UDC 519.2
DOI: https://doi.org/10.17721/1812-5409.2025/x.x

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LIMIT THEOREMS FOR GENERALISED CONVEX HULLS

The $(K,\mathbb{H})$***-hull of a set*** $A \subseteq \mathbb{R}^d$ ***is a generalization of the standard closed convex hull of*** A ***which can be defined as the intersection of all images that contain*** A ***of a closed convex subset*** $K \subset \mathbb{R}^d$ ***under the action of a subset*** $\mathbb{H}$ ***of the group of invertible affine transformations of*** $\mathbb{R}^d$***. We demonstrate that the scaled normalization of the*** $(K,\mathbb{H})$***-hull of a random sample, distributed according to a probability measure*** μ ***on*** K ***with a power-like behavior near the boundary*** ∂K ***of*** K***, converges in distribution to a random closed set which can be viewed as the zero cell of a certain Poisson hyperplane tessellation with explicit intensity measure.***

K e y w o r d s: ***Generalized convexity, Poisson hyperplane tessellation, zero cell, curvature measure.***

AMS 2020 classification: 60D05

Introduction

A random closed set in a finite-dimensional Euclidean space is a random element taking values in the family of closed sets in this space equipped with the Fell topology. More information on the Fell topology can be found in Appendix B in (Molchanov, 2005). Over the past decades, considerable attention has been paid to the problem of generalized convexity and the related constructions emerging from random point samples. For instance, recall that for a subset A of the unit Euclidean ball, the ball hull of A is defined as the intersection of all translated unit balls that contain A. The paper (Fodor, Kevei, & Vígh, 2014) investigates ball hulls of random samples in $\mathbb{R}^2$, that is, the case when $A = \Xi_n = \{\xi_1, \dots, \xi_n\}$ consists of independent points uniformly distributed in the unit Euclidean disk. This study has been subsequently extended to ball hulls in $\mathbb{R}^d$ in (Fodor, 2020). This concept was further generalized by replacing the unit Euclidean ball with an arbitrary closed convex body K in (Fodor, Papvári, & Vígh, 2020) and (Marynych & Molchanov, 2022). The approach evolved further in a recent study (Kabluchko, Marynych, & Molchanov, 2025), introducing the concept of a $(K,\mathbb{H})$-hull in the following manner. Let K be a closed convex subset of $\mathbb{R}^d$, distinct from the entire space. Let $\mathbb{H}$ be a nonempty subset of $\mathbb{R}^d \times \mathbb{GL}_d$, where $\mathbb{GL}_d$ represents the group of all invertible linear transformations in $\mathbb{R}^d$. The $(K,\mathbb{H})$ hull of a set $A \subseteq \mathbb{R}^d$ is defined in (Kabluchko et al., 2025) as follows:

$$\mathrm{conv}_{(K,\mathbb{H})}(A) = \bigcap_{(x,g)\in\mathbb{H}:\, A\subseteq g(K+x)} g(K+x),$$

where $g(B) = \{gz : z \in B\}$ and $B + x = \{z + x : z \in B\}$, for $(g, x) \in \mathbb{H}$ and $B \subseteq \mathbb{R}^d$.

We assume that $K \in \mathcal{K}^d_{(0)}$, where $\mathcal{K}^d_{(0)}$ is the family of compact convex sets that contain the origin in their interior. Let μ be a probability measure supported by K and $\Xi_n = \{\xi_1, \dots, \xi_n\}$ a set of n independent random points with the common distribution μ. Let M_d denote the space of all real-valued $d \times d$ matrices, and let $\exp : \mathsf{M}_d \mapsto \mathbb{GL}_d$ denote the standard matrix exponent. The following set

$$\mathfrak{X}_n := \{(x, C) \in \mathbb{R}^d \times \mathsf{M}_d : \Xi_n \subseteq \exp(C)(K + x)\},$$

which is the main object of investigation in the present paper, has been introduced and studied in (Kabluchko et al., 2025) in the case when μ is the uniform distribution on K. The importance of this set and its connections to $\mathrm{conv}_{(K,\mathbb{H})}(\Xi_n)$ and also to the Lie algebra of $\mathbb{R}^d \times \mathbb{GL}_d$ are explained in (Kabluchko et al., 2025).

Assuming that μ is the uniform distribution on K, it was shown in (Kabluchko et al., 2025) that for any closed set $\mathfrak{D} \subset \mathbb{R}^d \times \mathsf{M}_d$, the random closed set $n\mathfrak{X}_n \cap \mathfrak{D} \subset \mathbb{R}^d \times \mathsf{M}_d$ converges in distribution, in the Fell topology, to a random closed set defined as a certain intersection of half-spaces in $\mathbb{R}^d \times \mathsf{M}_d$. These half-spaces are derived from a Poisson point process whose intensity measure is determined by the curvature measure of K. The present study aims to determine the limiting behavior of an appropriately rescaled $\mathfrak{X}_n$ when the measure μ is not uniform. Our main result (Theorem 2 below) states that, under suitable normalization, the intersection $\mathfrak{X}_n \cap \mathfrak{D}$ converges in distribution to a random closed set, which is also defined as an intersection over the points of a Poisson point process. However, in this case, the intensity measure of the Poisson process depends not only on the curvature measure of K but also on the behavior of the measure μ near the boundary of K. The limiting random closed set can be interpreted as the zero cell of a Poisson hyperplane tessellation in this space. When $\mathbb{R}^d \times \mathsf{M}_d$ is replaced by $\mathbb{R}^d \times \{0\}$, a similar result has been established in (Visona, 2024).

The remainder of the paper is structured as follows. In Section 1, we review relevant definitions from convex geometry and introduce our assumptions on the distribution μ. Section 2 is dedicated to the proof of Lemma 2, which is a variant of Proposition 2.2 from (Visona, 2024), as well as results from (Kiderlen & Rataj, 2006). This lemma plays a central role in the proof of our main theorem. In Section 3, we state and prove our main result concerning the convergence of a suitably rescaled $\mathfrak{X}_n$. Several examples are presented at the end of Section 3.

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1. Preliminaries and necessary definitions

Let us introduce some necessary definitions. Int A denotes topological interior of a set $A \subset \mathbb{R}^d$. The Minkowski sum of two sets K and L is the following set, see Section 3.1 in (Schneider, 2013):

$$K \oplus L = \{k + l : k \in K, l \in L\}.$$

Minkowski difference of two sets K and L is defined as follows:

$$K \ominus L = \{x : x + L \subset K\}.$$

Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$. By $B_r(a)$ we denote the closed ball in $\mathbb{R}^d$ of radius r and center $a \in \mathbb{R}^d$.

Even though in the present work we are interested in convex closed sets K, we find it useful to formulate some auxiliary results for a more general class of closed sets called gentle sets. To this end, we need to introduce some additional notation. For an arbitrary closed (and not necessarily convex) set K, the exoskeleton $\mathrm{exo}(K)$ is the set of points that do not admit a unique nearest point in K. It is known that the set $\mathrm{exo}(K)$ is Lebesgue measurable and, with V_d denoting the d-dimensional Lebesgue measure, $V_d(\mathrm{exo}(K)) = 0$, see Section 2 in (Kiderlen & Rataj, 2006). The metric projection $\xi_K : \mathbb{R}^d \backslash \mathrm{exo}(K) \to K$ is defined by letting $\xi_K(a)$ be a unique point in K closest to a.

Let $K \subseteq \mathbb{R}^d$ be a closed set. The reduced normal bundle of K is

$$N(K) := \left\{ \left(\xi_K(z), \frac{z - \xi_K(z)}{\|z - \xi_K(z)\|} \right) : z \notin K \cup \mathrm{exo}(K) \right\}.$$

The set

$$\hat{N}(K) := N(\partial K) \setminus N(K)$$

is called the inner reduced normal bundle of K. The reach function of K is a mapping $\delta(K, \cdot, \cdot) : N(K) \to [0, +\infty]$ defined by

$$\delta(K, a, u) := \inf\{t \geqslant 0 : a + tu \in \mathrm{exo}(K)\}, \quad (a, u) \in N(K),$$

assuming that $\delta(K, a, u) = +\infty$ if the set $\{t \geqslant 0 : a + tu \in \mathrm{exo}(K)\}$ is empty.

Let $\mathcal{H}^{d-1}$ denote $(d-1)$-dimensional Hausdorff measure. A closed set K is said to be gentle, see (Kiderlen & Rataj, 2006), if the following holds:

1. $\mathcal{H}^{d-1}(\{a \in B : (a, u) \in N(\partial K), u \in S^{d-1}\}) < \infty$ for all bounded Borel sets $B \subset \mathbb{R}^d$;
2. For $\mathcal{H}^{d-1}$-almost all $a \in \partial K$, there are non-degenerate balls B_i and B_o containing a such that $B_i \subset K$ and $\mathrm{Int}\, B_o \subset K^c$, where K^c denotes the set complement.

It is known, see p. 107 in (Kiderlen & Rataj, 2006), that for $\mathcal{H}^{d-1}$-almost all $a \in \partial K$, there exists a unique outer normal vector $u = u(a)$ such that $(a, u) \in N(K)$. Let $C_{d-1}(K, \cdot)$ be a measure defined as the image of $\mathcal{H}^{d-1}$ on ∂K under the map

$$\partial K \ni a \mapsto (a, u) \in N(K).$$

As explained above, this map is defined $\mathcal{H}^{d-1}$-almost everywhere on ∂K and is measurable. The measure $C_{d-1}(K, \cdot)$ is called the curvature measure of K. Let $C^*_{d-1}(K, \cdot)$ be the image measure of $C_{d-1}(K, \cdot)$ under the reflection $(a, u) \in N(K) \mapsto (a, -u) \in \hat{N}(K)$, this reflection is well-defined almost everywhere.

We will use the abbreviation for almost all $(a, u) \in N(K)$ instead of for $C_{d-1}(K, \cdot)$-almost all $(a, u) \in N(K)$. The same convention is used for $\hat{N}(K)$ equipped with the measure $C^*_{d-1}(K, \cdot)$.

The following assumptions on the measure μ on K are borrowed from (Visona, 2024). In what follows we shall work under these assumptions.

Let μ be a finite measure, supported by K, and satisfying:

(M1) The measure μ has a compact support and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and with density f.

(M2) There exists an $\alpha > -1$ such that, for almost all $(a, u) \in \hat{N}(K)$,

$$\lim_{t \downarrow 0} \frac{f(a + tu)}{t^\alpha} = \hat{g}(a, u) = g(a) \in [0, +\infty), \tag{1}$$

where the function $\hat{g}$ is strictly positive on a subset of ∂K of positive measure and is bounded almost everywhere.

Note that if K is a compact set, then μ always has a compact support.

We will also need some definitions for a convex compact set S. The support plane of S with outer normal vector $u \neq 0$ is the following set:

$$H(S, u) := \{x \in \mathbb{R}^d : \langle x, u \rangle = h(S, u)\},$$

where $h(S, \cdot)$ is the support function of S in the direction u defined by[1]

$$h(S, u) := \sup\{\langle x, u \rangle \, : \, x \in S\}, \quad u \in S^{d-1}.$$

[1] Observe that $h(S, \cdot)$ is well-defined for any compact set S (not necessarily convex) and $h(S, \cdot) \equiv h(\mathrm{conv}\,(S), \cdot)$.

Let $H_u^-(t)$ denote the closed half-space defined by:

$$H_u^-(t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \leqslant t\}, \quad t \in \mathbb{R}, \quad u \in S^{d-1}.$$

For v, which belong to the boundary ∂S of a convex compact set S, the normal cone $N(S, v)$ to S at v is defined by

$$N(S, v) := \{u \in \mathbb{R}^d \setminus \{0\} : v \in H(S, u)\} \cup \{0\}.$$

Let $Nor(S)$ denote the normal bundle of the closed convex S, that is, a subset of $\partial S \times S^{d-1}$, which is the collection $(x, N(S, x) \cap S^{d-1})$ for $x \in \partial S$. For the closed convex set S, the reduced normal bundle $N(S)$ coincides with $Nor(S)$, the normal bundle.

2. Measure change after taking Minkowski difference

Our proof uses a version of the Proposition 2.2 in (Visona, 2024). This theorem describes the asymptotic properties of a measure satisfying (M1) and (M2) with respect to the Minkowski difference between the gentle set K and a scaled compact set L. While in (Visona, 2024) the theorem is proven for the measure of $K \setminus (K \ominus \varepsilon L)$, as $\varepsilon \downarrow 0$, we extend this result in Lemma 2 to the case when the measured set under the limit is $\xi_{\partial K}^{-1}(C) \cap (K \setminus K \ominus \varepsilon L)$, where C is a bounded Borel set.

The following lemma was formulated and proven in (Visona, 2024), see Lemma 2.1 therein. We provide only the formulation of this lemma.

Lemma 1. *Let K be a gentle set and let L be a compact set in $\mathbb{R}^d$. Let f be a nonnegative integrable function and let $h(L, -u)^+$ denote positive part of the support function. Then, for almost all $(a, u) \in \hat{N}(K)$, there exist functions t_+ and t_-, which satisfy*

$$\lim_{\varepsilon \downarrow 0} \frac{t_\pm(\varepsilon)}{\varepsilon} = h(L, -u)^+,$$

and such that

$$\int_0^{t_+(\varepsilon)} f(a + tu)\, dt \leqslant \int_0^{\delta(\partial K, a, u)} f(a + tu)\mathbf{1}_{K \setminus K \ominus \varepsilon L}(a + tu)\, dt \leqslant \int_0^{t_-(\varepsilon)} f(a + tu)\, dt$$

for all sufficiently small ε.

Lemma 2. *Let K be a gentle set, let C be a bounded Borel set from $\mathbb{R}^d$, and let L be a compact set in $\mathbb{R}^d$. Suppose that μ is a finite measure on K which has compact support, is absolutely continuous in a neighborhood of ∂K with density f that satisfies (M2). Put $\gamma := (\alpha + 1)^{-1}$. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\mu(\xi_{\partial K}^{-1}(C) \cap (K \setminus K \ominus \varepsilon^\gamma L))}{\varepsilon} = \int_{N(K)} \mathbf{1}_C(a)\, g(a) \frac{(h(L, u)^+)^{\alpha+1}}{\alpha + 1} C_{d-1}(K, d(a, u)).$$

Proof. Our demonstration closely mirrors the proof of Proposition 2.2 in (Visona, 2024). There exists a certain constant $s > 0$ such that the finite measure μ is absolutely continuous on the boundary $\partial K \oplus B_s(0)$. Additionally, there exists a constant $r > 0$ with $L \subseteq B_r(0)$.

For $\varepsilon < (s/r)^{1/\gamma}$, Proposition 4 from (Kiderlen & Rataj, 2006) implies that

$$\mu(\xi_{\partial K}^{-1}(C) \cap (K \setminus K \ominus \varepsilon^\gamma L)) = \int_{\mathbb{R}^d} f(x)\mathbf{1}_C(\xi_{\partial K}(x))\mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(x)\, dx =$$

$$= \sum_{i=1}^{d} i\kappa_i \int_{N(\partial K)} \mathbf{1}_C(a) \int_0^{\delta(\partial K, a, u)} t^{i-1} f(a + tu)\mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a + tu)\, dt\, \nu_{d-i}(\partial K, d(a, u)), \tag{2}$$

where the measures on ∂K, denoted as $\nu_0(\partial K, \cdot), \ldots, \nu_{d-1}(\partial K, \cdot)$ are called support measures, with κ_i representing the volume of the i-dimensional unit ball for $i = 1, \ldots, d$. According to Corollary 2.5 and equation (2.13) in (Hug, Last, & Weil, 2004), these support measures possess locally finite total variation. It is also important to highlight that the measures are signed for convex bodies.

The conditions outlined in Proposition 4 of (Kiderlen & Rataj, 2006) necessitate that f is bounded. However, the proposition can be applied to integrable functions that are not necessarily bounded. Consider the sequence of functions defined by $f_n := f\mathbf{1}_{\{f \leqslant n\}}$, which is both monotonic and composed of bounded elements. Due to the principle of monotone convergence, equation (2) holds true even for integrable functions that are unbounded.

We next verify that for $i \geqslant 2$, the terms in (2) are on the order of $o(\varepsilon)$ as $\varepsilon \to 0$. The function $\mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}$ is supported within $\partial K \oplus B_{2r\varepsilon^\gamma}(0)$, implying that

$$\left|\frac{1}{\varepsilon}\int_{N(\partial K)}\mathbf{1}_C(a)\int_0^{\delta(\partial K,a,u)}t^{i-1}f(a+tu)\ \times\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\,\nu_{d-i}(\partial K,d(a,u))\right|\leqslant$$
$$\leqslant\frac{1}{\varepsilon}\int_0^{2r\varepsilon^\gamma}\int_{N(\partial K)}\mathbf{1}_C(a)t^{i-1+\alpha}\frac{f(a+tu)}{t^\alpha}\,dt\,|\nu_{d-i}|(\partial K,d(a,u))\leqslant$$
$$\leqslant\frac{(2r\varepsilon^\gamma)^{i+\alpha}}{(i+\alpha)\varepsilon}\sup_{((a,u),t)\in N(\partial K)\times[0,2r\varepsilon^\gamma)}\frac{f(a+tu)}{t^\alpha}\times|\nu_{d-i}|(\partial K,\partial K(\varepsilon)\times S^{d-1})\to 0,$$

as $\varepsilon\to 0$, for all $i\neq 1$, with $\gamma=(\alpha+1)^{-1}$ and $\alpha>-1$, where

$$\partial K(\varepsilon):=\{x\in\partial K:x+y\in\text{supp}\,f\text{ for }y\in B_{2r\varepsilon^\gamma}(0)\},$$

which remains compact for all sufficiently small ε because the support of f is compact on ∂K. Consequently, the measure $|\nu_{d-i}|(\partial K,\partial K(\varepsilon)\times S^{d-1})$ is bounded for each integer i in the range 1 to $d-1$, due to $\partial K(\varepsilon)$ being bounded and these measures having a locally finite total variation. Additionally, from assumption (M2), the expression $f(a+tu)/t^\alpha$ remains constrained for almost every $((a,u),t)$ within the set $N(\partial K)\times[0,2r\varepsilon^\gamma)$ when ε is sufficiently minimal.

Propositions 4.1 and 5.1 in (Hug et al., 2004) imply the equation:

$$2\nu_{d-1}(\partial K,\cdot)=C_{d-1}(K,\cdot)+C^*_{d-1}(K,\cdot).$$

For almost every $(a,u)\in N(K)$, we define$\delta_\pm:=\delta(\partial K,a,\pm u)$. Using that $\kappa_1=2$ we conclude

$$\lim_{\varepsilon\downarrow0}\frac{\mu(\xi_{\partial K}^{-1}(C)\cap(K\setminus K\ominus\varepsilon^\gamma L))}{\varepsilon}=$$
$$=\lim_{\varepsilon\downarrow0}\frac{2}{\varepsilon}\int_{N(\partial K)}\mathbf{1}_C(a)\int_0^{\delta(\partial K,a,u)}f(a+tu)\times\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\,\nu_{d-1}(\partial K,d(a,u))=$$
$$=\lim_{\varepsilon\downarrow0}\frac{2}{2\varepsilon}\int_{N(K)}\mathbf{1}_C(a)\int_0^{\delta_+}f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\,C_{d-1}(K,d(a,u))+$$
$$+\frac{2}{2\varepsilon}\int_{\hat N(K)}\mathbf{1}_C(a)\int_0^{\delta_-}f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\,C^*_{d-1}(K,d(a,u))=$$
$$=\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_{\hat N(K)}\mathbf{1}_C(a)\int_0^{\delta_-}f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\,C^*_{d-1}(K,d(a,u)),$$

since the density function f vanishes outside K. Applying Lemma 1, we obtain for almost all $(a,u)\in\hat N(K)$,

$$\begin{aligned}g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}&=\lim_{\varepsilon\downarrow0}\inf_{s\in[0,t_+(\varepsilon^\gamma))}\frac{f(a+su)}{s^\alpha}\frac{1}{\varepsilon}\int_0^{t_+(\varepsilon^\gamma)}t^\alpha\,dt\leqslant\\&\leqslant\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_0^{t_+(\varepsilon^\gamma)}f(a+tu)\,dt\leqslant\\&\leqslant\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_0^{\delta_-}f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\leqslant\\&\leqslant\lim_{\varepsilon\downarrow0}\frac{1}{\varepsilon}\int_0^{t_-(\varepsilon^\gamma)}f(a+tu)\,dt\leqslant\\&\leqslant\lim_{\varepsilon\downarrow0}\sup_{s\in[0,t_-(\varepsilon^\gamma))}\frac{f(a+su)}{s^\alpha}\frac{1}{\varepsilon}\int_0^{t_-(\varepsilon^\gamma)}t^\alpha\,dt=\\&=g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}.\end{aligned}$$

Thus,

$$F_\varepsilon(a,u):=\frac{1}{\varepsilon}\int_0^{\delta_-}f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,dt\to g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}\quad\text{as }\varepsilon\downarrow0$$

for almost all $(a,u)\in\hat N(K)$. The limiting function is bounded and has compact support, because g is a bounded function with compact support and L is compact.

The sequence F_ε is bounded for all $\varepsilon>0$ and for almost all $(a,u)\in\hat N(K)\cap((\text{supp}\,f\cap\partial K)\times S^{d-1})$, which is compact. We can then find an upper bound and apply the dominated convergence theorem.

We conclude by noticing that

$$\int_{N(\hat K)}\mathbf{1}_C(a)g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}C^*_{d-1}(K,d(a,u))=\int_{N(K)}\mathbf{1}_C(a)g(a)\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1}C_{d-1}(K,d(a,u)).$$

□

The following corollary follows from the fact that $K \subseteq \mathcal{K}^d_{(0)}$ is also a gentle set, see Section 2 in (Kiderlen & Rataj, 2006).

Corollary 1. *The limit relation in Lemma 2 holds true if* $K \in \mathcal{K}^d_{(0)}$.

3. Limit theorem for $\mathfrak{X}_n$

From now on, we will assume that $K \in \mathcal{K}^d_{(0)}$. Each set from $\mathcal{K}^d_{(0)}$ is a convex body (a compact convex set with a nonempty interior).

Let $\mathcal{P}_K$ denote Poisson point process on $(0,\infty) \times N(K)$, with the intensity measure ν defined as a product of an absolutely continuous measure on $(0,\infty)$ with density t^α and a measure ν^* on $N(K)$, defined by

$$\nu^*(K, W \cap N(K)) = \int_{N(K)} \mathbf{1}_{(a,u)\in W} g(a) C_{d-1}(K, d(a,u)),$$

where W is a Borel subset of $N(K)$.

We start with the lemma which is the main technical result, needed for the proof of our main result, Theorem 2 below. The proof follows the same reasoning as in Lemma 4.4 in (Kabluchko et al., 2025), with only minor modifications for our specific setting.

Lemma 3. *Let* K *be an element of* $\mathcal{K}^d_{(0)}$*, and assume* μ *meets conditions (M1) and (M2). It follows that*

$$\sum_{k=1}^{n} \delta_{(n^{-\gamma}(K-\xi_k)^o, \xi_k)} \xrightarrow{\mathsf{d}} \sum_{(t,a,u)\in\mathcal{P}_K} \delta_{([0,t^{-1}u],a)} \quad \textit{as} \quad n \to \infty, \tag{3}$$

with the convergence interpreted as convergence in distribution within the space of point measures on $(\mathcal{K}^d_0 \setminus \{0\}) \times \mathbb{R}^d$ *furnished with the vague topology. Here,* $\mathcal{K}^d_0$ *represents the set of compact convex sets that include the origin (which may not be interior). As previously, we define* $\gamma = (\alpha+1)^{-1}$.

Proof. For $L \in \mathcal{K}^d_0 \setminus \{0\}$ let L^o denote its polar set defined by:

$$L^o := \{x \in \mathbb{R}^d : h(L,x) \leqslant 1\}.$$

To begin, we observe that the subsequent equality is valid for the limiting Poisson process. $L \in \mathcal{K}^d_0 \setminus \{0\}$ and every Borel $R \subseteq \mathbb{R}^d$

$$\begin{aligned}
&\mathbf{P}\big\{[0,t^{-1}u] \subseteq L \text{ or } a \notin \xi_{\partial K}(R) \text{ for all } (t,a,u) \in \mathcal{P}_K\big\} = \\
&= \exp\big(-\nu(\{(t,a,u) : [0,t^{-1}u] \not\subseteq L, a \in \xi_{\partial K}(R)\}\big) = \\
&= \exp\big(-\nu(\{(t,a,u) : L^o \not\subseteq H^-_u(t), a \in \xi_{\partial K}(R)\}\big) = \\
&= \exp\big(-\nu(\{(t,a,u) : h(L^o,u) > t, a \in \xi_{\partial K}(R)\}\big) = \\
&= \exp\Big(-\int_{N(K)} \mathbf{1}_{\{a\in\xi_{\partial K}(R)\}} g(a) \frac{(h(L^o,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, d(a,u))\Big).
\end{aligned}$$

According to Proposition 7.9 in the Appendix of (Kabluchko et al., 2025), we need to show that, for every $L \in \mathcal{K}^d_0 \setminus \{0\}$ and every Borel $R \subseteq \mathbb{R}^d$, as $n \to \infty$,

$$n\mathbf{P}\{n^{-\gamma}(K-\xi)^o \not\subseteq L, \xi \in R\} \longrightarrow \int_{N(K)} \mathbf{1}_{\{a\in\xi_{\partial K}(R)\}} g(a) \frac{(h(L^o,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, d(a,u)). \tag{4}$$

Note that

$$\begin{aligned}
&\lim_{n\to\infty} n\mathbf{P}\{n^{-\gamma}(K-\xi)^o \not\subseteq L, \xi \in R\} = \lim_{n\to\infty} n\mathbf{P}\{n^{-\gamma}L^o \not\subseteq (K-\xi), \xi \in R\} = \\
&= \lim_{n\to\infty} n\mathbf{P}\{\xi \notin K \ominus n^{-\gamma}L^o, \xi \in R\} = \lim_{n\to\infty} n\mu(R \cap (K \setminus (K \ominus n^{-\gamma}L^o))) = \\
&= \int_{N(K)} \mathbf{1}_{\{a\in R\}} g(a) \frac{(h(L^o,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, d(a,u)),
\end{aligned}$$

where the last equality follows from Corollary 1, where $\varepsilon = n^{-1}$, L is replaced by L^o and $C = \xi_{\partial K}(R)$. Thus, (4) is verified and the proof is complete. □

Let us formulate now the main theorem of this work.

Theorem 2. *Assume that* $K \in \mathcal{K}^d_{(0)}$ *and the measure* μ *satisfies (M1) and (M2). Suppose that* $\mathfrak{D}$ *is a closed convex set in* $\mathbb{R}^d \times \mathsf{M}_d$ *which contains the origin and let* $\gamma = (\alpha+1)^{-1}$ *be a real parameter, with* $\alpha > -1$. *The sequence of random*

closed sets $((n^\gamma \mathfrak{X}_n) \cap \mathfrak{D})_{n\in\mathbb{N}}$ *converges in distribution in the space of closed subsets of* $\mathbb{R}^d \times \mathsf{M}_d$ *endowed with the Fell topology to a random closed convex set* $\check{\mathfrak{Z}}_K \cap \mathfrak{D}$*, where*

$$\mathfrak{Z}_K := \bigcap_{(t,\eta,u)\in\mathcal{P}_K} \Big\{(x,C) \in \mathbb{R}^d \times \mathsf{M}_d : \langle C\eta + x, u\rangle \leqslant t\Big\} \tag{5}$$

and $\check{\mathfrak{Z}}$ *is the reflection of* $\mathfrak{Z}_K$ *with respect to the origin in* $\mathbb{R}^d \times \mathsf{M}_d$*.*

Proof. The structure of this proof will closely resemble that of Theorem 4.1 in (Kabluchko et al., 2025). Referring to Lemma 7.3 in the same source, it is necessary to demonstrate that $(n^\gamma \mathfrak{X}_n) \cap \mathfrak{D} \cap \mathfrak{K}$ converges to $\check{\mathfrak{Z}}_K \cap \mathfrak{D} \cap \mathfrak{K}$. This should be shown for an arbitrary compact convex subset $\mathfrak{K}$ of $\mathbb{R}^d \times \mathsf{M}_d$ that contains the origin in its interior, and subsequently, one must take the limit as $\mathfrak{K}$ approaches $(\mathbb{R}^d \times \mathsf{M}_d)$. Consequently, it holds

$$\begin{aligned}
(n^\gamma \mathfrak{X}_n) \cap \mathfrak{K} &= \Big\{(x,C) \in \mathfrak{K} : \Xi_n \subseteq \exp(C/n^\gamma)(K + x/n^\gamma)\Big\} = \\
&= \bigcap_{k=1}^n \Big\{(x,C) \in \mathfrak{K} : \xi_k \in \exp(C/n^\gamma)(K + x/n^\gamma)\Big\} = \\
&= \bigcap_{k=1}^n \Big\{(x,C) \in \mathfrak{K} : \exp(-C/n^\gamma)\xi_k \in K + x/n^\gamma\Big\} = \\
&= \bigcap_{k=1}^n \Big\{(x,C) \in \mathfrak{K} : \big(n^\gamma(\exp(-C/n^\gamma) - \boldsymbol{I})\big)\xi_k - x \in n^\gamma(K - \xi_k)\Big\},
\end{aligned}$$

where I is the identity matrix in M_d. Let

$$a_m := \sup_{n\geqslant m} \sup_{(x,C)\in\mathfrak{K}, y\in K} \big\|\big(n^\gamma(\exp(-C/n^\gamma) - \boldsymbol{I}\big)y + Cy\big\|, \quad m \in \mathbb{N}.$$

Note that $a_m \to 0$ as $m \to \infty$, because $n^\gamma\big(\exp(-C/n^\gamma) - \boldsymbol{I}\big) \to -C$ locally uniformly in C as $n \to \infty$, the set $\mathfrak{K}$ is compact in $\mathbb{R}^d \times \mathsf{M}_d$, and K is compact in $\mathbb{R}^d$.

Recall that B_{a_m} denotes the closed ball of radius a_m in $\mathbb{R}^d$ centred at the origin. For each $m \in \mathbb{N}$ and $n^\gamma \geqslant m$, we have

$$\mathfrak{N}^-_{m,n} \subseteq \big((n^\gamma \mathfrak{X}_n) \cap \mathfrak{K}\big) \subseteq \mathfrak{N}^+_{m,n}, \tag{6}$$

where

$$\mathfrak{N}^+_{m,n} := \bigcap_{k=1}^n \big\{(x,C) \in \mathfrak{K} : -C\xi_k - x \in n^\gamma(K - \xi_k) + B_{a_m}\big\}$$

and

$$\mathfrak{N}^-_{m,n} := \bigcap_{k=1}^n \big\{(x,C) \in \mathfrak{K} : -C\xi_k - x + B_{a_m} \subseteq n^\gamma(K - \xi_k)\big\}. \tag{7}$$

The benefit of the lower and upper bounds in (6) is the convex nature of $\mathfrak{N}^\pm_{m,n}$, simplifying their analysis. We plan to utilize Lemma 7.8 from (Kabluchko et al., 2025) by setting $Y^\pm_{m,n} = \mathfrak{N}^\pm_{m,n} \cap \mathfrak{D}$ and $X_n = (n^\gamma \mathfrak{X}_n) \cap \mathfrak{K} \cap \mathfrak{D}$.

Let $\mathfrak{L}$ be a compact subset of $\mathfrak{K}$. Denote

$$\begin{aligned}
M^+_m(\mathfrak{L}) &:= \big\{(L,y) \in \mathcal{K}^d_0 \times \mathbb{R}^d : -Cy - x \in L^o + B_{a_m} \text{ for all } (x,C) \in \mathfrak{L}\big\}, \\
M^-_m(\mathfrak{L}) &:= \big\{(L,y) \in \mathcal{K}^d_0 \times \mathbb{R}^d : -Cy - x + B_{a_m} \subseteq L^o \text{ for all } (x,C) \in \mathfrak{L}\big\}, \\
M(\mathfrak{L}) &:= \big\{(L,y) \in \mathcal{K}^d_0 \times \mathbb{R}^d : -Cy - x \in L^o \text{ for all } (x,C) \in \mathfrak{L}\big\}.
\end{aligned}$$

Then

$$\mathbf{P}\big\{\mathfrak{L} \subseteq \mathfrak{N}^\pm_{m,n}\big\} = \mathbf{P}\big\{(n^{-\gamma}(K - \xi_i)^o, \xi_i) \in M^\pm_m(\mathfrak{L}), i = 1,\dots,n\big\}.$$

By Lemma 3, the point process $\{(n^{-\gamma}(K - \xi_i)^o, \xi_i), i = 1,\dots,n\}$ converges in distribution to the Poisson process $\big\{([0,t^{-1}u],\eta) : (t,\eta,u) \in \mathcal{P}_K\big\}$. The sets $M(\mathfrak{L})$, $M^\pm_m(\mathfrak{L})$ are continuity sets for the distribution of the limiting Poisson process. Indeed, for each $(t,\eta,u) \in (0,\infty) \times N(K)$,

$$\begin{aligned}
&\Big\{([0,t^{-1}u],\eta) \in \partial M^+_m(\mathfrak{L})\Big\} \\
&= \Big\{ -C\eta - x \in H^-_u(t + a_m) \text{ for all } (x,C) \in \mathfrak{L}\Big\} \setminus \Big\{ -C\eta - x \in \operatorname{Int} H^-_u(t + a_m) \text{ for all } (x,C) \in \mathfrak{L}\Big\} = \\
&= \Big\{\langle -C\eta - x, u\rangle \leqslant t + a_m \text{ for all } (x,C) \in \mathfrak{L}\Big\} \setminus \Big\{\langle -C\eta - x, u\rangle < t + a_m \text{ for all } (x,C) \in \mathfrak{L}\Big\} = \\
&= \Big\{\langle -C\eta - x, u\rangle \leqslant t + a_m \text{ for all } (x,C) \in \mathfrak{L}\Big\} \bigcap \Big\{\langle -C\eta - x, u\rangle = t + a_m \text{ for some } (x,C) \in \mathfrak{L}\Big\},
\end{aligned}$$

where Int denotes the topological interior. Since the probability of the latter event for some $(t,\eta,u) \in \mathcal{P}_K$ vanishes, it follows

that $M_m^+(\mathfrak{L})$ is a continuity set for the Poisson point process $\{([0,t^{-1}u],\eta):(t,\eta,u)\in\mathcal{P}_K\}$. Letting $a_m=0$, we obtain that $M(\mathfrak{L})$ is also a continuity set. The argument for $M_m^-(\mathfrak{L})$ is similar by replacing a_m with $(-a_m)$.

Thus, for all $m\in\mathbb{N}$,

$$\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{N}_{m,n}^+\right\}\to\mathbf{P}\left\{\left\{([0,t^{-1}u],\eta):(t,\eta,u)\in\mathcal{P}_K\right\}\subseteq M_m^+(\mathfrak{L})\right\}=\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{N}_m^+\right\}$$

as $n\to\infty$, where

$$\mathfrak{N}_m^+:=\bigcap_{(t,\eta,u)\in\mathcal{P}_K}\Big\{(x,C)\in\mathfrak{K}:-C\eta-x\in H_u^-(t+a_m)\Big\}.$$

The random closed sets $\mathfrak{N}_{m,n}^+$ and $\mathfrak{N}_m^+$ are convex and almost surely contain a neighbourhood of the origin in $\mathbb{R}^d\times\mathsf{M}_d$, hence, are regular closed, meaning that they coincide with the closure of their respective interiors. By Theorem 7.5 from (Kabluchko et al., 2025) applied to the space $\mathbb{R}^d\times\mathsf{M}_d$, the random convex set $\mathfrak{N}_{m,n}^+$ converges in distribution to $\mathfrak{N}_m^+$. Since $\mathfrak{N}_{m,n}^+\xrightarrow{\mathsf{d}}\mathfrak{N}_m^+$ as $n\to\infty$, and the involved sets almost surely contain the origin in their interiors, Corollary 7.7 from (Kabluchko et al., 2025) yields that $(\mathfrak{N}_{m,n}^+\cap\mathfrak{D})\xrightarrow{\mathsf{d}}(\mathfrak{N}_m^+\cap\mathfrak{D})$ as $n\to\infty$, for each closed convex set $\mathfrak{D}$ which contains the origin in $\mathbb{R}^d\times\mathsf{M}_d$ (not necessarily as an interior point). Thus, we have checked part (i) of Lemma 7.8 from (Kabluchko et al., 2025).

We proceed with checking part (ii) of Lemma 7.8 from (Kabluchko et al., 2025) with $Y_m^-:=\mathfrak{N}_m^-$, where

$$\mathfrak{N}_m^-:=\bigcap_{(t,\eta,u)\in\mathcal{P}_K}\Big\{(x,C)\in\mathfrak{K}:-C\eta-x\in H_u^-(t-a_m)\Big\}.$$

Note that the random sets $\mathfrak{N}_{m,n}^-$ and $\mathfrak{N}_m^-$ may be empty and otherwise not necessarily contain the origin. We need to check 7.14 from (Kabluchko et al., 2025), which in our case reads as follows

$$\mathbf{P}\left\{\mathfrak{N}_{m,n}^-\cap\mathfrak{D}\cap\mathfrak{L}\neq\varnothing,0\in\mathfrak{N}_{m,n}^-\right\}\to\mathbf{P}\left\{\mathfrak{N}_m^-\cap\mathfrak{D}\cap\mathfrak{L}\neq\varnothing,0\in\mathfrak{N}_m^-\right\}\quad\text{as}\quad n\to\infty,\tag{8}$$

for all compact sets $\mathfrak{L}$ which are continuity sets of $\mathfrak{N}_m^-\cap\mathfrak{D}$. We shall prove (8) for all compact sets $\mathfrak{L}$ in $\mathbb{R}^d\times\mathsf{M}_d$. To this end, we shall employ Lemma 7.6 (Kabluchko et al., 2025) and divide the derivation (8) into several steps, each devoted to checking one condition of Lemma 7.6 from (Kabluchko et al., 2025).

Step 1. Let us check that, for sufficiently large $n\in\mathbb{N}$,

$$\mathbf{P}\left\{(0,0)\in\mathfrak{N}_{m,n}^-\right\}=\mathbf{P}\left\{(0,0)\in\operatorname{Int}\mathfrak{N}_{m,n}^-\right\}>0,\quad m\in\mathbb{N}.$$

Since the interior of a finite intersection is the intersection of the interiors, and using independence, it suffices to check this for each of the sets which appear in the intersection in (7). If $(0,0)$ belongs to $Y_k:=\left\{(x,C)\in\mathfrak{K}:-C\xi_k-x+B_{a_m}\subseteq n^\gamma(K-\xi_k)\right\}$, then $B_{a_m}\subseteq n^\gamma(K-\xi_k)$. Since ξ_k is not concentrated on ∂K, we have

$$\mathbf{P}\left\{B_{a_m}\subseteq n^\gamma(K-\xi_k)\right\}=\mathbf{P}\left\{B_{a_m}\subseteq n^\gamma\operatorname{Int}(K-\xi_k)\right\}$$

for all n. If $B_{a_m}\subseteq n^\gamma\operatorname{Int}(K-\xi_k)$, then $-C\xi_k-x+B_{a_m}\subseteq n^\gamma(K-\xi_k)$ for all x and C from a sufficiently small neighbourhood of the origin in $\mathbb{R}^d\times\mathsf{M}_d$. Furthermore,

$$\mathbf{P}\left\{(0,0)\in\mathfrak{N}_{m,n}^-\right\}=\mathbf{P}\left\{B_{a_m}\subseteq n^\gamma(K-\xi_k),k=1,\dots,n\right\}=\mathbf{P}\left\{B_{a_m/n^\gamma}\subseteq K\ominus\Xi_n\right\}>0$$

for all sufficiently large n.

Step 2. Let us check that, for each $m\in\mathbb{N}$,

$$\mathbf{P}\left\{(0,0)\in\mathfrak{N}_m^-\right\}=\mathbf{P}\left\{(0,0)\in\operatorname{Int}\mathfrak{N}_m^-\right\}>0.$$

The equality above follows from the observation that the origin lies on the boundary of $\left\{(x,C)\in\mathfrak{K}:-C\eta-x\in H_u^-(t-a_m)\right\}$ only if $t=a_m$, which happens with probability zero. Furthermore, $(0,0)\in\mathfrak{N}_m^-$ if $t\geqslant a_m$ for all $(t,\eta,u)\in\mathcal{P}_K$, which has positive probability.

Step 3. By a similar argument as we have used for $\mathfrak{N}_{m,n}^+$, for every compact subset $\mathfrak{L}$ of $\mathfrak{K}$ and $m\in\mathbb{N}$, we have

$$\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{N}_{m,n}^-\right\}\to\mathbf{P}\left\{\left\{([0,t^{-1}u],\eta):(t,\eta,u)\in\mathcal{P}_K\right\}\subseteq M_m^-(\mathfrak{L})\right\}=\mathbf{P}\left\{\mathfrak{L}\subseteq\mathfrak{N}_m^-\right\}\quad\text{as}\quad n\to\infty.$$

In summary, we have checked all conditions of Lemma 7.6 (Kabluchko et al., 2025). This finishes the proof of (8) and shows that all the conditions of part (ii) of Lemma 7.8 from (Kabluchko et al., 2025) hold. It remains to note that

$$(\mathfrak{N}_m^+\cap\mathfrak{D})\downarrow(\check{\mathfrak{Z}}_K\cap\mathfrak{K}\cap\mathfrak{D}),\quad(\mathfrak{N}_m^-\cap\mathfrak{D})\uparrow(\check{\mathfrak{Z}}_K\cap\mathfrak{K}\cap\mathfrak{D})\quad\text{a.s. as}\quad m\to\infty$$

in the Fell topology, and

$$\lim_{m\to\infty}\mathbf{P}\left\{0\in\mathfrak{N}_m^-\right\}=1.$$

Thus, by Lemma 7.8 from (Kabluchko et al., 2025) $(n^\gamma\mathfrak{X}_n)\cap\mathfrak{K}\cap\mathfrak{D}$ converges in distribution to $\check{\mathfrak{Z}}_K\cap\mathfrak{K}\cap\mathfrak{D}$ as $n\to\infty$. By Lemma 7.3 from (Kabluchko et al., 2025), $(n^\gamma\mathfrak{X}_n)\cap\mathfrak{D}$ converges in distribution to $\check{\mathfrak{Z}}_K\cap\mathfrak{D}$. □

Note that $\mathfrak{Z}_K$ is almost surely a closed set as the intersection of half-spaces. The question of boundedness depends on the properties of measure ν^*. If it is supported by sets contained in $\partial K \times H \subset \partial K \times S^{d-1}$, where H denotes a closed hemisphere of S^{d-1}, then this set is almost surely unbounded and is almost surely bounded otherwise.

As set $\mathfrak{Z}_K$ is an element of space $\mathbb{R}^d \times \mathsf{M}_d$, the latter space can be turned into a Euclidian one, given the proper scalar product, see Section 5.1 in (Kabluchko et al., 2025). Then, the set $\mathfrak{Z}_K$ can be viewed as the zero cell of the Poisson hyperplane tessellation in space $\mathbb{R}^d \times \mathsf{M}_d$.

Corollary 3. *Convergence in distribution of random closed sets in Hausdorff metric also implies convergence in distribution of intrinsic volumes, so*

$$V_j(n^\gamma \mathfrak{X}_n \cap \mathfrak{D}) \xrightarrow{\mathsf{d}} V_j(\check{\mathfrak{Z}}_K \cap \mathfrak{D}) \quad \textit{as} \quad n \to \infty,\ j = 0, 1, \dots d,$$

where V_j denotes the $j-$th intrinsic volume.

Before we proceed with example, it is necessary to introduce some new definitions as well as some assumptions, borrowed from (Kabluchko et al., 2025). Let $\mathbb{V}$ be a sufficiently small neighbourhood of I in $\mathbb{GL}_d$, where the matrix exponent is bijective. Let $log : \mathbb{V} \mapsto \mathsf{M}_d$ be the inverse of the matrix exponent. Define mapping $\widetilde{\exp} : \mathbb{R}^d \times log\mathbb{V} \mapsto \mathbb{R}^d \times \mathbb{V}$ by $\widetilde{\exp}(x, h) = (x, exp\, h), x \in \mathbb{R}^d, h \in log\mathbb{V}$.

Assume that the following objects exist:

- a neighbourhood $\mathfrak{U} \subseteq \mathbb{R}^d \times log\, \mathbb{V}$ of $(0, 0)$ in $\mathbb{R}^d \times \mathsf{M}_d$;
- a neighbourhood $\mathbb{U} \subseteq \mathbb{R}^d \times \mathbb{V}$ of $(0, I)$ in $\mathbb{R}^d \times \mathbb{GL}_d$;
- a closed convex cone $\mathfrak{C}_\mathbb{H}$ in $\mathbb{R}^d \times \mathsf{M}_d$ with apex at $(0, 0)$ such that

$$\mathbb{H} \cap \mathbb{U} = \widetilde{exp}(\mathfrak{C}_\mathbb{H} \cap \mathfrak{U}).$$

Similarly to (Kabluchko et al., 2025), in our examples we will look at intersections of $\mathfrak{Z}_K$ with the cone $\mathfrak{C}_\mathbb{H}$. More information on properties of this cone can be found on pages 12-13 of (Kabluchko et al., 2025). In the following example, we examine properties of such $\mathfrak{Z}_K$ that arise as part of the limit in Theorem 2, where $\mathfrak{X}_n$ is defined using Ξ_n sampled according to a probability measure μ satisfying assumptions (M1) and (M2).

Example. Let K be a unit ball B_1 and $\mathbb{H} = \mathbb{T} \times \mathbb{G} = \{0\} \times \mathbb{GL}_d$. In this case, $\mathfrak{C}_\mathbb{H} = \{0\} \times \mathsf{M}_d$ and

$$\mathfrak{Z}_{B_1} \cap (\{0\} \times \mathsf{M}_d) = \{0\} \times \bigcap_{(t,u)\in\mathcal{P}} \{C \in \mathsf{M}_d : \langle Cu, u\rangle \leqslant t\},$$

where $\mathcal{P}$ is a Poisson point process on $\mathbb{R}_+ \times S^{d-1}$ with intensity measure being the product of an absolutely continuous measure with density t^α and measure ν^* on S^{d-1}, defined as

$$\nu^*(K, W \cap S^{d-1}) = \int_{N(B_1)} \mathbf{1}_{u\in W} g(a) C_{d-1}(B_1, d(a, u)),$$

where $g(a)$ is defined in the same way as in (1).

Discussion and conclusions

In this work, we studied the asymptotic behavior of generalized convex hulls defined via group actions when the underlying point sample is drawn from a non-uniform distribution on a convex body K. Although the construction as well as limit theorems for the $(K, \mathbb{H})$-hulls were introduced in (Kabluchko et al., 2025) under the assumption of uniform sampling, our main contribution lies in identifying how the local behavior of the sampling density near the boundary of K affects the limit. We proved that under power-like decay conditions, the scaled hull converges in distribution to a random closed set defined as the zero cell of a Poisson hyperplane tessellation, where the intensity measure now depends not only on the curvature measure of K, but also explicitly on the boundary behavior of the density. This result confirms and extends previous findings from (Visona, 2024) on randomly translated sets and shows that the limit structure is sensitive to the anisotropy introduced by the sampling scheme. Our approach demonstrates that non-uniformity introduces additional geometric information into the scaling limit, opening new perspectives for modeling and analysis. These insights are likely relevant in applications where data is naturally concentrated near certain regions of a domain such as in image processing, statistical physics, or non-homogeneous spatial processes.

Acknowledgments, funding sources. I express my deep gratitude to Professor Alexander Marynych for his invaluable help, insightful advice, and careful reading of the drafts of this paper. His suggestions and corrections greatly improved both the clarity and the quality of the final work.

References

Fodor, F. (2020). *Random ball-polytopes in smooth convex bodies.* https://doi.org/10.48550/arXiv.1906.11480

Fodor, F., Kevei, P., & Vígh, V. (2014). On random disc polygons in smooth convex discs. *Advances in Applied Probability*, 46(4), 899–918. https://doi.org/10.1239/aap/1418396236

Fodor, F., Papvári, D. I., & Vígh, V. (2020). On random approximations by generazlised disc-polygons. *Mathematika*, 66(2), 498–513. https://doi.org/10.1112/mtk.12027

Hug, D., Last, G., & Weil, W. (2004). A local steiner-type formula for general closed sets and applications. *Mathematische Zeitschrift*, 246, 237–272. https://doi.org/10.1007/s00209-003-0597-9

Kabluchko, Z., Marynych, A., & Molchanov, I. (2025). Generalised convexity with respect to families of affine maps. *Israel Journal of Mathematics*, 266, 131–175. https://doi.org/10.1007/s11856-025-2751-0

Kiderlen, M., & Rataj, J. (2006). On infinitesimal increase of volumes of morphological transforms. *Mathematika*, 53(1), 103–127. https://doi.org/10.1112/S002557930000005X

Marynych, A., & Molchanov, I. (2022). Facial structure of strongly convex sets generated by random samples. *Advances in Mathematics*, 395, 108086. https://doi.org/10.1016/j.aim.2021.108086

Molchanov, I. (2005). *Theory of random sets*. Springer London. https://doi.org/10.1007/1-84628-150-4

Schneider, R. (2013). *Convex bodies: The brunn-minkowski theory* (2nd ed.). Cambridge University Press. https://doi.org/10.1017/CBO9781139003858

Visona, T. (2024). Intersections of randomly translated sets. *Journal Of Theoretical Probability*, 38, 10. https://doi.org/10.1007/s10959-024-01371-z

Отримано редакцією журналу / Received: xx.01.25
Прорецензовано / Revised: xx.03.25
Схвалено до друку / Accepted: xx.05.25

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ГРАНИЧНІ ТЕОРЕМИ ДЛЯ УЗАГАЛЬНЕНИХ ОПУКЛИХ ОБОЛОНОК

$(K, \mathbb{H})$-оболонка множини $A \subseteq \mathbb{R}^d$ - це узагальнення стандартної замкненої опуклої оболонки A, яка може бути визначена як перетин усіх образів опуклої замкненої підмножини $K \subset \mathbb{R}^d$ під дією підмножини $\mathbb{H}$ групи усіх оборотних афінних перетворень в $\mathbb{R}^d$, що містять A. Ми доводимо, що масштабована нормалізація $(K, \mathbb{H})$-оболонки випадкової вибірки, розподіленої згідно з ймовірнісною мірою μ на K, яка веде себе як степінь біля межі ∂K, збігається за розподілом до випадкової замкненої множини, яку можна розглядати як нульову клітинку деякої пуасонівської гіперплощинної мозаїки з явною мірою інтенсивності.

Ключові слова: ***Узагальнена опуклість, пуасонівська гіперплощинна мозаїка, нульова клітинка, міра кривизни.***

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.