Problem 1

A non-empty family $\mathfrak{R}$ of sets is called **ring** if, for every $A, B \in \mathfrak{R}$, it holds $A \Delta B \in \mathfrak{R}$ and $A \cap B \in \mathfrak{R}$. Prove that

$$A, B \in \mathfrak{R} \implies A \cup B \in \mathfrak{R}, \quad A \setminus B \in \mathfrak{R}, \quad \varnothing \in \mathfrak{R}.$$

Let $\mathscr{A}$ be a collection of subsets of a set X with the following properties:

- $X \in \mathscr{A}$;
- If $A, B \in \mathscr{A}$, then $A \setminus B \in \mathscr{A}$.

Show that $\mathscr{A}$ is a ring.

Problem 2

Suppose $A \subseteq B$ are (Lebesgue) measurable subsets of $\mathbb{R}^2$ and $\mu(A) = \mu(B)$. Show that any set C such that $A \subseteq C \subseteq B$ is also measurable and that $\mu(C) = \mu(A)$.

Problem 3

Let $A = \{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y > 0, \ x + y < 1\}$ be a standard simplex in $\mathbb{R}^2$. Find $\mu(A)$ using only the definition of the outer measure μ^*.

Problem 4

Show that any convex set in $\mathbb{R}^2$ is (Lebesgue) measurable.

Problem 5

If A is (Lebesgue) measurable and $\mu(A) > 0$, then for every $\alpha \in (0, 1)$ there exists an open interval $I \subset \mathbb{R}$ such that $\mu(A \cap I) > \alpha \mu(I)$. Prove this.

Problem 6

Let A, B and C be (Lebesgue) measurable subsets of the real line such that $A + B \subseteq C$. Show that (a) $\mu(C) \geq \mu(A) + \mu(B)$; (b) if $\mu(A) > 0$, then $A - A$ contains an interval.

Problem 7

Let $\mathbb{Q}$ be the set of all rational numbers in $\mathbb{R}$. For any $\varepsilon > 0$, construct an open set $O \subset \mathbb{R}$ such that $\mathbb{Q} \subset O$ and $\mu^*(O) \leq \varepsilon$.

Problem 8

If $A \subseteq \mathbb{R}^2$ is (Lebesgue) measurable, then $\mu(A) = \inf \sum_n \mu(B_n)$, where the infimum is over all countable coverings of A by open disks $B_1, B_2, \ldots$, and $\mu(B_n)$ is the area of B_n equal (by definition to πr^2. Show from this that the Lebesgue measure on $\mathbb{R}^2$ is rotation-invariant.

Problem 9 (Fatou's lemma)

Let (A_n) be a sequence of measurable sets. Show that

$$\mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

and, if also $A_n \subseteq E$ for some measurable E such that $\mu(E) < \infty$, then

$$\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n).$$

Problem 10

Let $K \subset [0,1]$ be the standard Cantor set. Prove that K is uncountable, compact, measurable and has the Lebesgue measure zero.
