Problem 1

Let B_n be the number of partitions of $\{1,2,\dots,n\}$ into disjoint subsets. For example, $B_3 = 5$, since there are five different partitions of $\{1,2,3\}$ into disjoint subsets, namely

$$\{\{1\},\{2\},\{3\}\}, \quad \{\{1\},\{2,3\}\}, \quad \{\{2\},\{1,3\}\}, \quad \{\{3\},\{1,2\}\} \quad \text{and} \quad \{\{1,2,3\}\}.$$

The integer B_n is called n-th Bell number. Prove that the sequence $(B_n)_{n\geq 0}$ satisfies the recurrence

$$B_0 = B_1 = 1, \quad B_{n+1} = \sum_{k=0}^{n} C_n^k B_k, \quad n \geq 1,$$

and derive the exponential generating function of the sequence $(B_n)_{n\geq 0}$:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = e^{e^z - 1}, \quad z \in \mathbb{R}. \tag{1}$$

Problem 2

Put $x^{n\downarrow} = x(x-1)(x-2)\cdots(x-n+1)$ and prove that

$$x^n = \sum_{\pi} x^{|\pi|\downarrow}, \quad x, n \in \{0,1,2,\dots\}, \tag{2}$$

where the sum is taken over all partitions of the set $\{1,2,\dots,n\}$ and $|\pi|$ is the number of blocks in partition π. Prove this by counting in two ways the number of all mappings from a size-n set to a size-x set. By plugging in (2) instead of x a random variable X with the Poisson distribution with parameter 1, prove the Dobiński formula:

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n \geq 0.$$

Derive this formula also from (1) by calculating $\mathbb{E}[e^{zX}]$.

Problem 3

Let N be a Poisson process on a measurable space $(S, \mathscr{S})$ with the intensity measure μ. Let $A_1, A_2 \in \mathscr{S}$ be two measurable subsets of S (not necessarily disjoint!). Are $N(A_1)$ and $N(A_2)$ positively correlated? Prove your claim by calculating the covariance

$$\mathrm{Cov}(N(A_1), N(A_2)).$$

Problem 4

Suppose that N has the Poisson distribution with parameter $\mu > 0$ and that, given N, M has the binomial distribution with parameter (N, p) for some fixed constant $p \in (0,1)$, that is,

$$\mathbb{P}[M = k | N = n] = C_n^k p^k (1-p)^{n-k}, \quad k = 0, \dots, n.$$

Prove that M and $N - M$ are independent and have Poisson distributions with parameters μp and $\mu(1-p)$, respectively.

Problem 5

Let Π be a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$. Let $0 < S_1 < S_2 < \cdots$ be an increasing enumeration of the atoms of Π. Put $S_0 := 0$. Prove that:

- For all $i \geq 1$, $S_i - S_{i-1}$ has the exponential distribution with parameter λ, that is $\mathbb{P}\{S_i - S_{i-1} \geq x\} = 1 - e^{-\lambda x}$, $x \geq 0$.
- The random variables $(S_i - S_{i-1})_{i \geq 1}$ are mutually independent.

Problem 6

Let Π_1 and Π_2 be two independent homogeneous Poisson point processes on $[0, \infty)$ with intensities λ and μ.

- Find probability that there is exactly one atom of Π_2 on the interval $[0, S_1]$, where S_1 is the first atom of Π_1.
- The expected number of atoms of Π_2 on $[0, S_1]$.

Problem 7

A hen wants to cross a one-way road, where cars drive according to a homogeneous Poisson process with intensity λ and constant speed v. This means that at each moment of time the positions of the cars on the road (identified with $\mathbb{R}$) form a homogeneous Poisson process on $\mathbb{R}$ with intensity λ. It takes T time units for the hen to cross the road. Assume that after approaching the road the hen waits for the first car to pass by and starts crossing the road immediately after. Compute the expected total waiting and crossing time for the hen and the probability to successfully cross the road.

Problem 8

Suppose that accidents on a highway occur according to a homogeneous Poisson process. Assume that on average there are two accidents per months. Find the probability that there are 5 or more accidents on the highway in a period of six months.

Problem 9

Events occur according to a inhomogenous Poisson process on $[0,\infty)$, whose mean value function is given by $m(t) = t^2 + 2t$, $t \geq 0$. What is the probability n events occur between times $t = 4$ and $t = 5$?

Problem 10

Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $\mathbb{P}[X_j > t] = e^{-\lambda_j t}$, $t \geq 0$, $j = 1, 2, \ldots, n$.

- Calculate $\mathbb{P}[X_i < X_j]$, $i \neq j$.
- Calculate $\mathbb{P}[X_i = \min_{j=1,\ldots,n} X_j]$, $i = 1, \ldots, n$.
- Assume that $\lambda_1 = \cdots = \lambda_n = 1$. Prove that

$$\max_{i=1,\ldots,n} X_i \quad \text{has the same distribution as} \quad \sum_{i=1}^{n} i^{-1} X_i.$$

Problem 11

Customers arrive at a bank according to a Poisson point process of intensity λ (clients per hour). Suppose at least two customers arrived during the first hour.

- Find probability that both arrived during the first 20 minutes.
- Find probability that at least one arrived during the first 20 minutes.

Problem 12

Let $\Pi := \sum_{k\geq 1} \delta_{X_k}$ be a homogeneous Poisson process on $\mathbb{R}^2$ with intensity $\lambda > 0$. Consider a random set $Z := \cup_{k\geq 1} B_{R_k}(X_k)$, where $(R_k)_{k\geq 1}$ is a sequence of independent copies of a random variable R taking values in $[0,\infty)$ and $B_r(x)$ is closed ball of radius r centered at x.

- Find the probability $\mathbb{P}\{0 \notin Z\}$ and, more generally, $\mathbb{P}\{x \notin Z\}$ for a fixed $x \in \mathbb{R}^2$.
- Show that $\mathbb{E}[R^2] = \infty$ implies $\mathbb{P}[Z = \mathbb{R}^2] = 1$.
