Problem 1

Let $\mathbf{x}_0, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ be affinely independent, $V := \operatorname{conv}(\{\mathbf{x}_0, \ldots, \mathbf{x}_k\})$ and $\Delta := \{(\lambda_0, \ldots, \lambda_k) \in [0,1]^{k+1} : \sum_{i=0}^{k} \lambda_i = 1\}$. Prove that the mapping

$$\Delta \ni (\lambda_0, \ldots, \lambda_k) \stackrel{\phi}{\longmapsto} \sum_{i=0}^{k} \lambda_i \mathbf{x}_i \in V$$

is a bicontinuous bijection (homeomorphism). Prove that ϕ is globally Lipschitz continuous. Is ϕ^{-1} also globally Lipschitz continuous?

Problem 2

Find explicitly the metric projections:

- $p([0,1]^2, \cdot)$ in $\mathbb{R}^2$;
- $p(\Pi^{\leq}_{\mathbf{a},\beta}, \cdot)$ in $\mathbb{R}^d$, where $\beta \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ are fixed;
- $p(B_R(\mathbf{x_0}), \cdot)$ in $\mathbb{R}^d$, where $R > 0$ and $\mathbf{x_0} \in \mathbb{R}^d$ are fixed.

Problem 3

Prove that the following sets are convex, find their recession cones and lineality spaces:

- $\{(x,y) \in \mathbb{R}^2 : y \geq |x|\}$;
- $\{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$;
- $\{(x,y) \in \mathbb{R}^2 : y^2 - x^2 \geq 1,\ y \geq 0\}$;
- $\{(x,y,z) \in \mathbb{R}^3 : y^2 - x^2 \geq 1,\ y \geq 0\}$;

Problem 4

Let $P := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_i, \mathbf{x} \rangle \leq \beta_i \ \ \forall i = 1, \ldots, m\}$ be a polyhedron. Prove that $\operatorname{rec}(P) := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_i, \mathbf{x} \rangle \leq 0 \ \ \forall i = 1, \ldots, m\}$.

Problem 5

Construct an example of a closed convex set in $\mathbb{R}^2$ with a non-empty interior and a point $\mathbf{u} \in \partial K$ such that a support hyperplane of K at $\mathbf{u}$ is not unique.

Problem 6

Find *all* supporting hyperplanes for every $\mathbf{u} \in \partial K$ if

- K is the closed unit ball in $\mathbb{R}^d$;
- K is the unit cube in $\mathbb{R}^d$;
- K is the half-ellipse $\{(x,y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 \leq 1, y \geq 0\}$.

Problem 7

Recall that a convex subset F of a convex set V is called *face* if two inclusions $\mathbf{x},\mathbf{y} \in V$ and $(\mathbf{x}+\mathbf{y})/2 \in F$ together imply $\mathbf{x},\mathbf{y} \in F$. Suppose that V is closed. A subset F of V is called exposed face if $F = V \cap H$, for some support hyperplane H of V. Prove that that every exposed face is a face. Give an example of a face, which is not an exposed face.

Problem 8

Describe the faces of

- the unit ball in $\mathbb{R}^d$;
- the unit cube in $\mathbb{R}^d$;
- the standard d-dimensional simplex $\operatorname{conv}(\mathbf{0}, e_1, \ldots, e_d)$.

Problem 9

Let $A \subset \mathbb{R}^d$ be a set. Prove that $\mathbf{u}$ is an extreme point of $\operatorname{conv}(A)$ if and only if $\mathbf{u} \in A$ and $\mathbf{u} \in \operatorname{conv}(A \setminus \{\mathbf{u}\})$.

Problem 10

An extreme point of a polyhedron is called *vertex*. Let $P := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_i, \mathbf{x} \rangle \leq \beta_i \ \forall i = 1, \ldots, m\}$ be a polyhedron.

- Prove the following characterization of the vertices: For $\mathbf{u} \in P$, let

$$I(u) = \{i \in \{1,2,\ldots,m\} : \langle \mathbf{a}_i, \mathbf{u} \rangle = \beta_i\}$$

  be the set of the inequalities that are *active* on $\mathbf{u}$. Then $\mathbf{u}$ is a vertex of P if and only if the set of vectors $\{\mathbf{c}_i : i \in I(\mathbf{u})\}$ linearly spans the vector space $\mathbb{R}^d$. In particular, if $\mathbf{u}$ is a vertex of P, the set $I(\mathbf{u})$ contains at least d indices, that si $|I(\mathbf{u})| \geq d$.
- Prove that every bounded polyhedron is a polytope, that is, the convex hull of finitely many points.
