Problem 1

Prove that there exist Lebesgue measurable subsets of $[0,1]$ which are not Borel sets.

Problem 2

Let $([0,1],\mathfrak{M},\mu)$ be a measure space where μ is the Lebesgue measure and $\mathfrak{M}$ is the class of Lebesgue measurable subsets of $[0,1]$.

- give an example of $f:[0,1]\mapsto\mathbb{R}$ which is not Borel measurable;
- give an example of $f:[0,1]\mapsto\mathbb{R}$ such that $|f|$ is Lebesgue measurable but f is not.

Problem 3

Let $(X,\mathfrak{S},\mu)$ be a measure space and $f:X\mapsto\mathbb{R}$ be a measurable function. A function $g:X\mapsto\mathbb{R}$ is called simple if there exists a finite family of pairwise disjoint subsets $E_1,\dots,E_m\subset X$ and real numbers $\alpha_1,\dots,\alpha_m$ such that

$$g(x)=\sum_{i=1}^{m}\alpha_i\mathbb{1}[x\in E_i],\quad x\in X.$$

Prove that

- Every simple function is (Lebesgue) measurable;
- The product of two simple functions, and any finite linear combination of simple functions, are again simple functions.
- Every real-valued (Lebesgue) measurable junction f is the limit of a sequence (f_n) of simple functions. If f is nonnegative, then each f_n may be assumed nonnegative and the sequence (f_n) may be assumed increasing.

Problem 4

If (f_n) converges in measure μ to f, then (f_n) is fundamental in measure. If also (f_n) converges in measure g, then $f=g$ μ-almost surely.

Problem 5

Suppose that a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-integrable over a set A functions converges μ-almost everywhere to a μ-integrable over A function f. Prove that

$$\int_A|f_n(x)|\mu(\mathrm{d}x)\ \to\ \int_A|f(x)|\mu(\mathrm{d}x)\iff\int_A|f_n(x)-f(x)|\mu(\mathrm{d}x)\ \to\ 0.$$

Problem 6

Assume that f and f_n are μ-measurable functions on $[0,1]$, $f_n \geq 0$ and $f_n(x) \to f(x)$, $n \to \infty$, μ-almost everywhere on $[0,1]$. Prove that

$$\int_{[0,1]} f_n(x) e^{-f_n(x)} \mu(\mathrm{d}x) \ \to \ \int_{[0,1]} f(x) e^{-f(x)} \mu(\mathrm{d}x), \quad n \to \infty.$$

Problem 7

For which $\lambda \in \mathbb{R}$, does the limit exist

$$\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-\lambda x} \mathrm{d}x?$$

Find the limit for the corresponding λ's.

Problem 8

Let $(X, \mathscr{S}, \mu)$ be a measure space such that $\mu(X) = 1$, ϕ is a convex function on $(a,b) \subset \mathbb{R}$ and $f : X \to (a,b)$ is an integrable function on X. Then

$$\phi\left(\int_X f(x)\mu(\mathrm{d}x)\right) \leq \int_X \phi(f(x))\mu(\mathrm{d}x).$$

This is called Jensen's inequality. Formulate a counterpart for random variables.

Problem 9

Let μ by the Lebesgue measure one $\mathbb{R}$. Show that $f_n(x) = e^{-n|1-\sin x|}$ converges in μ-measure to $f(x) = 0$ on every compact subset $[a,b] \subset \mathbb{R}$. Does it converge to f in μ-measure on $\mathbb{R}$?

Problem 10

Let $(X, \mathscr{S})$ be a measurable space and μ, ν be measures on it. Recall that $\nu \ll \mu$ (ν is absolutely continuous with respect to μ) if $\mu(A) = 0$ implies $\nu(A) = 0$, for $A \in \mathscr{S}$. Prove the following:

(i) if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$, $A \in \mathscr{A}$, implies $\nu(A) < \varepsilon$, then $\nu \ll \mu$;

(ii) if $\nu \ll \mu$ and ν is finite, then (i) holds.

Give an example of an infinite measure ν such that $\nu \ll \mu$ but (i) does not hold.

Problem 11

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that its Riemann integral is equal to its Lebesgue integral with respect to Lebesgue measure on $[a,b]$.