Estimates of the convergence rate in a limit theorem for geometric sums and some of their applications

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Abstract

New non-uniform estimates of the convergence rate to the exponential distribution in the boundary theorem for geometric sums are established. Examples of their application to extrema of regenerative random birth and death processes are given.

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1 Introduction and main results

Let $\zeta, \zeta_1, \zeta_2, \dots$ be a sequence of positive independent identical distributed random variables (i.i.d.r.v.), with distribution function (d.f) $F(x) = \mathbf{P}(\zeta \le x)$, ν be a geometrically distributed r.v., independent of the sequence (ζ_i),

$$\mathbf{P}(\nu = k) = q(1-q)^{k-1}, \quad k = 1, 2, \dots, \quad 0 < q < 1.$$

Put

$$S_\nu = \sum_{k=1}^{\nu} \zeta_k.$$

Such sums S_ν are called geometric random sums. The research of asymptotic properties of geometric sums originates from Rényi's work [15], in which the following asymptotic relation was established:

$$\forall x > 0 \quad \lim_{q\to 0} \mathbf{P}\left(\frac{q}{a_1} S_\nu \le x\right) = 1 - \exp(-x), \tag{1}$$

where

$$a_1 = \mathbf{E}\zeta < \infty. \tag{2}$$

This topic is closely related to some problems of queuing systems (QS) and reliability theory (see, [11]).

Estimates of the rate of convergence in Rényi's theorem have been also studied in many other works ([4], [8], [11], [14], [18], [19]).

It is important to mention the book [11] and the article [14] that have a detailed review of the results of geometric sums and their applications as well as fairly complete bibliography of this topic.

So in [11], provided that r.v. qS_ν has a bounded density and (2) holds, the following estimate is established

$$\sup_{x\ge 0} |\Delta_0(x)| \le C_0 q, \tag{3}$$

where constant C_0 depends on distribution r.v. ζ,

$$\Delta_0(x) = 1 - \exp(-x) - \mathbf{P}\left(\frac{q}{a_1} S_\nu \le x\right).$$

Earlier, under the condition

$$a_2 = \mathbf{E}\zeta^2 < \infty \tag{4}$$

estimate (3) was obtained in article [4] with constant

$$C_0 = \frac{1}{1-q} \max\left(1 + \frac{\gamma}{1-q}, 2\gamma - 1\right), \quad \gamma = \frac{a_2}{2a_1^2}. \tag{5}$$

Moreover, as follows from [4], in the general case, this constant C_0 in the uniform estimate (3) cannot be significantly improved.

Remark 1. In proving equality (1) it is usually assumed (possibly implicitly) that the distribution of r.v. ζ doesn't depend on the parameter q. There are known examples [11] when the condition (2) is fulfilled, the distribution ζ depends on parameter q, and equality (1) is not true.

It should be noted that in works [4], [11] when obtaining estimate (3), nowhere does it say that the distribution ζ doesn't depend on the parameter q, which can be important for applications.

In this work we can find new (non-uniform) estimates for geometric sums based on one exact formula for distribution r.v. S_ν from [11] and known results of the renewal theory.

In this case we impose conditions close to (4). But unlike estimate (3), we give non-uniform estimates, that is, we assume that $x > 0$ is a fixed number, and $q \to 0$.

In addition, throughout the article, $o(1)$ means a value that tends to 0 at $q \to 0$.

Obviously, some non-uniform estimates were known previously [11], [18] (for example, in the case when in r.v. there exists exponential moments). But our estimates seem more convenient for possible applications and the methods used are simpler.

Let $q^* = -\ln(1-q)$. It will be more convenient to normalize geometric sum with value q^*, which is equivalent to q. More precisely

$$q \le q^* \le q + q^2 \quad \text{for} \quad 0 < q < 1/2.$$

It is obvious that estimates of type (3) make sense with small values q. Therefore, we will assume that everywhere $0 < q < 1/2$ and

$$\Delta(x) = 1 - \exp(-x) - G(x), \quad G(x) = \mathbf{P}\left(\frac{q^*}{a_1} S_\nu \le x\right).$$

Theorem 1. *Let the condition (4) be satisfied, $x > 0$ - fixed number and $q \to 0$.*

(i) If distribution ζ doesn't depend on the parameter q, then the following estimates hold

$$|\Delta(x)| \;\le\; \left\{ \begin{array}{ll} C_1(x,F)\, q^*, & \text{for} \quad 0 < x < 1, \\ C_2(x,F)\, q^*, & \text{for} \quad x \ge 1, \end{array} \right\} \tag{6}$$

where

$$C_1(x,F) = \frac{1}{2}\exp(-x)\left(1 + \frac{\sigma^2}{a_1^2} + o(1)\right), \quad C_2(x,F) = \frac{\exp(-x)}{2} + \frac{K(x)\sigma^2}{x a_1^2} + o(1),$$

$\sigma^2 = \mathbf{D}\zeta = a_2 - a_1^2$ *is the variance of a r.v.* ζ, $K(x) = 1 - e^{-x}(1+x)$.

(ii) If distribution ζ depends on the parameter q, $a_1 = a_1(q)$, $\sigma^2 = \sigma^2(q)$ *and*

$$q\frac{\sigma^4}{a_1^4} \to 0 \quad \textit{where} \quad q \to 0, \tag{7}$$

then estimate (6) is used if the constants $C_1(x,F), C_2(x,F)$ are changed to

$$C_3(x,F) = \exp(-x)\left((e-2)\left(\frac{4\sigma^2}{a_1^2} + 2x + 3\right) + 1 + o(1)\right)$$

and

$$C_4(x,F) = \exp(-x) + \frac{K(x)}{x}\left(\frac{4\sigma^2}{a_1^2} + 5 + o(1)\right),$$

respectively.

As will be clear below, the values $C_i(x,F)$, $i=\overline{1,4}$, in Theorem 1 depend on x, as well as on the moments of the distribution function F and q. But to shorten the notation of these values, we omit q.

In addition to the unusual geometric sums S_ν we consider geometric sums with delay $S_\nu^{(d)}$, that is

$$S_n^{(d)} = \zeta_1^{(d)} + \zeta_2 + \ldots\ldots + \zeta_n,$$

where the first term $\zeta_1^{(d)}$ has distribution $F^{(d)}(x) = \mathbf{P}(\zeta_1^{(d)} \le x)$ different from $F(x)$.

Let us assume that functions $F(x)$ and $F^{(d)}(x)$ depend on parameters q. Such models are of considerable interest for application (see.[11] and subsection 4).

Put

$$\Delta^{(d)}(x) = 1 - \exp(-x) - G^{(d)}(x), \quad G^{(d)}(x) = \mathbf{P}\left(\frac{q^*}{a_1} S_\nu^{(d)} \le x\right).$$

Theorem 2. *Let condition (7) of Theorem 1 be satisfied for the geometric sum with delay* $S_\nu^{(d)}$, $a_1^{(d)} = a_1^{(d)}(q) = \mathbf{E}\zeta_1^{(d)} < \infty$, $x > 0$ *- fixed number and*

$$\forall y > 0 \quad \frac{1}{q^*}\left(1 - F^{(d)}\left(\frac{a_1 y}{q^*}\right)\right) \to 0 \quad \text{for} \quad q \to 0. \tag{8}$$

Then for $q \to 0$ *the following estimates hold*

$$|\Delta^{(d)}(x)| \le \left\{ \begin{array}{lll} C_5(x,\mathbb{F})\ q^*, & \text{when} & 0 < x < 1, \\ C_6(x,\mathbb{F})\ q^*, & \text{when} & x \ge 1, \end{array} \right\} \tag{9}$$

where

$$C_5(x,\mathbb{F}) = \exp(-x)\left(\frac{(2x(e-2)+1)a_1^{(d)}}{a_1} + (e-2)\left(\frac{4\sigma^2}{a_1^2} + 3\right) + o(1)\right),$$

$$C_6(x,\mathbb{F}) = \exp(-x)\frac{a_1^{(d)}}{a_1} + \frac{K(x)}{x}\left(\frac{2a_1^{(d)}}{a_1} + \frac{4\sigma^2}{a_1^2} + 3 + o(1)\right), \quad \mathbb{F} = (F, F^{(d)}).$$

The following sections 2 and 3 give proof of theorems 1, 2.

Section 4 presents a number of comparisons of the obtained estimates with known results.

At the end of the article, in Section 5 , we will also consider applications of the above theorems for studying extreme of regenerative processes and birth and death processes.

2 Proof of theorem 1

(i) Let $N(t)$ be counting process built on sequence (S_k), that is

$$N(t) = \max(k \geq 1 : S_k \leq t).$$

In book ([11], chapter 3, theorem 1.1), the following exact formula was obtained

$$\mathbf{P}(S_\nu \leq x) = 1 - \mathbf{E}(1-q)^{N(x)}, \tag{10}$$

on which we will rely.

First we establish inequalities:

$$\Delta(x) \leq C_1(x,F)\, q^*, \quad \text{for} \quad 0 < x < 1, \tag{11}$$

$$\Delta(x) \leq C_2(x,F)\, q^*, \quad \text{for} \quad x \geq 1, \tag{12}$$

where constants $C_1(x,F), C_2(x,F)$ are defined in Theorem 1.

We introduce the following notation:

$$A = A(q,x) = -q^* N\left(\frac{a_1 x}{q^*}\right) + x, \quad I(B) - \text{random event indicator B.}$$

Then according to (10)

$$\Delta(x) = \mathbf{E}(1-q)^{N(a_1 x/q^*)} - \exp(-x).$$

Hence and definition A we have

$$\begin{aligned} \Delta(x) &= \exp(-x)\mathbf{E}(\exp(A)-1) = \exp(-x)(\mathbf{E}A + \mathbf{E}(\exp(A)-1-A) \\ &= \exp(-x)(\mathbf{E}A + M_1 + M_2 + M_3), \end{aligned} \tag{13}$$

where

$$M_1 = \mathbf{E}(\exp(A)-1-A)I(A \leq 0),$$
$$M_2 = \mathbf{E}(\exp(A)-1-A)I(A \in (0,1)),$$
$$M_3 = \mathbf{E}(\exp(A)-1-A)I(A \geq 1).$$

Next, we estimate the upper bounds of M_1, M_2, M_3. According to elementary inequality:

$$\exp(y) - 1 - y \leq \frac{y^2}{2}, \quad \text{for} \quad y \leq 0,$$

we get

$$M_1 \leq \mathbf{E}\left(\frac{A^2}{2}\right) I(A \leq 0). \tag{14}$$

The next value M_2 is also estimated

$$\begin{aligned} M_2 &\le \mathbf{E}\left(\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots\right) I(A\in(0,1)) \\ &\le \mathbf{E}A^2\left(\frac{1}{2!}+\frac{1}{3!}+\ldots\right) I(A\in(0,1)) = (e-2)\mathbf{E}A^2 I(A\in(0,1)). \end{aligned} \tag{15}$$

And finally for value M_3 we have

$$\begin{aligned} M_3 &\le \mathbf{E}\left(\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots\right) I(A\ge 1) \\ &\le \mathbf{E}A^2\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\ldots\right) I(A\ge 1) \\ &\le \frac{\exp(x)-1-x}{x^2}\mathbf{E}A^2 I(A\ge 1). \end{aligned} \tag{16}$$

Since the simple inequality

$$\exp(-x)(e-2) < \frac{K(x)}{x^2}, \quad x\ge 1,$$

is true, where $K(x)$ is defined in Theorem 1, and when $0<x<1$, $I(A\ge 1)=0$ a.s., then from here and (14)-(16) we obtain

$$\exp(-x)(M_1+M_2+M_3) \le \left\{ \begin{array}{rl} \exp(-x)(e-2)\mathbf{E}A^2, & \text{for} \quad 0<x<1, \\ \frac{K(x)}{x^2}\mathbf{E}A^2, & \text{for} \quad x\ge 1. \end{array} \right\} \tag{17}$$

Hence the problem is reduced to the estimation of values $\mathbf{E}A$ and $\mathbf{E}A^2$.

Let $H(t)=\mathbf{E}N(t)$ be a renewal function created by r.v. $\zeta_i, i\ge 1$. Under condition (4) it satisfies the following asymptotic relation:

$$H(t) = \frac{t}{a_1}+\frac{a_2}{2a_1^2}-1+\epsilon_1(t) = \frac{t}{a_1}+\frac{\sigma^2-a_1^2}{2a_1^2}+\epsilon_1(t), \tag{18}$$

$$\epsilon_1(t)\to 0, \quad t\to\infty$$

(see [17]).

Then, according to (18), we get

$$\mathbf{E}A = -q^* H\left(\frac{a_1 x}{q^*}\right)+x = -q^*\left(\frac{\sigma^2-a_1^2}{2a_1^2}+o(1)\right). \tag{19}$$

Then we proceed to estimate the value $\mathbf{E}A^2$.

Here we use the well-known asymptotic of variance of r.v. $N(t)$. Under the conditions of Theorem 1

$$\mathbf{D}N(t) = \frac{\sigma^2}{a_1^3}t(1+\epsilon_2(t)), \quad \epsilon_2(t)\to 0, \quad t\to\infty \tag{20}$$

([17]).

Hence

$$\begin{aligned}\mathbf{E}A^2 &= (q^*)^2\mathbf{D}N\left(\frac{a_1x}{q^*}\right)+(\mathbf{E}A)^2\\ &= q^*\frac{\sigma^2x}{a_1^2}(1+o(1)).\end{aligned}\tag{21}$$

Getting together (13) - (17), (19), (21), we have for $0<x<1$

$$\begin{aligned}\Delta(x) &\le \exp(-x)(\mathbf{E}A+(e-2)\mathbf{E}A^2)\\ &\le \exp(-x)q^*\left(\frac{a_1^2-\sigma^2}{2a_1^2}+(e-2)\frac{\sigma^2x}{a_1^2}\right)(1+o(1))\\ &\le \exp(-x)q^*\left(\frac{1}{2}+(e-2.5)\frac{\sigma^2}{a_1^2}\right)(1+o(1)).\end{aligned}\tag{22}$$

The same for $x\ge 1$ we have

$$\begin{aligned}\Delta(x) &\le \exp(-x)\mathbf{E}A+\frac{K(x)\mathbf{E}A^2}{x^2}\\ &\le q^*\left(\exp(-x)\frac{a_1^2-\sigma^2}{2a_1^2}+\frac{K(x)\sigma^2}{a_1^2x}\right)(1+o(1))\\ &\le q^*\left(\frac{\exp(-x)}{2}+\frac{K(x)\sigma^2}{a_1^2x}\right)(1+o(1)).\end{aligned}\tag{23}$$

From (22), (23) immediately follow the inequalities (11), (12).

Next, we obtain a lower bound for $\Delta(x)$. By Jensen's inequality and (19) we have

$$\begin{aligned}\Delta(x) &\ge \exp(-x)(\exp(\mathbf{E}A)-1)\ge\exp(-x)\mathbf{E}A\\ &\ge q^*\exp(-x)\frac{a_1^2-\sigma^2}{2a_1^2}(1+o(1)).\end{aligned}\tag{24}$$

It is easy to check that for $x>0$ the following estimate is true

$$\exp(-x)\le(ex)^{-1}.$$

Thus, inequality (24) together with (11), (12) give estimate (6) of Theorem 1. Thus item (i) of the Theorem 1 is established.

Item (ii) simply follows from Theorem 2.Indeed, for $a_2^{(d)}=a_2$, under condition (7), we have

$$\frac{1}{q^*}(1-F^{(d)}(a_1y/q^*))\le\frac{1}{q^*}\frac{a_2^{(d)}(q^*)^2}{(a_1y)^2}=q^*\frac{\sigma^2+a_1^2}{(a_1y)^2}\to 0\quad\text{for}\quad q\to 0,$$

that is condition (8) of Theorem 2 also holds. To get constants $C_3(x),C_4(x)$ we need to use the result of Theorem 2 and substitute a_1 instead of $a_1^{(d)}$.

□

3 Proof of Theorem 2

Denote by $N^{(d)}(t)$ the counting process created by sequence $(S_k^{(d)})$, that is

$$N^{(d)}(t) = \max(k \geq 1 : S_k^{(d)} \leq t).$$

By analogy with formula (10) there is equality

$$\mathbf{P}(S_\nu^{(d)} \leq x) = 1 - \mathbf{E}(1-q)^{N^{(d)}(x)},$$

(see. [11], Chapter 3, Theorem 1.2).

Therefore

$$\Delta^{(d)}(x) = \mathbf{E}(1-q)^{N^{(d)}(a_1 x/q^*)} - \exp(-x). \tag{25}$$

Put

$$A^{(d)} = A^{(d)}(q,x) = -q^* N^{(d)}\left(\frac{a_1 x}{q^*}\right) + x.$$

Repeating the arguments of Theorem 1, used in proving relations (13) - (16), (17), we get

$$\begin{aligned} \Delta^{(d)}(x) &\leq \exp(-x)(\mathbf{E}A^{(d)} + M_1^{(d)} + M_2^{(d)} + M_3^{(d)}) \\ &\leq \exp(-x)\mathbf{E}A^{(d)} + \left\{ \begin{array}{lr} \exp(-x)(e-2)\mathbf{E}(A^{(d)})^2, & \text{for} \quad 0 < x < 1, \\ \frac{K(x)}{x^2}\mathbf{E}(A^{(d)})^2, & \text{for} \quad x \geq 1, \end{array} \right\} \end{aligned} \tag{26}$$

where $M_i^{(d)}$ is introduced in the same way as M_i, $i = 1,2,3$, with A replaced by $A^{(d)}$.

Even if the distribution r.v. ζ depends on the parameter q, then inequality (26) remains true.

And we come to the problem of estimating the values $\mathbf{E}A^{(d)}$ and $\mathbf{E}(A^{(d)})^2$.

As mentioned above, we denote by $H(t)$ the regeneration function constructed from r.v. $\zeta_k, k \geq 2$ from c.d.f. $F(t)$, and $H_2(t) = \mathbf{E}N^2(t)$ be the corresponding second moment of the counting process $N(t)$. We introduce similar notations for the counting process $N^{(d)}(t)$: $H^{(d)}(t) = \mathbf{E}N^{(d)}(t)$, $H_2^{(d)}(t) = \mathbf{E}(N^{(d)})^2(t)$.

Using the total probability formula, we can write the following relations for these functions $H^{(d)}(t)$, $H_2^{(d)}(t)$:

$$H^{(d)}(t) = \int_0^t (\mathbf{E}N(t-x)+1)dF^{(d)}(x) = F^{(d)}(t) + \int_0^t H(t-x)dF^{(d)}(x) \tag{27}$$

$$\begin{aligned} H_2^{(d)}(t) &= \int_0^t (\mathbf{E}(N(t-x)+1)^2)dF^{(d)}(x) = F^{(d)}(t) + \int_0^t (H_2(t-x) + 2H(t-x))dF^{(d)}(x) \\ &\leq H_2(t) + 2H(t) + 1. \end{aligned} \tag{28}$$

In this case we can't use exact asymptotic formulas (18), (20), since the time $t = a_1x/q^*$ and c.d.f. $F(t)$ and $F^{(d)}(t)$ depend on the parameter q. And we replace (18) with the following uniform estimates

$$\frac{t}{a_1} - 1 \le H(t) \le \frac{t}{a_1} + \frac{a_2}{a_1^2} - 1 = \frac{t}{a_1} + \frac{\sigma^2}{a_1^2}, \tag{29}$$

(see. [10], [11], p.57, Proposition 4.2).

From the left inequalities in (29) and (27) we have

$$\begin{aligned} H^{(d)}(t) &\ge F^{(d)}(t) + \int_0^t \left(\frac{t-x}{a_1} - 1\right) dF^{(d)}(x) \\ &\ge F^{(d)}(t) + \left(\frac{t}{a_1} - 1\right) F^{(d)}(t) - \frac{1}{a_1}\int_0^t x dF^{(d)}(x) \\ &\ge \frac{t}{a_1} - (1 - F^{(d)}(t))\frac{t}{a_1} - \frac{a_1^{(d)}}{a_1}. \end{aligned} \tag{30}$$

The last inequality and condition (8) of Theorem 2 allow us the following estimate of the value $\mathbf{E}A^{(d)}$:

$$\begin{aligned} \mathbf{E}A^{(d)} &= -q^* H^{(d)}\left(\frac{a_1 x}{q^*}\right) + x \\ &\le -q^*\left(\frac{x}{q^*} - \left(1 - F^{(d)}\left(\frac{a_1 x}{q^*}\right)\right)\frac{x}{q^*} - \frac{a_1^{(d)}}{a_1}\right) + x \\ &= q^*\left(\frac{a_1^{(d)}}{a_1} + o(1)\right). \end{aligned} \tag{31}$$

Since

$$H^{(d)}(t) \le H(t) + 1,$$

then using the right inequality from (29), we obtain

$$\mathbf{E}A^{(d)} \ge -q^*\left(\frac{a_1 x}{a_1 q^*} + \frac{\sigma^2}{a_1^2} + 1\right) + x = -q^*\left(\frac{\sigma^2}{a_1^2} + 1\right). \tag{32}$$

Turning to estimating the value $\mathbf{E}(A^{(d)})^2$. We have

$$\begin{aligned} \mathbf{E}\left(A^{(d)}\right)^2 &= \mathbf{E}\left(-q^* N^{(d)}\left(\frac{a_1 x}{q^*}\right) + x\right)^2 \\ &= (q^*)^2 H_2^{(d)}\left(\frac{a_1 x}{q^*}\right) - 2xq^* H^{(d)}\left(\frac{a_1 x}{q^*}\right) + x^2 = D_1 + D_2, \end{aligned} \tag{33}$$

where

$$D_1 = -2xq^* H^{(d)}\left(\frac{a_1 x}{q^*}\right) + 2x^2, \quad D_2 = (q^*)^2 H_2^{(d)}\left(\frac{a_1 x}{q^*}\right) - x^2.$$

It remains to calculate the values D_1, D_2. The value D_1 is estimated simply. According to inequality (31), we have

$$D_1 = 2x\mathbf{E}A^{(d)} \le 2q^* x\left(\frac{a_1^{(d)}}{a_1} + o(1)\right). \tag{34}$$

To calculate D_2 we need the upper bound of the function $H_2(t)$. For this we use the well-known result (see. [17], chapter.2, §6):

$$H_2(t) = 2\int_0^t H(t-s)dH(s) + H(t). \tag{35}$$

From the right estimate in (29) and (35) we have

$$\begin{aligned} H_2(t) &\le 2\int_0^t \left(\frac{t-s}{a_1} + \frac{\sigma^2}{a_1^2}\right) dH(s) + H(t) \\ &\le 2\left(\frac{t}{a_1} + \frac{\sigma^2}{a_1^2}\right) H(t) - \frac{2}{a_1}\int_0^t s\, dH(s) + H(t) \\ &\le 2\frac{\sigma^2}{a_1^2} H(t) + \frac{2}{a_1}\int_0^t H(s)ds + H(t) \\ &\le \left(\frac{2\sigma^2}{a_1^2} + 1\right) H(t) + \frac{t^2}{a_1^2} + \frac{2\sigma^2 t}{a_1^3}. \end{aligned} \tag{36}$$

Under Theorem 2 for $q \to 0$

$$q\frac{\sigma^4}{a_1^4} \to 0, \quad q\frac{\sigma^2}{a_1^2} \to 0.$$

Hence and (28), (29), (36) we get upper bound for D_2

$$\begin{aligned} D_2 &\le (q^*)^2\left(H_2\left(\frac{a_1 x}{q^*}\right) + 2H(\frac{a_1 x}{q^*}) + 1\right) - x^2 \\ &\le (q^*)^2\left(\left(\frac{2\sigma^2}{a_1^2} + 3\right) H\left(\frac{a_1 x}{q^*}\right) + \left(\frac{x}{q^*}\right)^2 + \frac{2\sigma^2 x}{a_1^2 q^*} + 1\right) - x^2 \\ &\le q^* x\left(\frac{4\sigma^2}{a_1^2} + 3 + o(1)\right). \end{aligned} \tag{37}$$

Thus, putting together the estimates (34), (37), we have

$$\mathbf{E}(A^{(d)})^2 = D_1 + D_2 \leq q^* x \left(\frac{2a_1^{(d)}}{a_1} + \frac{4\sigma^2}{a_1^2} + 3 + o(1) \right). \tag{38}$$

The next step, returning to estimates (26), (31) and (38), we can obtain an upper bound for $\Delta^{(d)}(x)$. Indeed, for $0 < x < 1$

$$\begin{aligned} \Delta^{(d)}(x) &\leq \exp(-x)(\mathbf{E}A^{(d)} + (e-2)\mathbf{E}(A^{(d)})^2) \\ &\leq q^* \exp(-x) \left(\frac{a_1^{(d)}}{a_1} + x(e-2) \left(\frac{a_1^{(d)}}{a_1} + \frac{4\sigma^2}{a_1^2} + 3 + o(1) \right) \right) \leq q^* C_5(x). \end{aligned} \tag{39}$$

Similarly for $x \geq 1$ we have

$$\begin{aligned} \Delta^{(d)}(x) &\leq \exp(-x)\mathbf{E}A^{(d)} + \frac{K(x)\mathbf{E}(A^{(d)})^2}{x^2} \\ &\leq q^* \left(\exp(-x)\frac{a_1^{(d)}}{a_1} + \frac{K(x)}{x} \left(\frac{2a_1^{(d)}}{a_1} + \frac{4\sigma^2}{a_1^2} + 3 + o(1) \right) \right) \leq q^* C_6(x) \end{aligned} \tag{40}$$

(constants $C_5(x), C_6(x)$ are defined in Theorem 2).

Thus, it remains to find a lower bound for $\Delta^{(d)}(x)$. Here considerations are similar to those given above in proving inequality (24). It is only necessary to replace the asymptotic relation (19) with the inequality (32). Then

$$\Delta^{(d)}(x) \geq \exp(-x)\mathbf{E}A^{(d)} \geq -q^* \exp(-x) \left(\frac{\sigma^2}{a_1^2} + 1 \right).$$

But obviously

$$\exp(-x) \left(\frac{\sigma^2}{a_1^2} + 1 \right) \leq C_5(x) \quad \text{for} \quad 0 < x < 1,$$

and

$$\exp(-x) \left(\frac{\sigma^2}{a_1^2} + 1 \right) \leq C_6(x) \quad \text{for} \quad x \geq 1,$$

that completes the proof of Theorem 2.

□

4 Some examples of estimates comparison for geometric sums

In this section, it is assumed that the conditions of item (i) of Theorem 1 are satisfied.

Example 1. (Uniform estimate (3), see [4]). It is easy to see that the constant C_0 from formula (5) satisfies the inequality:

$$C_0 \geq 1 + \gamma = 1 + \frac{a_2}{2a_1^2} = \frac{1}{2}\left(3 + \frac{\sigma^2}{a_1^2}\right).$$

Then for fixed $x \in (0,1)$ and sufficiently small q:

$$C_1(x, F) \leq \frac{1}{2}\left(1 + \frac{\sigma^2}{a_1^2} + o(1)\right) \leq C_0.$$

Since at $x \geq 1$

$$\frac{K(x)}{x} \leq \frac{1}{2},$$

then

$$C_2(x, F) \leq \frac{e^{-1}}{2} + \frac{K(x)}{x}\frac{\sigma^2}{a_1^2} + o(1) \leq C_0.$$

Thus, the non-uniform estimate (6) from Theorem 1 asymptotically more accurately estimates the value of $|\Delta(x)|$ than the uniform estimate (3).

Note that in article [18], estimate (7), a more precise uniform estimate is given than estimate (3) from [4]. But it contradicts the following rather simple example.

It is known and simply verified that in case ζ has a standard exponential distribution, then r.v. νS_ν has a standard exponential distribution. Therefore, $\Delta_0(x) = 0$. The estimate given below shows that by adjusting the r.v. ζ moment conditions or condition $0 \leq \zeta \leq C < \infty$ a.s., it is impossible to obtain a uniform estimate of the type: for $\epsilon > 0$, $\quad 0 < q < 1/2$

$$\sup_{x \geq 0} |\Delta_0(x)| \leq (1 - \epsilon)\left(1 + \frac{\sigma^2}{a_1^2}\right) q$$

Let $\zeta = 1$ a.s., then $S_\nu = \nu$, $\sigma^2 = 0$. Let us show that the estimates

$$q - 3q^2 \leq \sup_{x \geq 0} |\Delta_0(x)| \leq q + 2q^2 \tag{41}$$

are true.

To obtain the left inequality, we put $x = 2q - q^2$. Obviously,

$$q < x < 2q, \qquad \left[\frac{x}{q}\right] = 1.$$

It follows that

$$\begin{aligned} \Delta_0(x) &= (1-q)^{[\frac{x}{q}]} - \exp(-x) = 1 - q - \exp(-(2q - q^2)) \\ &\geq 1 - q - \left(1 - 2q + q^2 + \frac{(2q - q^2)^2}{2}\right) \\ &\geq q - 3q^2, \end{aligned}$$

that is the left inequality in (41) is established.

The proof of the right inequality in (41) is simple and we omit it.

Interestingly, this example performs a non-uniform evaluation of (6) with

$$C_1(x,F) = C_2(x,F) = \exp(-x)\left(\frac{1}{2} + o(1)\right).$$

Remark 2. It can be assumed that in many practical problems at $q \to 0$, $a_1(q) \to a_1$, $a_1^{(d)}(q) \to \infty$. Then the non-uniform estimate (9) from Theorem 2 in such cases will be asymptotically more accurate than some uniform estimates for geometric sum with delay (see, for example, Theorem 2.2 from [4]).

Example 2. (Non-uniform top estimate, see [18]). From the point of view of possible applications, non-uniform upper bounds of the following type are of considerable interest:

$$\Delta(x) \approx \Delta_0(x) \le C_0(x,F)\cdot q. \tag{42}$$

One such estimate is given in [18]. Under the conditions of item (i) of Theorem 1, with $a_1 = 1$ and $q \to 0$, the value $C_0(x,F)$ from [18] can be written as follows:

$$C_0(x,F) = (\sigma^2+1)\frac{e^{-x}(e^2-1)}{2(1-q)}(1+o(1)) + I(x \ge 2)\frac{e}{x^2}\left(1 - \frac{2\ln x}{x-1} + o(1)\right)^{-2}.$$

Hence we have for a fixed x and for sufficiently small q:

$$C_1(x,F) \le C_0(x,F) \quad \text{for} \quad x \in (0,1).$$

These elementary calculations are derived from the following estimates:

$$\frac{K(x)}{x} \le \frac{e^{-x}(e^2-1)}{2} \quad \text{for} \quad x \in [1,2]$$

and

$$\frac{K(x)}{x} \le \frac{e}{x^2}\left(1 - \frac{2\ln x}{x-1}\right)^{-2} \quad \text{at} \quad x \in [2,10].$$

From the last inequalities follows the following estimate:

$$C_2(x,F) \le C_0(x,F) \quad \text{for} \quad x \in [1,10].$$

Since $\exp(-10) \approx 4,5\cdot 10^{-5}$, we can assume that almost always the upper estimate from Theorem 1 is asymptotically more accurate than the estimate (42) from [18].

Example 3. (Non-uniform top estimate, see [11]). A sufficiently accurate non-uniform upper bound was obtained in the book [11] (Theorem 3.2, pp. 118-119): for sufficiently small q:

$$\mathbf{P}(S_\nu > x) \le \frac{1}{1-q}\exp(-\varepsilon(b)x) + \frac{1 - F\left(\frac{b}{q}\right)}{q}, \tag{43}$$

where

$$\varepsilon(b) = \frac{1}{a_2 M(b)}\left(-1 + \sqrt{1 + 2qa_2 M(b)}\right), \quad M(b) = \frac{2}{b^2}(e^b - 1 - b), \quad a_1 = 1,$$

$b > 0$ - some parameter.

Unfortunately, we often do not know exactly the function $F(x)$ (since in the case of a regenerative process it coincides with the distribution function of the length of the regeneration cycle, $F(x) = F^-(x)$, see Section 5). In reality, in practice, we will be able to estimate the first few moments of the function $F(x)$. If we use the well-known estimate

$$1 - F\left(\frac{b}{q}\right) \leq \frac{a_2 q^2}{b^2},$$

then (43) can be rewritten as follows:

$$\Delta_0(x, F) \leq \frac{1}{1-q}\exp\left(-\varepsilon(b)\frac{x}{q}\right) + \frac{a_2 q^2}{b^2} - e^{-x} = C_0(x, b, F) \cdot q. \tag{44}$$

We note that in the proof of estimate (43) a number of additional conditions were imposed.

In addition, when using estimates (43) or (44), one should also choose the parameter b in an optimal way and write the value $C_0(x, F)$ in a simple form.

Put

$$B = \frac{1}{1-q}\left[\exp\left(-\varepsilon(b)\frac{x}{q}\right) - e^{-x}\right].$$

Then, given the equality

$$\varepsilon(b) = \frac{2q}{1 + \sqrt{1 + 2qa_2 M(b)}}$$

we have:

$$B = \frac{1}{1-q}\left[\exp\left(\frac{-2x}{1 + \sqrt{1 + 2qa_2 M(b)}}\right) - \exp(-x)\right] =$$

$$= \frac{1}{1-q}\exp(-x)\left[\exp\left(-x\left(\frac{2}{1 + \sqrt{1 + 2qa_2 M(b)}} - 1\right)\right)\right] =$$

$$= \frac{1}{1-q}\exp(-x)\left[\exp\left(\frac{2xqa_2 M(b)}{\left(1 + \sqrt{1 + 2qa_2 M(b)}\right)^2}\right) - 1\right].$$

We choose the parameter b so that $qa_2 M(b) \to 0$ with $q \to 0$, since otherwise $B \nrightarrow 0$.

Since $e^y - 1 + y \to 0$ with $y \to 0$, then

$$B = \frac{1}{1-q}\exp(-x)\left[\frac{xqa_2 M(b)}{2 + o(1)}\right].$$

Further, it is already possible to write down the value of $C_0(x,b,F)$ with (44):

$$C_0(x,b,F) = \frac{1}{1-q}\exp(-x)\left[1+\frac{xa_2M(b)}{2+o(1)}\right]+\frac{a_2}{b^2}. \tag{45}$$

Our attempts to compare the upper bound (6) of Theorem 1 with the bound (44) did not give an unambiguous answer for all cases. Below, we present the ranges of parameter changes for which the following inequalities hold:

$$\min_b C_0(x,b,F) \geq C_1(x,F) \quad \text{for} \quad 0 < x \leq 1 \tag{46}$$

$$\min_b C_0(x,b,F) \geq C_2(x,F) \quad \text{for} \quad x \geq 1. \tag{47}$$

We programmatically solved the optimization and comparison problems in estimates (46), (47) (recall that in examples 2 and 3 $a_1 = 1$):

$$\text{Domain} \quad D_1 = \left\{\begin{array}{c} 1 \leq a_2 \leq 2, \\ 0,001 \leq x \leq 1, \Rightarrow (46) \\ 1 \leq x \leq 7,3 \Rightarrow (47) \end{array}\right\}, \quad e^{-7,3} \approx 0,0007$$

$$\text{Domain} \quad D_2 = \left\{\begin{array}{c} 2 \leq a_2 \leq 5, \\ 0,1 \leq x \leq 1, \Rightarrow (46) \\ 1 \leq x \leq 5 \Rightarrow (47) \end{array}\right\}, \quad e^{-5} \approx 0,0067$$

$$\text{Domain} \quad D_3 = \left\{\begin{array}{c} 5 \leq a_2 \leq 10, \\ 0,2 \leq x \leq 1, \Rightarrow (46) \\ 1 \leq x \leq 4 \Rightarrow (47) \end{array}\right\}, \quad e^{-4} \approx 0,0183.$$

Note that region D_1 is the domain of distributions whose tails decrease faster than the tails of the exponential distribution.

Therefore, for such distributions, the upper bounds from Theorem 1 are almost always asymptotically more accurate than the bound (44), where the right-hand side is:

$$\inf_b (C,x,b,F)\cdot q.$$

For the case when the parameters lie outside the domains D_1, D_2, D_3, inequalities (46), (47) may not be satisfied.

5 Estimates in limit theorems for extreme values

The extreme values of the processes that describe the queue length or the waiting time in the queue for queuing systems have been studied in many works (see, for example, [6], [9] and reviews [2], [3]). Similar problems were considered for birth and death processes [16].

But as it turned out, in many important cases, under linear normalizations of extreme values, does not exist non-degenerate boundary distribution. Therefore, in article [19] , another method was proposed, which is based on a non-linear time transformation.

In this section, we use this method and the above theorems to estimate the rate of convergence in boundary theorems for extremal values of some random processes.

1. Regenerative random processes.

Let's start from the definition of the regenerative process (see, for example, [17], p. II, ch.2 , [7], chapter.11,§8).

By a cycle of duration T we mean an ordered pair $\pounds = (T, \xi(t))$, in which T is a non-negative r.v., and $\xi(t)$ is a random process defined on [0, T),

$$\mathbf{P}(T = 0) < 1, \quad \mathbf{P}(T < \infty) = 1.$$

In the general case, r.v. T and process $\xi(t)$ are dependent.

Suppose $\pounds_i = (T_i, \xi_i(t))$, $i \geq 1$, is an infinite sequence of independent cycles equally distributed with $\pounds$. Let's define a random process $X(t)$, $t \geq 0$, by the formula

$$X(t) = \xi_i(t - S_{i-1}), \quad \text{at} \quad t \in [S_{i-1}, S_i),$$

where $S_i = T_1 + \ldots + T_i$, $i \geq 1$, $S_0 = 0$.

Then we will call process $X(t)$ regenerative, points S_i - moments of regeneration, and the interval $[S_{i-1}, S_i)$ - the i^{th} regeneration period.

We will impose the condition of separability on r.p. $\xi_i(t)$. Then r.p. $X(t)$ will also be separable. We introduce r.p.

$$\bar{X}(t) = \sup_{0 \leq s < t} X(s), \quad t \geq 0$$

and sequence of i.i.d.r.v.

$$\bar{X}_k = \sup_{S_{k-1} \leq s < S_k} X(s), \quad k = 1, 2, \ldots.$$

Let's consider that $m_1 = \mathbf{E}T < \infty$ and for all $u \in \mathbf{R}$

$$q(u) = \mathbf{P}(\bar{X}_k \geq u) > 0 \quad \text{and}$$

$$q(u) \downarrow 0 \quad \text{for} \quad u \uparrow \infty,$$

that is $\bar{X}_k$ a.s. is finite and unlimited r.v.

We introduce some necessary notations in the future. Let T_k^- be the length of cycles where the event $\{\bar{X}_k < u\}$ occurred (cycles of type 1),

$$F^-(x) = \mathbf{P}(T_k^- < x) = \mathbf{P}(T_k < x / \bar{X}_k < u),$$

$$m_1^- = \int_0^\infty x dF^-(x), \quad m_2^- = \int_0^\infty x^2 dF^-(x).$$

Similar notations

$$T_k^+, \quad F^+(x), \quad m_1^+, \quad m_2^+$$

we refer to cycles of the 2-nd type where the opposite event $\{\bar{X}_k \geq u\}$ took place.

Then Theorem 2 implies the following

Statement 1. Let $x > 0, \quad \hat{t} = \hat{t}(x,u) = x m_1^- / q(u)$,

$$\Delta(\bar{X}, x) = \mathbf{P}(\bar{X}(\hat{t}) \geq u) - 1 + \exp(-x).$$

If

$$m_2 = \mathbf{E}T^2 < \infty, \tag{48}$$

then

$$|\Delta(\bar{X}, x)| \;\leq\; \left\{ \begin{array}{lll} C_7(x) q(u), & \text{for} & 0 < x < 1, \\ C_8(x) q(u), & \text{for} & x \geq 1. \end{array} \right\} \tag{49}$$

where

$$C_7(x, \mathbb{F}) = \exp(-x) \left((2x(e-2)+1) \frac{m_1^+}{m_1^-} + (e-2) \left(\frac{4 m_2}{(m_1^-)^2} - 1 \right) + o(1) \right),$$

$$C_8(x, \mathbb{F}) = \left(\exp(-x) + \frac{2K(x)}{x} \right) \frac{m_1^+}{m_1^-} + \frac{K(x)}{x} \left(\frac{4 m_2}{(m_1^-)^2} - 1 + o(1) \right), \quad \mathbb{F} = (F^-, F^+).$$

The above statement clarifies the results of [19].

Proof of Statement 1.

Let

$$\epsilon_k(u) = I(\bar{X}_k \geq u),$$

$$\nu(u) = \min(k \geq 1 : \quad \epsilon_k(u) = 1).$$

For cycles of the 2-nd type we put

$$\eta_k = \inf(t \geq 0 : \xi_k(t) \geq u), \quad \hat{m}_1^+ = \mathbf{E}(\eta_k / \epsilon_k(u) = 1),$$

that is $\hat{m}_1^+$ be an average time to reach level u in cycles of the 2-nd type.

Let

$$\hat{T}_k \;=\; \left\{ \begin{array}{ll} \eta_k, & \text{for} \quad \epsilon_k(u) = 1, \\ T_k, & \text{otherwise.} \end{array} \right\}$$

Random events

$$(\bar{X}(t) \geq u) \quad \text{and} \quad (\sum_{k=1}^{\nu(u)} \hat{T}_k \leq t)$$

are equivalent (see. [19])), and then

$$\mathbf{P}(\bar{X}(\hat{t}) \geq u) = \mathbf{P}\left(\sum_{k=1}^{\nu(u)} \hat{T}_k \leq \hat{t}\right) = \mathbf{P}\left(\frac{q(u)}{a^-}\sum_{k=1}^{\nu(u)} \hat{T}_k \leq x\right). \tag{50}$$

The definition immediately implies the equality of distributions:

$$\sum_{k=1}^{\nu(u)-1} \hat{T}_k \overset{def}{=} \sum_{k=1}^{\nu(u)-1} T_k^-$$

and

$$\hat{T}_{\nu(u)} \overset{def}{=} \eta_{\nu(u)}.$$

Moreover, if the r.v. $\nu(u)$ and $\hat{T}_k$ are generally dependent, then $\nu(u)$ and $T_k^-, k = 1, 2, \ldots, \nu(u) - 1$, will be independent. Also r.v. $\eta_{\nu(u)}$ will not depend on $\sum_{k=1}^{\nu(u)-1} T_k^-$.

That is we have geometric sum with delay

$$\sum_{k=1}^{\nu(u)} \hat{T}_k \overset{def}{=} \zeta_1^d + \sum_{k=2}^{\nu(u)} \zeta_k,$$

where $\zeta_1^d = \eta_{\nu(u)}$, $\zeta_k = T_k^-, k \geq 2$.

To use Theorem 2, we need to check conditions (7) and (8).

Directly from the definition, we have the equations [19]

$$m_1 = (1 - q(u))m_1{}^- + q(u)m_1{}^+, \quad m_2 = (1 - q(u))m_2{}^- + q(u)m_2{}^+. \tag{51}$$

Further, from them it is not difficult to derive the following estimates

$$(\sigma^-)^2 = DT_k^- \leq m_2{}^- \leq \frac{m_2}{1 - q(u)}, \quad m_1^+ \leq \sqrt{\frac{m_2}{q(u)}}.$$

Thus, for $u \to \infty$

$$q(u)m_1^+ \to 0, \quad m_1^- \to m_1.$$

From here the relation (7) follows immediately.

Let us establish validity (8). We have for $y > 0$

$$\begin{aligned}
\frac{1}{q(u)}\left(1 - F^d\left(\frac{m_1^- y}{q(u)}\right)\right) &\leq \frac{1}{q(u)}\mathbf{P}\left(T_{\nu(u)} > \frac{m_1^- y}{q(u)}\right) \\
&= \frac{1}{q(u)}\sum_{k=1}^{\infty} \mathbf{P}(\nu(u) = k)\mathbf{P}\left(T_k > \frac{m_1^- y}{q(u)} \quad / \quad \nu(u) = k\right).
\end{aligned} \tag{52}$$

Further

$$\mathbf{P}\left(T_k > \frac{m_1^- y}{q(u)} \quad / \quad \nu(u) = k\right) = \mathbf{P}\left(T_k > \frac{m_1^- y}{q(u)} \quad / \quad \epsilon_1(u) = 0, \dots, \epsilon_{k-1}(u) = 0, \epsilon_k(u) = 1\right)$$
$$= \mathbf{P}\left(T_k > \frac{m_1^- y}{q(u)} \quad / \quad \epsilon_k(u) = 1\right) = \frac{1}{q(u)}\mathbf{P}\left(T_k > \frac{m_1^- y}{q(u)}, \quad \epsilon_k(u) = 1\right)$$
$$\leq \frac{1}{q(u)}\mathbf{P}\left(T_1 > \frac{m_1^- y}{q(u)}\right). \tag{53}$$

Since $\mathbf{E}T^2 = m_2 < \infty$, then

$$x^2\mathbf{P}(T_1 > x) \to 0 \quad \text{for} \quad x \to \infty.$$

Therefore, taking into account estimates (52) and (53), we get (8).

□

Denote by $\mathfrak{T}_X(u)$ the 1st moment when the process $X(t)$ reaches the level u, i.e.

$$\mathfrak{T}_X(u) = \inf(t \geq 0 : \quad X(t) \geq u).$$

Since

$$\forall t > 0 \quad \{\bar{X}(t) \geq u\} \Leftrightarrow \{\mathfrak{T}_X(u) \leq t\},$$

then the following corollary follows from Statement 1.

Corollary 1. *In the conditions and notation of Statement 1* $\quad \forall x > 0$

$$\left|\mathbf{P}\left(\frac{q(u)}{a^-}\mathfrak{T}_X(u) < x\right) - 1 + \exp(-x)\right| \leq \left\{ \begin{array}{ll} C_7(x, \mathbb{F})\ q(u), & \textit{for} \quad 0 < x < 1, \\ C_8(x, \mathbb{F})\ q(u), & \textit{for} \quad x \geq 1. \end{array} \right\} \tag{54}$$

Remark 3. If in Statement 1 we strengthen condition (48) to the following :

$$\exists \quad \alpha > 0, \quad \beta > 0 \quad \mathbf{E}\exp(\alpha T^\beta) < \infty, \tag{55}$$

then

$$m_1^+ \leq \left(\frac{1}{\alpha}\ln\frac{1}{q(u)}\right)^{1/\beta} \tag{56}$$

and

$$m_1^- = m_1 + o(1).$$

An estimate close to (56) at $\beta = 1$ is given in article [1]. Generalizing it to arbitrary $\beta > 0$ is not difficult.

Thus, under condition (55), inequality (49) can be rewritten as

$$|\Delta(\bar{X}, x)| \leq \left\{ \begin{array}{ll} C_9(x, \mathbb{F})\, q(u)(\ln \frac{1}{q(u)})^{1/\beta}, & \text{for} \quad 0 < x < 1, \\ C_{10}(x, \mathbb{F})\, q(u)(\ln \frac{1}{q(u)})^{1/\beta}, & \text{for} \quad x \geq 1, \end{array} \right\} \tag{57}$$

where

$$C_9(x, \mathbb{F}) = \frac{\exp(-x)(2x(e-2)+1)}{\alpha^{1/\beta} m_1}(1 + o(1)),$$

$$C_{10}(x, \mathbb{F}) = \frac{\exp(-x) + \frac{2K_0}{x}}{\alpha^{1/\beta} m_1}(1 + o(1)).$$

2. Birth and death processes.

Consider the birth and death process $X(t)$ with parameters

$$\lambda_0 > 0, \quad \mu_0 = 0, \quad \lambda_n > 0, \quad \mu_n > 0, \quad n = 1, 2, \ldots, \tag{58}$$

where λ_n and μ_n are the intensities of transitions from state n, respectively, up and down (see [12, ch.7 , §6]).

Let $X(0) = 0$ a.s. and put

$$\theta_0 = 1, \quad \theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \quad \alpha_k = \prod_{i=1}^{k} \frac{\mu_i}{\lambda_i}, \quad k \geq 1.$$

Let us assume that the following relations hold

$$\sum_{k \geq 1} \theta_k < \infty, \tag{59}$$

$$\sum_{k \geq 1} \alpha_k = \infty. \tag{60}$$

Conditions (59), (60), as is known (see [12] [13]), as is known, are sufficient for the existence of stationary state probabilities

$$\lim_{t \to \infty} \mathbf{P}(X(t) = k) = \lim_{t \to \infty} p_k(t) = p_k,$$

moreover

$$p_k = \theta_k p_0, \quad p_0 = \left(\sum_{k=0}^{\infty} \theta_k \right)^{-1}. \tag{61}$$

In addition, $X(t)$ will be a regenerative process with moments of regeneration $S_0 = 0, S_1, S_2, \ldots$, where S_k is the first moment of getting into state 0 after the k-th exit from it, $T_k = S_k - S_{k-1}$, is the length of the kth regeneration cycle $T_1 = T$.

Therefore, to estimate the value of $\Delta(\bar{X}, x)$, it is natural to use Statement 1. We will use the notation from the previous item 1.

Article [19] establishes the following formula for the birth and death processes

$$m_2 = \mathbf{E}T^2 = \mathbf{D}\mathcal{T} + \frac{1}{(\lambda_0)^2}\left(\frac{1}{p_0^2} + 1\right), \tag{62}$$

where

$$\mathbf{D}\mathcal{T} = \sum_{i=1}^{\infty}\left(\frac{1}{\mu_i(\mu_i+\lambda_i)} + \frac{\lambda_i}{\mu_i+\lambda_i}(m_{i+1,i}+m_{i,i-1})^2\right)\prod_{j=1}^{i-1}\frac{\lambda_j}{\mu_j}, \tag{63}$$

$$m_{k,k-1} = \sum_{i=k}^{\infty}\frac{1}{\mu_i}\prod_{j=k}^{i-1}\frac{\lambda_j}{\mu_j}, \quad k \geq 1, \tag{64}$$

$\mathcal{T}$ is the duration of the busy period on one regeneration cycle (that is, the period when $X(t) \neq 0$).

Statement 2. Let $X(t)$ be a birth and death process that satisfies conditions (59), (60) and series (63) converges, $u > 0$ is an integer, $x > 0$. Then inequality (49) from Statement 1 holds with constants $C_7(x,\mathbb{F})$, $C_8(x,\mathbb{F})$, in which m_2 is given by formula (62),

$$q(u) = \left(\sum_{k=0}^{u-1}\alpha_k\right)^{-1} = \left(\sum_{k=0}^{u-1}\frac{\lambda_0}{\lambda_k\theta_k}\right)^{-1}, \tag{65}$$

$$m_1^- = \frac{1}{\lambda_0}\left(1 + \frac{q(u)}{1-q(u)}\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\theta_k\left(1 - \frac{q(u)}{q(k)}\right)\right), \tag{66}$$

$$m_1^+ \leq \frac{1}{\lambda_0}\left(2\sum_{n=0}^{u-1}\alpha_n\sum_{k=n+1}^{\infty}\theta_k + 1\right). \tag{67}$$

Proof of Statement 2. In fact, here it is only necessary to check formulas (65)-(67). But equality (65) has long been known (see [5, Vol.1, §12, p.106-113], [12, Ch.3, Task 7], where it was established for the Markov chain embedded in the birth and death process). Thus, it remains only to prove formulas (66), (67).

Let's start with equality (66). We introduce the following notatio:

$$\mathbb{T}_{i,0} = \min(t > 0 : X(t) = 0/X(0) = i),$$

$$\mathbb{T}_{i,0}^{(u)} = \mathbb{T}_{i,0}I(\bar{X}(\mathbb{T}_{i,0}) < u), \quad i = 1, 2, \dots, u-1,$$

i.e., $\mathbb{T}_{i,0}^{(u)}$ is the time until process $X(t)$ reaches state 0 from state i for the first time for paths where level u is not reached, and is 0 otherwise.

And let $M_i = M_{i,0}^{(u)} = \mathbf{E}\mathbb{T}_{i,0}^{(u)}$, and τ_i is the time spent by process $X(t)$ in state i, $\mathbf{E}\tau_i = 1/(\lambda_i + \mu_i)$.

Note that Chun's results [5] imply the following formula

$$\mathbf{P}(\bar{X}(\mathbb{T}_{i,0}) \geq u), = 1 - f_{i,0}^{(u)} = \frac{q(u)}{q(i)}, \quad i = 1, 2, \ldots, u-1, \tag{68}$$

($f_{i,0}^{(u)}$ is the probability of the transition of the nested Markov chain from the state i to the state 0 without falling into the state u; $f_{i,0}^{(u)}$ and it was calculated in [5, Vol.1, §12]).

From the definition we have

$$\mathbb{T}_{i,0} = \tau_i + I(i \to i-1)\mathbb{T}_{i-1,0} + I(i \to i+1)\mathbb{T}_{i+1,0}, \quad i = 1, 2, \ldots, u-1,$$

here we denote by $I(i \to i-1), I(i \to i+1)$ the indicators of transitions from state i down and up, respectively.

Next, we multiply the right and left parts of the last equality by $I(\bar{X}(\mathbb{T}_{i,0}) < u)$ and proceed to the mathematical expectation. Get

$$\begin{aligned} M_i = \frac{1}{\lambda_i + \mu_i}\mathbf{P}(\bar{X}(\mathbb{T}_{i,0}) < u) \quad &+ \quad \frac{\mu_i}{\lambda_i + \mu_i}\mathbf{E}(\mathbb{T}_{i-1,0} I(\bar{X}(\mathbb{T}_{i,0}) < u)/i \to i-1) \\ &+ \quad \frac{\lambda_i}{\lambda_i + \mu_i}\mathbf{E}(\mathbb{T}_{i+1,0} I(\bar{X}(\mathbb{T}_{i,0}) < u)/i \to i+1). \end{aligned} \tag{69}$$

If condition $(i \to i-1)$is satisfied, that is, the 1st transition from state i is made to state $i-1$, then at $1 \leq i < u$

$$I(\bar{X}(\mathbb{T}_{i,0}) < u) = I(\bar{X}(\mathbb{T}_{i-1,0}) < u).$$

Similarly, under the condition $(i \to i+1)$

$$I(\bar{X}(\mathbb{T}_{i,0}) < u) = I(\bar{X}(\mathbb{T}_{i+1,0}) < u).$$

Thus, equalities (68), (69) imply the following recurrence relation:
for $1 \leq i < u$

$$M_i = b(i) + (1 - p_i)M_{i-1} + p_i M_{i+1}, \tag{70}$$

where $b_i = (1 - \frac{q(u)}{q(i)})/(\lambda_i + \mu_i), \quad p_i = \frac{\lambda_i}{\lambda_i + \mu_i}$.

Put $d_i = M_i - M_{i+1}$, $i = 0, 1, 2, \ldots, u-1$. It is easy to see that (70) implies the following equality

$$d_i = \frac{b_i}{p_i} + \frac{1 - p_i}{p_i} d_{i-1}, \quad i = 1, 2, \ldots, u-1,$$

(close reflections can be found in Karlin ([12], Ch.7, §7).

Then, iterating the last relation, we obtain

$$\begin{aligned} d_n &= \frac{b_n}{p_n} + \frac{(1-p_n)b_{n-1}}{p_n p_{n-1}} + \frac{(1-p_n)(1-p_{n-1})}{p_n p_{n-1}} d_{n-2} = \dots \\ &= \sum_{i=1}^{n} \frac{b_i}{p_i} \prod_{j=i+1}^{n} \frac{1-p_j}{p_j} + \prod_{j=1}^{n} \frac{1-p_j}{p_j} d_0. \end{aligned} \tag{71}$$

Since $M_0 = 0$, then $d_0 = -M_1$. And thus equality (71) can be written in the following form:

$$M_n - M_{n+1} = \sum_{i=1}^{n} \frac{b_i}{p_i} \prod_{j=i+1}^{n} \frac{\mu_j}{\lambda_j} - \alpha_n M_1. \tag{72}$$

Next, we use the equalities

$$\sum_{i=1}^{n} \frac{b_i}{p_i} \prod_{j=i+1}^{n} \frac{\mu_j}{\lambda_j} = \alpha_n \sum_{i=1}^{n} \frac{b_i}{p_i \alpha_i} = \frac{\alpha_n}{\lambda_0} \sum_{i=1}^{n} \left(1 - \frac{q(u)}{q(i)}\right) \theta_i$$

and once again rewrite (72)

$$M_n - M_{n+1} = \frac{\alpha_n}{\lambda_0} \sum_{i=1}^{n} \left(1 - \frac{q(u)}{q(i)}\right) \theta_i - \alpha_n M_1, \tag{73}$$

for $n = 1, 2, \dots, u-1$.

Considering that $M_0 = M_u = 0$, $\alpha_0 = 1$, $\sum_{i=1}^{0} = 0$, we will have that relation (73) is true and for $n = 0$.

Therefore, summing over n from 0 to $u-1$ the right and left parts of relation (73), we obtain

$$0 = \frac{1}{\lambda_0} \sum_{n=0}^{u-1} \alpha_n \sum_{i=1}^{n} \left(1 - \frac{q(u)}{q(i)}\right) \theta_i - \sum_{n=0}^{u-1} \alpha_n M_1$$

or

$$M_1 = \frac{q(u)}{\lambda_0} \sum_{n=0}^{u-1} \alpha_n \sum_{i=1}^{n} \left(1 - \frac{q(u)}{q(i)}\right) \theta_i.$$

From here it is not difficult to derive equality (66):

$$\begin{aligned} m_1^- &= \mathbf{E} T_k^- = \mathbf{E}(\tau_0 + T_{1,0} / \bar{X}(\mathbb{T}_{1,0}) < u) = \frac{1}{\lambda_0} + \frac{M_1}{1 - q(u)} \\ &= \frac{1}{\lambda_0} \left(1 + \frac{q(u)}{1 - q(u)} \sum_{n=0}^{u-1} \alpha_n \sum_{i=1}^{n} \left(1 - \frac{q(u)}{q(i)}\right) \theta_i \right). \end{aligned}$$

Let us pass to inequality (67). It is known that

$$m_1 = \mathbf{E}T = \frac{1}{\lambda_0 p_0} = \frac{1}{\lambda_0}\sum_{k=0}^{\infty}\theta_k, \tag{74}$$

(see, for example, [19]).

Then, taking into account also equalities (51), (65) and (66), we have

$$\begin{aligned} q(u)m_1^+ &= m_1 - (1-q(u))m_1^- \\ &= \frac{1}{\lambda_0}\sum_{k=0}^{\infty}\theta_k - \frac{1-q(u)}{\lambda_0}\left(1 + \frac{q(u)}{1-q(u)}\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\left(1-\frac{q(u)}{q(k)}\right)\theta_k\right) \\ &= \frac{1}{\lambda_0}\left(\sum_{k=1}^{\infty}\theta_k + q(u) - q(u)\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\left(1-\frac{q(u)}{q(k)}\right)\theta_k\right) \\ &= \frac{1}{\lambda_0}\left(\frac{\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{\infty}\theta_k}{\sum_{n=0}^{u-1}\alpha_n} + q(u) - \frac{\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\theta_k}{\sum_{n=0}^{u-1}\alpha_n} + q^2(u)\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\frac{\theta_k}{q(k)})\right) \\ &= \frac{q(u)}{\lambda_0}\left(\sum_{n=0}^{u-1}\alpha_n\sum_{k=n+1}^{\infty}\theta_k + 1 + q(u)\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\frac{\theta_k}{q(k)}\right). \end{aligned} \tag{75}$$

The last term in (75) can be estimated as follows

$$q(u)\sum_{n=0}^{u-1}\alpha_n\sum_{k=1}^{n}\frac{\theta_k}{q(k)} \le \sum_{k=0}^{u-1}\theta_k\sum_{i=0}^{k-1}\alpha_i \le \sum_{i=0}^{u-1}\alpha_i\sum_{k\ge i+1}\theta_k.$$

Putting together the last estimate and (75) we have

$$m_1^+ \le \frac{1}{\lambda_0}\left(2\sum_{n=0}^{u-1}\alpha_n\sum_{k=n+1}^{\infty}\theta_k + 1\right),$$

that is, inequality (67) is established. □

Remark 4. The value of m_1^- for birth and death process was found earlier in [19]. Unfortunately, the calculations given there contain an error.

Corollary 2. *Let $X(t)$ be a birth and death process that satisfies the following conditions:*

$$\forall \quad 0 \le i \le j, \quad 1 \le j, \quad \frac{\lambda_i}{\mu_j} \le \rho < 1, \tag{76}$$

$$\exists\delta > 0, \quad \forall j \ge 1, \quad \mu_j \ge \delta. \tag{77}$$

Then

$$|\Delta(\bar{X},x)| \;\leq\; \left\{ \begin{array}{ll} C_{11}(x)\, u\rho^u, & for \quad 0<x<1, \\ C_{12}(x)\, u\rho^u, & for \quad x \geq 1, \end{array} \right\} \tag{78}$$

where

$$C_{11}(x) = \exp(-x)(2p_0(e-1)+o(1)),$$
$$C_{12}(x) = \left(\exp(-x) + \frac{2K(x)}{x}\right)(2p_0(e-1)+o(1)).$$

Proof of Corollary 2. If conditions (76), (77) are satisfied, then it is easy to check that process $X(t)$ satisfies conditions (59), (60) and the series (63) converges.

In addition, under conditions (76), (77), the following inequalities hold:

$$q(u) \leq \left(\sum_{n=0}^{u-1} \frac{1}{\rho^i}\right)^{-1} = \frac{1/\rho - 1}{1/\rho^u - 1} \approx \rho^u \left(\frac{1}{\rho} - 1\right), \tag{79}$$

$$\alpha_n \sum_{i=n+1}^{\infty} \theta_i = \frac{\lambda_0}{\mu_{n+1}} + \frac{\lambda_0 \lambda_{n+1}}{\mu_{n+1}\mu_{n+2}} + \ldots \leq \rho + \rho^2 + \ldots = \frac{\rho}{1-\rho}.$$

And so

$$\sum_{n=0}^{u-1} \alpha_n \sum_{i=n+1}^{\infty} \theta_i \leq \frac{\rho u}{1-\rho}. \tag{80}$$

Note also that for $u \to \infty$

$$q(u) m_1^+ \to 0, \quad m_1^- \to m_1 = \frac{1}{\lambda_0 p_0}.$$

Hence, substituting estimates (79), (80) into Statement 2, we obtain estimate (78). □

Remark 5. If in Corollary 2 we put

$$\lambda_n = \lambda, \quad \mu_n = \mu, \quad n \geq 1, \quad \lambda_0 = \lambda, \quad \mu_0 = 0,$$

and

$$\frac{\lambda}{\mu} = \rho < 1,$$

then equality (78) holds with $p_0 = 1 - \rho$.

Such a birth and death process is equivalent to the process that describes the length of the queue in the classical queueing system $(M/M/1)$.

Let $\mathfrak{T}_X(u)$ be the 1st moment when the birth and death process $X(t)$ reaches level u.

Corollary 3. *Let $X(t)$ be the birth and death process. Then under the conditions of Corollary 2* $\quad \forall x > 0$

$$\left|\mathbf{P}(\frac{q(u)}{a^-}\mathfrak{T}_X(u) < x) - 1 + \exp(-x)\right| \;\leq\; \left\{ \begin{array}{ll} C_{11}(x)\, u\rho^u, & for \quad 0<x<1, \\ C_{12}(x)\, u\rho^u, & for \quad x \geq 1, \end{array} \right\}$$

where $C_{11}(x), C_{12}(x)$ are defined in Corollary 2.

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