$ satisfying

$$t(x) = (r_{\alpha,\beta}^{-1}(x - b\gamma_\alpha) + o(x^{-\alpha/(1-\alpha)}))^{\alpha/(1-\alpha)}, \quad x \to \infty. \tag{4.6}$$

Proof. Fix any $\delta > \beta/(\alpha\beta - 1) > 1$. According to Lemma 4.3, $\psi'(t) \sim c\beta(\beta-1)^{-1}\Gamma(2-\beta)t^{\beta-1}$ as $t \to 0+$. Therefore,

$$\sum_{k\ge\lfloor t^\delta\rfloor+1} k^{-\alpha}\psi'(t/k^\alpha) = O\left(t^{\beta-1}\sum_{k\ge\lfloor t^\delta\rfloor+1} k^{-\alpha\beta}\right) = O(t^{\beta-1-(\alpha\beta-1)\delta}) = o(1/t), \quad t \to \infty.$$

Note that $\psi'(t) = b + L'(t)$, with $L(t) = \log\mathbb{E}[\mathrm{e}^{-t(b-\eta)}]$ for $t \ge 0$, and thus

$$\sum_{k=1}^{\lfloor t^\delta\rfloor} k^{-\alpha}\psi'(t/k^\alpha) = b\sum_{k=1}^{\lfloor t^\delta\rfloor} k^{-\alpha} + \sum_{k=1}^{\lfloor t^\delta\rfloor} k^{-\alpha}L'(t/k^\alpha) := A_\alpha(t) + B_\alpha(t).$$

By Theorem 3.2 (a,b) on p. 55 of [2], we have, in the case $\alpha = 1$,

$$A_1(t) = b\log(\lfloor t^\delta\rfloor) + b\gamma_1 + O(t^{-\delta}), \quad t \to \infty, \tag{4.7}$$

whereas for $\alpha \in (1/\beta, 1)$,

$$A_\alpha(t) = b(1-\alpha)^{-1}(\lfloor t^\delta\rfloor)^{1-\alpha} + b\gamma_\alpha + O(t^{-\alpha\delta}), \quad t \to \infty, \tag{4.8}$$

where γ_α denotes the Euler–Mascheroni constant (see (1.1) and (1.2)).

We intend to use Euler-Maclaurin summation formula. Let $f : [m,n] \to \mathbb{R}$ be a twice continuously differentiable function. A specialization of formula (9.78) on p. 460 in [7] yields

$$\sum_{j=m}^{n} f(j) = \int_m^n f(x)\mathrm{d}x + (f(n)+f(m))/2 + (f'(n)-f'(m))/12 + R_{m,n}, \tag{4.9}$$

where $|R_{m,n}| \le (1/12)\int_m^n |f''(x)|\mathrm{d}x$. For each $t > 0$, put $f_t(x) := x^{-\alpha}L'(t/x^\alpha)$ for $x > 0$. For future references, we note that

$$f_t'(x) = -\alpha\Big(x^{-(\alpha+1)}L'(t/x^\alpha) + (t/x^{2\alpha+1})L''(t/x^\alpha)\Big)$$

and

$$f_t''(x) = \alpha\Big((\alpha+1)x^{-(\alpha+2)}L'(t/x^\alpha) + (3\alpha+1)tx^{-(2\alpha+2)}L''(t/x^\alpha) + \alpha t^2x^{-(3\alpha+2)}L'''(t/x^\alpha)\Big).$$

According to (4.9) with $f = f_t$, $m = 1$ and $n = \lfloor t^\delta \rfloor$,

$$B_\alpha(t) = \int_1^{\lfloor t^\delta \rfloor} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x + \big(L'(t) + (\lfloor t^\delta \rfloor)^{-\alpha} L'(t/(\lfloor t^\delta \rfloor)^\alpha)\big)/2 \tag{4.10}$$
$$+\alpha\Big(L'(t)+tL''(t)-(\lfloor t^\delta \rfloor)^{-(\alpha+1)} L'(t/(\lfloor t^\delta \rfloor)^\alpha)-(t/(\lfloor t^\delta \rfloor)^{2\alpha+1})L''(t/(\lfloor t^\delta \rfloor)^\alpha)\Big)/12+R(t),$$

where $R(t) \le (1/12)\int_1^{\lfloor t^\delta \rfloor} |f_t''(x)|\mathrm{d}x$.

Using the fact that $\lim_{u\to 0+} L'(u) = -b$ and recalling that the choice of δ guarantees $\alpha\delta > (\alpha\beta - 1)\delta > \beta > 1$ (since $\alpha \in (1/\beta, 1]$), we obtain

$$(\lfloor t^\delta \rfloor)^{-\alpha} L'\big(t/(\lfloor t^\delta \rfloor)^\alpha\big) = O(t^{-\alpha\delta}) = o(1/t), \quad t \to \infty.$$

As a consequence,

$$(\lfloor t^\delta \rfloor)^{-(\alpha+1)} L'\big(t/(\lfloor t^\delta \rfloor)^\alpha\big) = o(1/t), \quad t \to \infty.$$

According to Lemma 4.3,

$$(t/(\lfloor t^\delta \rfloor)^{2\alpha+1})L''(t/(\lfloor t^\delta \rfloor)^\alpha) = (t/(\lfloor t^\delta \rfloor)^{2\alpha+1})\psi''(t/(\lfloor t^\delta \rfloor)^\alpha)$$
$$= O(t^{\beta-1-\delta(\alpha\beta+1)}) = o(1/t), \quad t \to \infty.$$

The last equality is justified as follows:

$$\beta - 1 - \delta(\alpha\beta+1) < \beta - 1 - \beta(\alpha\beta+1)/(\alpha\beta-1)$$
$$= (1-\alpha\beta-2\beta)/(\alpha\beta-1) < (1-\alpha\beta)/(\alpha\beta-1) = -1.$$

Substituting $y = t/x^\alpha$ yields

$$\alpha\int_1^\infty x^{-(\alpha+2)} L'(t/x^\alpha)\mathrm{d}x = t^{-(1+1/\alpha)}\int_1^\infty (t/x^\alpha)^{1/\alpha}(\alpha t/x^{\alpha+1})L'(t/x^\alpha)\mathrm{d}x$$
$$= t^{-(1+1/\alpha)}\int_0^t y^{1/\alpha} L'(y)\mathrm{d}y.$$

If $\int_0^\infty y^{1/\alpha}L'(y)\mathrm{d}y > -\infty$, then trivially the right-hand side is $o(1/t)$ as $t \to \infty$. If the integral diverges, then using L'Hôpital's rule together with Lemma 4.1(b) we infer

$$t^{-(1+1/\alpha)}\int_0^t y^{1/\alpha}L'(y)\mathrm{d}y \sim \alpha(\alpha+1)^{-1}L'(t) = o(1/t), \quad t \to \infty.$$

Thus,

$$\alpha\int_1^\infty x^{-(\alpha+2)} L'(t/x^\alpha)\mathrm{d}x = o(1/t), \quad t \to \infty.$$

Analogously,

$$\alpha t\int_1^\infty x^{-(2\alpha+2)} L''(t/x^\alpha)\mathrm{d}x = t^{-(1+1/\alpha)}\int_1^\infty (t/x^\alpha)^{1+1/\alpha}(\alpha t/x^{\alpha+1})L''(t/x^\alpha)\mathrm{d}x$$
$$= t^{-(1+1/\alpha)}\int_0^t y^{1+1/\alpha} L''(y)\mathrm{d}y = o(1/t), \quad t \to \infty$$

and

$$\alpha t^2\int_1^\infty x^{-(3\alpha+2)} L'''(t/x^\alpha)\mathrm{d}x = t^{-(1+1/\alpha)}\int_1^\infty (t/x^\alpha)^{2+1/\alpha}(\alpha t/x^{\alpha+1})L'''(t/x^\alpha)\mathrm{d}x$$
$$= t^{-(1+1/\alpha)}\int_0^t y^{2+1/\alpha} L'''(y)\mathrm{d}y = o(1/t), \quad t \to \infty$$

having utilized Lemma 4.1(c) in the cases where the corresponding integrals diverge. Combining fragments together we infer $R(t) = o(1/t)$ as $t \to \infty$ and thereupon

$$B_\alpha(t) = \int_1^{\lfloor t^\delta \rfloor} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x + o(1/t), \quad t \to \infty.$$

Noting that $\alpha\delta > 1$ and $\lim_{u\to 0+} L'(u) = -b$, we deduce that

$$\int_{\lfloor t^\delta \rfloor}^{t^\delta} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x = O(t^{-\alpha\delta}) = o(1/t).$$

This entails

$$B_\alpha(t) = \int_1^{t^\delta} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x + o(1/t), \quad t \to \infty. \tag{4.11}$$

Using Lemma 4.3, we have $\lim_{y\to 0+} y^{1-\beta}\psi'(y) = \frac{c\beta\Gamma(2-\beta)}{\beta-1}$. Therefore, the integral $c_0(\alpha) := \int_0^1 y^{-1/\alpha}\psi'(y)\mathrm{d}y$ is well defined and

$$\int_0^{t^{1-\delta\alpha}} y^{-1/\alpha}\psi'(y)\mathrm{d}y = O(t^{-(\delta\alpha-1)(\beta-1/\alpha)}) = o(t^{-1/\alpha}), \quad t \to \infty \tag{4.12}$$

since by the choice of δ the inequality $(\delta\alpha - 1)(\beta - 1/\alpha) > 1/\alpha$ holds. Furthermore, noting that $L'(y) = o(1/y)$ as $y \to \infty$ (see Lemma 4.1(b)), we deduce that the integral $c_1(\alpha) = -\int_1^\infty y^{-1/\alpha} L'(y)\mathrm{d}y = -\int_1^\infty y^{-1/\alpha}(\psi'(y) - b)\mathrm{d}y$ is also well defined and $-\int_t^\infty y^{-1/\alpha} L'(y)\mathrm{d}y = o(t^{-1/\alpha})$ as $t \to \infty$. Finally,

$$\begin{aligned}\int_1^{t^\delta} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x &= \alpha^{-1}t^{-1+1/\alpha}\Big(\int_{t^{1-\delta\alpha}}^1 y^{-1/\alpha} L'(y)\mathrm{d}y + \int_1^t y^{-1/\alpha} L'(y)\mathrm{d}y\Big) \\ &= \alpha^{-1}t^{-1+1/\alpha}\Big(-b\int_{t^{1-\delta\alpha}}^1 y^{-1/\alpha}\mathrm{d}y + \int_0^1 y^{-1/\alpha}\psi'(y)\mathrm{d}y \\ &\quad - \int_0^{t^{1-\delta\alpha}} y^{-1/\alpha}\psi'(y)\mathrm{d}y + \int_1^\infty y^{-1/\alpha} L'(y)\mathrm{d}y - \int_t^\infty y^{-1/\alpha} L'(y)\mathrm{d}y\Big).\end{aligned} \tag{4.13}$$

When $\alpha = 1$, we obtain

$$\int_1^{t^\delta} x^{-1} L'(t/x)\mathrm{d}x = b\log t - b\log(\lfloor t^\delta \rfloor) + c_0(1) - c_1(1) + o(1/t), \quad t \to \infty$$

having utilized $0 \le \log t^\delta - \log\lfloor t^\delta \rfloor = O(t^{-\delta}) = o(1/t)$ as $t \to \infty$ (recall that $\delta > 1$). Combining this with (4.7) and (4.11) gives (4.3). For $\alpha \in (1/\beta, 1)$,

$$\begin{aligned}\int_1^{t^\delta} x^{-\alpha} L'(t/x^\alpha)\mathrm{d}x &= (b(1-\alpha)^{-1} + \alpha^{-1}(c_0(\alpha) - c_1(\alpha)))t^{-1+1/\alpha} \\ &\quad - b(1-\alpha)^{-1}(\lfloor t^\delta \rfloor)^{1-\alpha} + o(1/t)\end{aligned}$$

as $t \to \infty$. Here, we have used $0 \le (t^\delta)^{1-\alpha} - (\lfloor t^\delta \rfloor)^{1-\alpha} = O(t^{-\alpha\delta}) = o(1/t)$ as $t \to \infty$, which is secured by $\alpha\delta > 1$. Recall that $\sigma^2_{\alpha,\beta} = (1-\alpha)\alpha^{-3}\int_0^\infty y^{-1-1/\alpha}\psi(y)\mathrm{d}y$. This entails

$$\begin{aligned}&b(1-\alpha)^{-1} + \alpha^{-1}(c_0(\alpha) - c_1(\alpha)) \\ &\quad = \alpha^{-1}\int_0^\infty y^{-1/\alpha}\psi'(y)\mathrm{d}y = \alpha^{-2}\int_0^\infty y^{-1-1/\alpha}\psi(y)\mathrm{d}y = \alpha\sigma^2_{\alpha,\beta}/(1-\alpha) = r_{\alpha,\beta}.\end{aligned} \tag{4.14}$$

Combining these results with (4.8) and (4.11), we obtain (4.5).

The function $t \to \sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha)$ is continuous, strictly increasing on $[0,\infty)$ and equal to 0 at 0. Hence, for each $x \geq 0$, the equation $\sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha) = x$ admits a unique solution $t = t(x)$. To analyze its asymptotic behavior when $\alpha = 1$, we replace $\sum_{k\geq 1} k^{-1}\psi'(t/k)$ with its asymptotic expansion derived in formula (4.3), thereby obtaining

$$b\log t + q + o(1/t) = x. \tag{4.15}$$

Then

$$t(x) = \exp((x-q)/b)(1+\varepsilon(x)),$$

where ε satisfies $\lim_{x\to\infty}\varepsilon(x) = 0$. Substituting this expression into (4.15) we obtain $\varepsilon(x) = o(\mathrm{e}^{-x/b})$ as $x \to \infty$ which in turn yields (4.4).

Now let $\alpha \in (1/\beta, 1)$. Using (4.5) and writing

$$r_{\alpha,\beta}t^{-1+1/\alpha} + b\gamma_\alpha + o(1/t) = x \tag{4.16}$$

we deduce that $t(x) = (r^{-1}_{\alpha,,\beta}(x - b\gamma_\alpha) + \delta(x))^{\alpha/(1-\alpha)}$ for some function δ satisfying $\lim_{x\to\infty}\delta(x) = 0$. Substituting this into (4.16), we find that $\delta(x) = o(x^{-\alpha/(1-\alpha)})$ as $x \to \infty$. Hence, representation (4.6) holds true.

□

Proposition 4.5. Assume, that $\eta \leq b$ a.s. for some $b > 0$, $\mathbb{P}\{b - \eta \leq x\} \sim \lambda x^r$ as $x \to 0+$ for some positive λ and r, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ for some $c > 0$ and $\beta \in (1,2)$.
If $\alpha = 1$, then

$$\sum_{k\geq 1} k^{-1}\psi'(t/k) = b\log t + q + rt^{-1}/2 + o(1/t), \quad t \to \infty \tag{4.17}$$

with the same q as in Proposition 4.4. For each $x \geq 0$, the equation $\sum_{k\geq 1} k^{-1}\psi'(t/k) = x$ has a unique solution $t = t(x)$ satisfying

$$t(x) = \exp((x-q)/b) - r/(2b) + o(1), \quad x \to \infty. \tag{4.18}$$

If $\alpha \in (1/\beta, 1)$, then

$$\sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha) = r_{\alpha,\beta}t^{-1+1/\alpha} + b\gamma_\alpha + rt^{-1}/2 + o(1/t), \quad t \to \infty \tag{4.19}$$

with the same $r_{\alpha,\beta}$ as in Proposition 4.4. For each $x \geq 0$, the equation $\sum_{k\geq 1} k^{-\alpha}\psi'(t/k^\alpha) = x$ has a unique solution $t = t(x)$ satisfying

(4.20)

$$t(x) = (r^{-1}_{\alpha,\beta}(x - b\gamma_\alpha) - (rr^{(2\alpha-1)/(1-\alpha)}_{\alpha,\beta}/2)x^{-\alpha/(1-\alpha)} + o(x^{-\alpha/(1-\alpha)}))^{\alpha/(1-\alpha)}, \quad x \to \infty.$$

The proof of Proposition 4.5 is similar to the proof of Proposition 5.2 on p. 2721 in [10], hence omitted.

We now move to Step 2.

Proposition 4.6. Assume that $\eta \leq b$ a.s. with $\mathbb{P}\{\eta = b\} = \theta \in (0,1)$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1,2)$. Then for $\alpha \in (1/\beta, 1]$, as $t \to \infty$,

$$\sum_{k\geq 1}\big(\psi(t/k^\alpha) - (t/k^\alpha)\psi'(t/k^\alpha)\big) = -\alpha\sigma^2_{\alpha,\beta}t^{1/\alpha} - (1/2)\log\theta + o(1) \tag{4.21}$$

with $\sigma^2_{\alpha,\beta}$ as defined in Theorem 1.2 (in particular, $\sigma^2_1 = b$).

Proof. For fixed $t>0$, put $f_t(x):=\psi(t/x^\alpha)-(t/x^\alpha)\psi'(t/x^\alpha)$ for $x>0$. Then,

$$\lim_{x\to\infty} f_t(x)=\lim_{x\to\infty} f_t'(x)=0. \tag{4.22}$$

The first relation follows from $\psi(0)=0$ and $\lim_{y\to 0+} y\psi'(y)=0$ (see Lemma 4.3), whereas the second one follows from

$$f_t'(x)=\frac{\alpha t^2}{x^{2\alpha+1}}\psi''(t/x^\alpha)\sim \frac{\alpha t^2 c\beta\Gamma(2-\beta)t^{\beta-2}}{x^{2\alpha+1}(x^\alpha)^{\beta-2}}=\frac{\alpha t^\beta c\beta\Gamma(2-\beta)}{x^{\alpha\beta+1}}\to 0,\quad x\to\infty$$

by another appeal to Lemma 4.3.

The remainder of the proof of Proposition 4.6 is similar to the corresponding fragment of the proof of Proposition 5.3 on pp. 2721-2723 in [10], hence omitted. The constant σ_α^2 in [10] has to be replaced by our $\sigma_{\alpha,\,\beta}^2$.

□

Proposition 4.7. Assume, that $\eta\le b$ a.s., $\mathbb{P}\{b-\eta\le x\}\sim\lambda x^r$ as $x\to 0+$ for positive λ and r, $\mathbb{E}[\eta]=0$ and $\mathbb{P}\{-\eta>x\}\sim cx^{-\beta}$ as $x\to\infty$ for some $c>0$ and $\beta\in(1,2)$. Then for $\alpha\in(1/\beta,1]$, as $t\to\infty$,

$$\begin{aligned}&\sum_{k\ge 1}\big(\psi(t/k^\alpha)-(t/k^\alpha)\psi'(t/k^\alpha)\big)\\ &\quad=-\alpha\sigma_{\alpha,\,\beta}^2 t^{1/\alpha}+(r/2)\log t+(r/2)(\alpha\log(2\pi)-1)-(1/2)\log(\lambda\Gamma(r+1))+o(1)\end{aligned} \tag{4.23}$$

with $\sigma_{\alpha,\,\beta}^2$ as defined in Theorem 1.2.

Proof. Fix an arbitrary $\delta>\beta/(\alpha\beta-1)$. By Lemma 4.3, as $t\to 0+$, $\psi(t)\sim\frac{c\Gamma(2-\beta)t^\beta}{\beta-1}$ and $\psi'(t)\sim\frac{c\beta\Gamma(2-\beta)t^{\beta-1}}{\beta-1}$. Using the inequalities

$$-t\sum_{k\ge\lfloor t^\delta\rfloor+1}k^{-\alpha}\psi'(t/k^\alpha)\le\sum_{k\ge\lfloor t^\delta\rfloor+1}\big(\psi(t/k^\alpha)-(t/k^\alpha)\psi'(t/k^\alpha)\big)\le\sum_{k\ge\lfloor t^\delta\rfloor+1}\psi(t/k^\alpha)$$

we obtain

$$\sum_{k\ge\lfloor t^\delta\rfloor+1}\big(\psi(t/k^\alpha)-(t/k^\alpha)\psi'(t/k^\alpha)\big)=o(1),\quad t\to\infty.$$

Indeed,

$$\sum_{k\ge\lfloor t^\delta\rfloor+1}\psi(t/k^\alpha)=O\left(t^\beta\sum_{k\ge\lfloor t^\delta\rfloor+1}k^{-\alpha\beta}\right)=O(t^{\beta-\delta(\alpha\beta-1)})=o(1),\quad t\to\infty,$$

and a similar asymptotic estimate applies to the lower bound involving ψ'.

The remainder of the proof of Proposition 4.6 is identical, up to some notation, to the corresponding part of the proof of Proposition 5.4 on pp. 2723-2725 in [10], hence omitted.

□

Step 3 is carried out with the help of the following result.

Proposition 4.8. Assume, that $\eta\le b$ a.s. for some $b>0$, $\mathbb{E}[\eta]=0$ and $\mathbb{P}\{-\eta>x\}\sim cx^{-\beta}$ as $x\to\infty$ for some $c>0$ and $\beta\in(1,2)$. Then

$$\lim_{t\to\infty}t^{1/(2\alpha)}\mathbb{E}^{(t)}\left[\mathrm{e}^{-tS_0^{(t)}(\alpha)}\,\mathbb{1}_{\{S_0^{(t)}(\alpha)>0\}}\right]=(2\pi\sigma_{\alpha,\,\beta}^2)^{-1/2}.$$

A slight extension of formula (2.1) is given below: for any bounded measurable function $g : \mathbb{R} \to \mathbb{C}$

(4.24)
$$\mathbb{E}^{(t)}[g(\eta_k)] = \frac{\mathbb{E}[\mathrm{e}^{tS(\alpha)} g(\eta_k)]}{\mathbb{E}[\mathrm{e}^{tS(\alpha)}]} = \frac{\mathbb{E}[\mathrm{e}^{t\eta_k/k^\alpha} g(\eta_k)] \prod_{j\neq k} \mathbb{E}[\mathrm{e}^{t\eta_j/j^\alpha}]}{\mathbb{E}[\mathrm{e}^{t\eta_k/k^\alpha}] \prod_{j\neq k} \mathbb{E}[\mathrm{e}^{t\eta_j/j^\alpha}]} = \frac{\mathbb{E}[\mathrm{e}^{t\eta_k/k^\alpha} g(\eta_k)]}{\mathbb{E}[\mathrm{e}^{t\eta_k/k^\alpha}]},$$

where the second equality follows from the independence of $\eta_1, \eta_2, \ldots$. Recall that $S(\alpha) = \sum_{k\geq 1} k^{-\alpha}\eta_k$ and that, under $\mathbb{P}^{(t)}$, we have $S_0^{(t)}(\alpha) = S(\alpha) - \mathbb{E}^{(t)}[S(\alpha)]$.

Lemma 4.9. *Under the assumptions of Theorem 1.2,*

$$\lim_{t\to\infty} \mathbb{E}^{(t)}[\mathrm{e}^{iut^{1-1/(2\alpha)} S_0^{(t)}(\alpha)}] = \mathrm{e}^{-\sigma^2_{\alpha,\beta} u^2/2}, \quad u \in \mathbb{R} \tag{4.25}$$

with $\sigma^2_{\alpha,\beta}$ as defined in Theorem 1.2.

Proof. It suffices to show that, for every $u \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}^{(t)}[\mathrm{e}^{ut^{1-1/(2\alpha)} S_0^{(t)}(\alpha)}] = \mathrm{e}^{\sigma^2_{\alpha,\beta} u^2/2}. \tag{4.26}$$

It was shown in the proof of Lemma 5.6 in [10] (pp. 2726-2727), that the relation (4.26) follows if we can prove that

$$\lim_{t\to\infty} t^{2-1/\alpha} \sum_{k\geq 1} k^{-2\alpha} \psi''(t/k^\alpha) = \sigma^2_{\alpha,\beta} \in (0,\infty) \tag{4.27}$$

and that, for each fixed $u \in \mathbb{R}$,

$$\lim_{t\to\infty} t^{3-3/(2\alpha)} \sum_{k\geq 1} k^{-3\alpha} |\psi'''((t + v_k u t^{1-1/(2\alpha)})/k^\alpha)| = 0. \tag{4.28}$$

Proof of (4.27). Define the function h by

$$h(x) = \begin{cases} x^{-2\alpha}\psi''(x^{-\alpha}), & x > 0, \\ 0, & x = 0. \end{cases}$$

We intend to show that the function h is directly Riemann integrable (dRi) on $[0,\infty)$. The function ψ'' is continuous on $(0,\infty)$. In view of Lemma 4.1(c), $\lim_{x\to 0+} h(x) = 0$. This entails that the function h is continuous on $[0,\infty)$. By the same lemma, $h(x) \leq C_1$ for some $C_1 > 0$ and all $x \in (0,1]$. By Lemma 4.3, $h(x) \leq C_2 x^{-\alpha\beta}$ for some $C_2 > 0$ and all $x > 1$. Hence, the function h is bounded on $[0,\infty)$ and $0 \leq h(x) \leq h_1(x)$ for $x > 0$, where $h_1(x) := C_1 \mathbb{1}_{[0,1]}(x) + C_2 x^{-\alpha\beta} \mathbb{1}_{(1,\infty)}(x)$ for $x \geq 0$. Being the sum of two nonnegative dRi functions, h_1 is dRi on $[0,\infty)$. Hence, so is h as a nonnegative Riemann integrable function bounded from above by a dRi function.

As a result, by a Riemann approximation,

$$\begin{aligned} \lim_{t\to\infty} t^{-1/\alpha} \sum_{k\geq 1} t^2 k^{-2\alpha} \psi''(tk^{-\alpha}) &= \int_0^\infty x^{-2\alpha}\psi''(x^{-\alpha})\mathrm{d}x \\ &= \alpha^{-1} \int_0^\infty x^{1-1/\alpha}\psi''(x)\mathrm{d}x = \sigma^2_{\alpha,\beta} \in (0,\infty), \end{aligned}$$

which proves (4.27).

Proof of (4.28). Fix any $u \geq 0$; the argument for $u < 0$ is analogous. By Lemma 4.1(c), for sufficiently large $t_1 > 0$ there exists $c_1 > 0$ such that $|\psi'''(t)| \leq c_1 t^{-3}$ whenever $t \geq t_1$. Hence, for any positive integer $k \leq (t/t_1)^{1/\alpha} \leq ((t + v_k u t^{1-1/(2\alpha)})/t_1)^{1/\alpha}$ we have

$$|\psi'''((t + v_k u t^{1-1/(2\alpha)})/k^\alpha)| \leq c_1 k^{3\alpha}/(t + v_k u t^{1-1/(2\alpha)})^3 \leq c_1 k^{3\alpha} t^{-3}.$$

This implies

$$t^{3-3/(2\alpha)} \sum_{1\le k\le (t/t_1)^{1/\alpha}} k^{-3\alpha}|\psi'''((t+v_k u t^{1-1/(2\alpha)})/k^\alpha)| \\ \le c_1 t^{-3/(2\alpha)}(t/t_1)^{1/\alpha} \to 0, \quad t\to\infty.$$

Set $t_2 := 2t_1$. By Lemma 4.3, there exists a constant $c_2 > 0$ such that $|\psi'''(t)| \le c_2 t^{\beta-3}$ for all $t \in (0, t_2]$. For sufficiently large t and any $k \in \mathbb{N}$, we have $t + v_k u t^{1-1/(2\alpha)} \le 2t$. Therefore, for such t and any $k \ge (2t/t_2)^{1/\alpha} \ge ((t+v_k u t^{1-1/(2\alpha)})/t_2)^{1/\alpha}$

$$|\psi'''((t+v_k u t^{1-1/(2\alpha)})/k^\alpha)| \le c_2 k^{-\alpha(\beta-3)}(t+v_k u t^{1-1/(2\alpha)})^{\beta-3} \le c_2 k^{-\alpha(\beta-3)} t^{\beta-3}.$$

As a result,

$$t^{3-3/(2\alpha)} \sum_{k\ge (t/t_1)^{1/\alpha}} k^{-3\alpha}|\psi'''((t+v_k u t^{1-1/(2\alpha)})/k^\alpha)| \le c_2 t^{\beta-3/(2\alpha)} \sum_{k\ge (t/t_1)^{1/\alpha}} k^{-\alpha\beta} \\ = t^{\beta-3/(2\alpha)} O(t^{1/\alpha-\beta}) \to 0, \quad t\to\infty.$$

This completes the proof of (4.28), hence also, of the lemma.

□

Next, we present a local limit theorem.

Theorem 4.10. *Assume that $\eta \le b$ a.s. for some $b > 0$, $\mathbb{E}[\eta] = 0$ and $\mathbb{P}\{-\eta > x\} \sim cx^{-\beta}$ as $x \to \infty$ for some $c > 0$ and $\beta \in (1, 2)$. Then, for each $h > 0$,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| t^{1/(2\alpha)} \mathbb{P}^{(t)}\{S_0^{(t)}(\alpha) \in (xt^{-1+1/(2\alpha)}, xt^{-1+1/(2\alpha)} + ht^{-1}]\} - h n_{\alpha,\beta}(x) \right| = 0,$$

where $n_{\alpha,\beta}(x) := (2\pi\sigma^2_{\alpha,\beta})^{-1/2} \mathrm{e}^{-x^2/(2\sigma^2_{\alpha,\beta})}$ for $x \in \mathbb{R}$.

We need an auxiliary result.

Lemma 4.11. *Under the assumptions of Theorem 4.10, for all $t > 0$, there exist positive constants c and ρ such that*

$$\sup_{k\ge t^{1/\alpha}} \left| \mathbb{E}^{(t)}[\mathrm{e}^{\mathrm{i}u\eta_k}] \right| \le \mathrm{e}^{-cu^2}$$

for all $u \in \mathbb{R}$ satisfying $|u| \le \rho$.

Proof. It was shown in the proof of Lemma 5.8 in [10] that, under the assumptions $\mathbb{E}[\eta] = 0$ and $\mathbb{E}[\eta^2] < \infty$, there exist positive constants c_0 and ρ such that

$$\sup_{s\in[0,\,1]} \frac{|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}]|}{\mathbb{E}[\mathrm{e}^{s\eta}]} \ \le\ 1 - c_0 u^2 \tag{4.29}$$

for all $u \in \mathbb{R}$ satisfying $|u| \le \rho$.

Observe that $\mathbb{P}\{|\eta| \le b\} \in (0, 1)$. Indeed, this probability is not equal to 1 because $\mathbb{P}\{\eta \le -x\} > 0$ for all $x > 0$. Also, it is not equal to 0, for otherwise the variable η would be negative. This would contradict to $\mathbb{E}[\eta] = 0$. Put

$$\gamma := \inf_{s\in[0,1]} \frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]}$$

and note that $\gamma \in (0, 1)$. We will show that, under the present assumptions (actually, just under $\mathbb{P}\{|\eta| \le b\} \in (0, 1)$), that

$$\sup_{s\in[0,\,1]} \frac{|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}]|}{\mathbb{E}[\mathrm{e}^{s\eta}]} \ \le\ 1 - c_0\gamma u^2 \tag{4.30}$$

for all $u \in \mathbb{R}$ satisfying $|u| \le \rho$.

As a preparation for the proof of (4.30), let η_b be a random variable with the distribution function $\mathbb{P}\{\eta_b \le x\} = \mathbb{P}\{\eta \le x \big| |\eta| \le b\}$ for $x \in \mathbb{R}$. Put $\overline{\eta}_b := \eta_b - \mathbb{E}[\eta_b]$. Then $\mathbb{E}[\overline{\eta}_b] = 0$ and also $\mathbb{E}[\overline{\eta}_b^2] < \infty$ as a consequence of $\eta_b \in [-b, b]$ a.s. Hence, according to (4.29),

$$\sup_{s\in[0,\,1]} \frac{|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\overline{\eta}_b}]|}{\mathbb{E}[\mathrm{e}^{s\overline{\eta}_b}]} \;\le\; 1 - c_0\gamma u^2$$

for all $u \in \mathbb{R}$ satisfying $|u| \le \rho$. Since $\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta_b}] = \frac{1}{\mathbb{P}\{|\eta|\le b\}}\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]$ and $|\mathrm{e}^{-\mathrm{i}u\mathbb{E}[\eta_b]}| = 1$, the last supremum is equal to

$$\sup_{s\in[0,\,1]} \frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta_b}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta_b}]} = \sup_{s\in[0,\,1]} \frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]} \le 1 - c_0u^2, \quad |u| \le \rho.$$

Further, for $s \in [0,1]$ and $|u| \le \rho$,

$$\begin{aligned}
\frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta}]} &\le \frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]\big| + \big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}\,\mathbb{1}_{\{|\eta|> b\}}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta}]} \\
&\le \frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]} \cdot \frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]} + 1 - \frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]} \\
&\le (1 - c_0u^2)\frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]} + 1 - \frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]} = 1 - c_0\frac{\mathbb{E}[\mathrm{e}^{s\eta}\,\mathbb{1}_{\{|\eta|\le b\}}]}{\mathbb{E}[\mathrm{e}^{s\eta}]}u^2.
\end{aligned}$$

Taking $\sup_{s\in[0,\,1]}$ on both sides of the inequality we arrive at (4.30).

Finally, for $|u| \le \rho$,

$$\sup_{k\ge t^{1/\alpha}} \big|\mathbb{E}^{(t)}[\mathrm{e}^{\mathrm{i}u\eta_k}]\big| \le \sup_{s\in[0,\,1]} \frac{\big|\mathbb{E}[\mathrm{e}^{(\mathrm{i}u+s)\eta}]\big|}{\mathbb{E}[\mathrm{e}^{s\eta}]} \le 1 - cu^2 \le \mathrm{e}^{-cu^2},$$

where $c := c_0\gamma$.

□

The proof of Theorem 4.10 mimics the proof of Theorem 5.7 in [10]. One should just invoke Lemmas 4.9 and 4.11 of the present paper instead of Lemmas 5.6 and 5.8 in [10]. Proposition 4.8 follows easily from Theorem 4.10, details can be found in the proof of Proposition 5.5 in [10].

5. Proofs of the main results

Proof of Theorem 1.2. We only consider the case $\alpha \in (1/\beta, 1)$, as the case $\alpha = 1$ is simpler.

We begin with representation (2.4). Applying Proposition 4.8 together with the fact that $\lim_{x\to\infty} t(x) = +\infty$ we obtain

$$\mathbb{E}^{(t(x))}\Big[\mathrm{e}^{-t(x)S_0^{(t)}(\alpha)}\,\mathbb{1}_{\{S_0^{(t)}(\alpha)>0\}}\Big] \sim \frac{1}{(2\pi\sigma^2_{\alpha,\,\beta}(t(x))^{1/\alpha})^{1/2}}, \quad x \to \infty.$$

Using (4.6) we deduce

$$\begin{aligned}
(5.1)\quad &(t(x))^{1/\alpha} - (r^{-1}_{\alpha,\,\beta}(x - b\gamma_\alpha))^{1/(1-\alpha)} \\
&= (r^{-1}_{\alpha,\,\beta}(x - b\gamma_\alpha) + o(x^{-\alpha/(1-\alpha)}))^{1/(1-\alpha)} - (r^{-1}_{\alpha,\,\beta}(x - b\gamma_\alpha))^{1/(1-\alpha)} \\
&= O(x^{1/(1-\alpha)})o(x^{-1/(1-\alpha)}) = o(1), \quad x \to \infty.
\end{aligned}$$

This entails

$$\mathbb{E}^{(t(x))}\Big[\mathrm{e}^{-t(x)S_0^{(t)}(\alpha)}\,\mathbb{1}_{\{S_0^{(t)}(\alpha)>0\}}\Big] \sim \left(\frac{r_{\alpha,\beta}^{1/(1-\alpha)}}{2\pi\sigma_{\alpha,\beta}^2 x^{1/(1-\alpha)}}\right)^{1/2}, \quad x\to\infty. \tag{5.2}$$

According to Proposition 4.6,

$$\exp\left(\sum_{k\ge 1}\big(\psi(t(x)/k^\alpha)-(t(x)/k^\alpha)\psi'(t(x)/k^\alpha)\big)\right) \sim \frac{1}{\theta^{1/2}}\exp\Big(-\alpha\sigma_{\alpha,\beta}^2(t(x))^{1/\alpha}\Big)$$
$$\sim \frac{1}{\theta^{1/2}}\exp\left(-\alpha\sigma_{\alpha,\beta}^2\Big(\frac{x-b\gamma_\alpha}{r_{\alpha,\beta}}\Big)^{1/(1-\alpha)}\right), \quad x\to\infty.$$

Combining this with (5.2) establishes Theorem 1.2 for the case $\alpha\in(1/\beta,1)$. □

Proof of Theorem 1.3. The proof follows the same reasoning as that of Theorem 1.2. We only present the analogue of (5.1) for $t=t(x)$ satisfying (4.20):

$$(t(x))^{1/\alpha} = (r_{\alpha,\beta}^{-1}(x-b\gamma_\alpha))^{1/(1-\alpha)} - (rr_{\alpha,\beta}^{(2\alpha-1)/(1-\alpha)})/(2(1-\alpha)) + o(1), \quad x\to\infty.$$

□

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