Problem 1

Let $A \subseteq \mathbb{R}^d$ be a convex set and $T : \mathbb{R}^d \to \mathbb{R}^m$ be a linear transform. Prove that $T(A) \subseteq \mathbb{R}^m$ is convex.

Problem 2

Prove that the following sets are convex in $\mathbb{R}^d$: standard simplex, cube, octahedron, open and closed balls.

Problem 3

Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be a non-zero vectors in $\mathbb{R}^d$, $\beta_1, \dots, \beta_m \in \mathbb{R}$. Prove that a polyhedron

$$P := \left\{ \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_i, \mathbf{x} \rangle \leq \beta_i, \ \forall \ i = 1, \dots, m \right\}$$

is a convex set.

Problem 4

Let $n = 2m$ be an even integer. Let $\mathbb{R}_n[x]$ be the space of polynomials in x with real coefficients of degree at most n. This space is naturally isomorphic to $\mathbb{R}^{n+1}$.

- Prove that the set $K := \{p \in \mathbb{R}_n[x] : \ p(x) \geq 0 \ \forall x \in \mathbb{R}\}$ is convex.
- The set K coincides with the set of polynomials expressible as a finite sum of perfect squares of polynomials of degree at most m:

$$K = \left\{ \sum q_i^2 : \ \deg q_i \leq m \right\}.$$

Problem 5

'Counterexamples' related to Helly's 'theorem':

- Give a counterexample showing that Helly's theorem does not hold for non-convex sets.
- Give a counterexample showing that Helly's theorem does not hold for infinite families.
- Give a counterexample showing that Helly's theorem does not hold if parameter $d+1$ is replaced by d (or any smaller one).

Problem 6

Let $\{K_\alpha : \alpha \in A\}$ be a family of compact sets in $\mathbb{R}^d$. Prove that if $\cap_\alpha K_\alpha = \varnothing$, then there exists a finite subfamily $K_{\alpha_1}, \dots, K_{\alpha_m}$ such that $\cap_{i=1}^m K_{\alpha_i} = \varnothing$.
Using this fact prove generalized Helly's theorem:
Theorem: Let $\mathscr{F}$ be an arbitrary family of compact convex sets in $\mathbb{R}^d$. If every $(d+1)$-tuple of sets in $\mathscr{F}$ has a non-empty intersection, then all sets in $\mathscr{F}$ have a non-empty intersection.

Problem 7

Prove that if a convex set is contained in the union of a finite family of halfspaces in $\mathbb{R}^d$, then it is contained in the union of some $d+1$ (or fewer) halfspaces from the family.

Problem 8

Let $A \subset \mathbb{R}^d$ be a compact convex set and $B = (-1/d)A$. Prove that there exists $b \in \mathbb{R}^d$ such that $b + B \subseteq A$. Is it possible to drop the compactness assumption?

Problem 9

Prove the **Gauss-Lucas Theorem**: Let f be a nonconstant polynomial in $\mathbb{C}[z]$ in a complex variable z and roots $z_1, \ldots, z_m$. Then each root of the derivative f' lies in the convex hull $\operatorname{conv}(z_1, \ldots, z_m)$.

Problem 10

Suppose that there is a finite set R of red points in $\mathbb{R}^d$ and a finite set B of blue points in $\mathbb{R}^d$. A hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle = \alpha\} \subset \mathbb{R}^d$ strictly separates red and blue points if $\langle \mathbf{a}, \mathbf{x} \rangle < \alpha$ for all $x \in R$ and $\langle \mathbf{a}, \mathbf{x} \rangle > \alpha$ for all $x \in B$.

Prove **Kirchberger's Theorem**: Suppose that for any set $S \subset \mathbb{R}^d$ of $d+2$ or fewer points there exists a hyperplane which strictly separates the sets $S \cap R$ and $S \cap B$ of red, resp. blue, points in S. Then there exists a hyperplane which strictly separates the sets R and B.

Problem 11

Let $P \subset \mathbb{R}^d$, $P = \{\mathbf{x} \in \mathbb{R}^d : \ \langle \mathbf{a}_i, \mathbf{x} \rangle \leq \beta_i, \ i = 1, \ldots, m\}$ be a polyhedron and let $T : \mathbb{R}^d \to \mathbb{R}^n$ be an invertible linear transformation. Prove that $Q = T(P)$ is a polyhedron in $\mathbb{R}^n$ defined by $Q = \{\mathbf{x} \in \mathbb{R}^n : \ \langle \mathbf{c}_i, \mathbf{x} \rangle \leq \beta_i, \ i = 1, \ldots, m\}$, where $\mathbf{c}_i = (T^*)^{-1}\mathbf{a}_i$ and T^* is the conjugate linear transformation.

Problem 12

Let P be a polyhedron in $\mathbb{R}^d$ and $\pi_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ the projection $\pi_d(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1})$. Prove that $\pi_d(P)$ is a polyhedron in $\mathbb{R}^{d-1}$. Using this fact and Problem 11 show that $Q(P) \subset \mathbb{R}^n$ is a polyhedron for an arbitrary linear transformation $Q : \mathbb{R}^d \to \mathbb{R}^n$.
