Asymptotic behavior of maxima of independent random variables. Discrete case

Kateryna Akbash[a], Natalia Doronina[b], Ivan Matsak[b]

[a] Volodymyr Vynnychenko Central Ukrainian State Pedagogical University,
1, Shevchenko street, Kropyvnytskyi 25006, Ukraine

[b] Taras Shevchenko National University of Kyiv,
2/6, Academician Glushkov avenue, Kyiv 03127, Ukraine

kateryna.akbash@gmail.com (K.S.Akbash), amalphela@gmail.com (N.V. Doronina),
ivanmatsak@univ.kiev.ua (I.K.Matsak)

Abstract

The asymptotic behavior of almost surely extreme values of discrete random variables has been studied. Applications to birth and death processes and processes describing the length of the queue have been given.

MSC: Primary 60K25, 60F15, 60G70

Keywords: extreme values, discrete random variables, almost sure limit theorems

1 Introduction and main results

Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a random variable (r.v.) ξ with cumulative distribution function (c.d.f.) $F(x) = \mathbb{P}\{\xi < x\}$, $x \in \mathbb{R}$. For $n \in \mathbb{N}$ put

$$z_n = \max_{1\le i\le n} \xi_i. \tag{1}$$

Almost surely (a.s.) asymptotic behavior of z_n has been studied very widely (see, for example, [1]–[2], [4], [8], [12], [14], [19]). A detailed bibliography can be found, for example, in the books [7] and [11].

For example, it is known, see [1] and [8], that for a quite large class of random variables ξ with unbounded support and differentiable c.d.f. F, z_n satisfies a law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf.

We shall note the recent article [17], where these laws for continuous r.v. were significantly strengthened.

We are mainly interested in asymptotic behavior of z_n for discrete case that has been studied much less. It is well known that asymptotics in continuous and discrete cases can be significantly different (see [2], [16], [18]).

Let ξ be discrete r.v. with distribution $(i, p_i), i \geq 0$, more precisely let us assume that

$$\mathbf{P}(\xi = i) = p_i > 0, \quad \sum_{i=0}^{\infty} p_i = 1.$$

For such r.v. we will use the notation:

$$R(n) = -\ln \mathbf{P}(\xi \geq n) = -\ln \left(\sum_{i \geq n} p_i \right),$$

$$r(n) = R(n) - R(n-1).$$

Let us define the following functions for sufficiently large $t > 0$:

$$L_0(t) = t, \quad L_m(t) = \ln L_{m-1}(t), \quad m \in \mathbb{N}.$$

And let $\xi, \xi_1, \xi_2, \ldots,$ be a sequence of discrete independent identical distributed random variables (i.i.d.r.v.), z_n defined by equality (1). It was noticed quite a while ago that asymptotic behavior of z_n in discrete case is closely related to sequence

$$a_n = \max \left(k \geq 0 : \sum_{i \geq k} p_i \geq \frac{1}{n} \right). \tag{2}$$

The case when r.v. ξ has Poisson distribution, $p_i = \frac{\lambda^i}{i!} \exp(-\lambda), i \geq 0$, or in some sense is similar to Poisson distribution has been studied in articles [16], [18].

In article [18] has been proven the following

Theorem A. *Let ξ be discrete r.v. with distribution (i, p_i), $i \geq 0$, $\beta > 0$ an arbitrary number, a_n is given by equality (2). If the earlier defined function $r(n)$ satisfies the condition:*
when $n \to \infty$

$$r(n) = \beta \ln n + o(L_2(n)), \tag{3}$$

then

$$\mathbf{P}(\exists n_0, \forall n \geq n_0 \quad z_n \in J_n = \{a_n + m, \quad m \in I_\beta\}) = 1, \tag{4}$$

$$\forall m \in I_\beta \quad \mathbf{P}(z_n = a_n + m \quad \text{i.o.}) = 1, \tag{5}$$

and

$$a_n = \frac{\ln n}{\beta L_2(n)}(1 + o(1)), \tag{6}$$

where $I_\beta = \{-1, 0, 1, \ldots, [1 + 1/\beta]\}$, *("i.o". – abbreviation "infinitely often")*.

For Poisson distribution with parameter $\lambda > 0$ (in this case $r(n) = \ln n + o(1), \beta = 1$) equalities (4), (5) hold when $I_\beta = I_1 = \{-1, 0, 1, 2\}$ and

$$a_n = \frac{\ln n}{L_2(n)} \left(1 + \frac{L_3(n) + \ln \lambda + 1 + o(1)}{L_2(n)}\right).$$

When function $r(n)$ increases a bit slower than (3), for example, if the following conditions hold

$$r(n) = o(\ln n), \tag{7}$$

$$\sum_{n \geq 1} \exp(-e^{r(n)}) < \infty, \tag{8}$$

then equalities (4), (5) still be valid for $\beta = 0$ [18].

We shall note that under the following condition

$$r(n) = v_n \ln n, \quad v_n \to \infty, \quad n \to \infty, \tag{9}$$

equalities (4), (5) are also true when $\beta = \infty$.

Although not stated in [16], [18], this statement is in fact a simple consequence of these works (see Theorems 1, 2 in [16] and Lemma 3 in [18]).

Thus, under conditions (3), (7)-(9) we can describe the asymptotic of the value z_n with sufficient accuracy.

At the same time for discrete r.v. a number of related problems remained open. For example, geometric distribution which is important for probability theory and its application doesn't satisfy any condition (3), (7)-(9) and the natural question arises: are the equations for type (4), (5) hold for him?

In this paper, in contrast to [18], the focus will be on the geometric distribution and random variables, whose distribution tails drop more slowly than the tails of the geometric distribution.

Moreover, we present a discrete variant of some of the results of the paper [17] and consider some applications.

Let's formulate the main results of the work.

Theorem 1. *Let* $(\xi_k)_{k \in \mathbb{N}}$ *be a sequence of independent copies of a discrete random variable* ξ *with distribution* $(i, p_i), i \geq 0$, a_n *defined by equality (2) and for each fixed* m

$$\lim_{n \to \infty} \frac{r(n + m)}{r(n)} = 1, \tag{10}$$

$$\exists C_0 < \infty, \quad \forall n \geq 1 \quad r(n) \leq C_0. \tag{11}$$

(i) If

$$\sum_{n \geq 1} r^2(n) = \infty, \tag{12}$$

then for any integer m

$$\mathbf{P}(z_n = a_n + m \quad i.o.) = \mathbf{P}(z_n = \xi_n = a_n + m \quad i.o.) = 1. \tag{13}$$

(ii) If the condition (12) does not hold, then for any integer $m > 0$

$$\mathbf{P}(z_n = \xi_n \in (a_n - m, a_n + m) \quad i.o.) = \mathbf{P}(\xi_n \in (a_n - m, a_n + m) \quad i.o.) = 0. \tag{14}$$

It should be noted that equalities (13), (14) describe asymptotic z_n at the moments of “high jumps”, that is $\xi_n \geq z_{n-1}$. Such ”high jumps” seem the most interesting for applications. Whether equality (14) will be true for all z_n if condition (12) is not satisfied we do not know.

To formulate the following result, we introduce some necessary notation. For the sequence $(r(n))$ we define its extension to the function $r : (0, \infty) \to \mathbb{R}$ by setting $r(x) = r(\lceil x \rceil)$, ($\lceil x \rceil$ - the least integer $\geq x$).

Let

$$R(x) = \int_0^x r(y)dy.$$

The function R is a piecewise linear extension of the sequence $R(n)$.

Given a function $H : \mathbb{R} \to \mathbb{R}$ we denote by H^{-1} its generalized inverse defined by

$$H^{-1}(y) = \inf \{x \in \mathbb{R} : H(x) \geq y\}, \quad y \in \mathbb{R}. \tag{15}$$

Put

$$\alpha_m(t) = \sum_{i=1}^{m} L_i(t), \quad a_m(t) = R^{-1}(\alpha_m(t)),$$

$$d(n) = R^{-1}(L_1(n) - L_3(n)).$$

Theorem 2. *Let $(\xi_k)_{k \in \mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $m \geq 1$ some fixed integer. Let the following condition be satisfied:* $\forall x > 0$

$$\lim_{t \to \infty} \frac{r(tx)}{r(t)} = x^{\rho}, \quad \rho > -1. \tag{16}$$

Then

$$\mathbf{P}\left(\limsup_{n \to \infty} \frac{r(a_1(n))(z_n + \theta_n - a_m(n))}{L_{m+1}(n)} = 1\right) = 1, \tag{17}$$

$$\mathbf{P}\left(\liminf_{n \to \infty} \frac{L_2(n) r(a_1(n))(z_n + \theta_n - d(n))}{2L_3(n)} = -1\right) = 1, \tag{18}$$

where θ_n some r.v., $0 \leq \theta_n \leq 1$.

Corollary 1. *(i) If the condition (11) holds, then value θ_n in formula (17) can be omitted. (ii) The same way can be omitted θ_n in formula (18), if*

$$\frac{L_2(n)r(a_1(n))}{L_3(n)} \to 0, \quad n \to \infty.$$

(iii) If

$$\frac{L_2(n)r(a_1(n))}{L_3(n)} \to \infty, \quad n \to \infty,$$

then

$$\mathbf{P}(\liminf_{n\to\infty}(z_n - d(n)) = \kappa) = 1, \tag{19}$$

where $\kappa \in [-1, 0]$. Here and further by κ we denote nonrandom constant, not necessarily the same in different parts of the article.

In the statements mentioned above, it was assumed that the functions $F(x)$ and $r(x)$ are known exactly. Unfortunately in many important practical cases this is not it. More often only their asymptotic is known when $x \to \infty$. Consider one such example.

Proposition 1. *Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $m \geq 1$ - some fixed integer.*

If for some $\gamma > 0$, $C_1 < \infty$, the next asymptotic relation holds

$$R(n) = \gamma n + C_1 + o(1), \tag{20}$$

then for any integer m, equality (13) holds with $a_n = \left[\frac{\ln n - C_1 + o(1)}{\gamma}\right]$,
and

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{\gamma z_n - \alpha_m(n)}{L_{m+1}(n)} = 1\right) = 1, \tag{21}$$

$$\mathbf{P}(\liminf_{n\to\infty}(z_n - (L_1(n) - L_3(n))/\gamma) = \kappa) = 1, \tag{22}$$

where $\quad \kappa \in [-1 - C_1/\gamma, \ -C_1/\gamma]$.

In the end of the paper, the examples of application of the obtained results of asymptotics of extreme values of birth and death processes and processes in queuing systems (QS) are considered.

Such problems were studied in many works ([2]-[3], [10], [13], [21], [24]), however it was mainly a case of weak convergence.

2 Proof of Theorem 1

Let's start with the auxiliary lemmas. For the sequence of discrete i.i.d.r.v ξ, ξ_1, ξ_2, ..., with distribution $(i, p_i), i \geq 0$, we create random events A_n, $A_n{}'$ as following:

$$A_n = \{\xi_n = z_n = a_n + m\}, \quad A_n{}' = \{\xi_n \in [a_n - m, a_n + m)\}, \tag{23}$$

where a_n is defined by the formula (2), m - some integer.

Lemma 1. *Let random events A_n be given by (23), m - arbitrary fixed integer. If, under Theorem 1, the function $r(n)$ satisfies equality (12), then*

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty, \tag{24}$$

Proof. In lemma 3 of the work [18] the following lower bounds for the series in (24) obtained:

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}(A_n) &= \sum_{n=1}^{\infty} \mathbf{P}(\xi = a_n + m)(1 - \mathbf{P}(\xi \geq a_n + m + 1))^{n-1} \\ &\geq \sum_{k=0}^{\infty} \mathbf{P}(\xi = k + m)(1 - \mathbf{P}(\xi \geq k + m + 1))^{\exp(R(k+1))} \sum_{n:a_n=k} 1, \end{aligned} \tag{25}$$

$$\sum_{n:a_n=k} 1 \;=\; \exp(R(k+1))(1 - \exp(-r(k+1))) + \theta_k, \tag{26}$$

where $|\theta_k| \leq 1$,

$$\begin{aligned} \mathbf{P}(\xi = k + m) &= \exp(-R(k+m)) - \exp(-R(k+m+1)) \\ &= \exp(-R(k+m))(1 - \exp(-r(k+m+1))). \end{aligned} \tag{27}$$

Obviously,

$$\sum_{k=0}^{\infty} \mathbf{P}(\xi = k + m)(1 - \mathbf{P}(\xi \geq k + m + 1))^{\exp(R(k+1))} |\theta_k| \leq 1. \tag{28}$$

Putting (25) - (27), (28) together, we get

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbf{P}(A_n) \geq \sum_{k=0}^{\infty} \exp(-R(k+m) + R(k+1))(1 - \exp(-r(k+1))) \\ &\times \;\; (1 - \exp(-r(k+m+1)))(1 - \exp(-R(k+m+1)))^{\exp(R(k+1))} - 1. \end{aligned} \tag{29}$$

Further, we note that if condition (10) of Theorem 1 is satisfied, only two cases are possible:
a) $\exists \delta > 0$ and subsequence (k_i) such as $r(k_i) \geq \delta$, $r(k_i + 1) \geq \delta, \ldots, r(k_i + m + 1) \geq \delta$;
b) $r(k) \to 0$ when $k \to \infty$.

Let's start with case a). According to estimate (29) we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}(A_n) &\geq \sum_{i} \exp(-|m|C_0)(1 - \exp(-\delta))^2 \\ &\times \;\; (1 - \exp(-R(k_i + m + 1)))^{\exp(R(k_i+1))} - 1. \end{aligned} \tag{30}$$

As it is well known

$$\left(1-\frac{1}{x}\right)^{x} \uparrow \frac{1}{e} \quad \text{when} \quad x \uparrow \infty \tag{31}$$

Suppose that $m \geq 0$. Then $R(k+m+1) \geq R(k+1)$ and the series on the right-hand side of inequality (30) diverges and therefore the series on the left-hand side also diverges.

Let $m < 0$. The same as mentioned above the inequality (30) holds. Given the inequality

$$|R(k+1) - R(k+m+1)| \leq |m|C_0 \tag{32}$$

and asymptotic relation (31) obtained for sufficiently large k and $\epsilon \leq 0.1$

$$(1-\exp(-R(k+m+1)))^{\exp(R(k+1))} \geq \left(\frac{1}{e}-\epsilon\right)^{\exp(mC_0)},$$

that is again series in the right-hand and left-hand sides (30) diverge.

Let us turn to case b). Again, we use the estimate (29) and elementary inequality

$$1-\exp(-x) \geq x - \frac{x^2}{2}, \quad x \geq 0.$$

Then for sufficiently large k the corresponding item in the sum on the right-hand side is estimated lower by value

$$\exp\left(-R(k+m)+R(k+1)\right)\left(r(k+m+1)-\frac{r(k+m+1)^2}{2}\right)\times$$
$$\times\left(r(k+1)-\frac{r(k+1)^2}{2}\right)(1-\exp(-R(k+m+1)))^{\exp(R(k+1))}.$$

Since $r(k) \to 0$ when $k \to \infty$, then for any whole fixed m

$$-R(k+m)+R(k+1) \to 0,$$

and

$$(1-\exp(-R(k+m+1)))^{\exp(R(k+1))} \to \frac{1}{e}.$$

If we add more condition (12), it is easy to see that the series (24) diverges. □

Lemma 2. *Let random events $A_n{}'$ be given by equality (23), m - arbitrary fixed integer and the conditions of Theorem 1 hold except (12), that is*

$$\sum_{n\geq 1} r^2(n) < \infty, \tag{33}$$

then

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n{}') < \infty. \tag{34}$$

Proof. Just as in lemma 1 we have the equality

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{P}(A_n{}') &= \sum_{n=1}^{\infty}\mathbf{P}(\xi_n \in [a_n - m, a_n + m)) \\ &= \sum_{k=0}^{\infty}\mathbf{P}(\xi \in [k-m, k+m)) \sum_{n:a_n=k} 1 \\ &= \sum_{j=-m}^{m}\sum_{k=0}^{\infty}\mathbf{P}(\xi = k+j) \sum_{n:a_n=k} 1. \end{aligned} \tag{35}$$

Clearly, to prove inequality (34), it suffices to establish the boundedness of the sum

$$S_j = \sum_{k=0}^{\infty}\mathbf{P}(\xi = k+j) \sum_{n:a_n=k} 1 \tag{36}$$

for any $j \in [-m, m]$.

Under condition (10) we have $\quad r(k+m) = r(k)(1+o(1)), \quad k \to \infty.$

Therefore,

$$\begin{aligned}\mathbf{P}(\xi = k+j) &= \exp(-R(k+j)) - \exp(-R(k+j+1)) \\ &= \exp(-R(k+j))r(k)(1+o(1)). \end{aligned} \tag{37}$$

Further we put estimates (37) and (26) into the formula (36)

$$\begin{aligned} S_j &\leq \sum_{k=0}^{\infty}\exp(R(k+1) - R(k+j))r(k)^2(1+o(1)) \\ &+ \sum_{k=0}^{\infty}\mathbf{P}(\xi = k+j)|\theta_k|. \end{aligned}$$

Obviously, in the last estimate, the second sum on the right ≤ 1. Hence, if we take into account condition (11) of theorem 1 and condition (33), we obtain the boundedness of the value S_j.

□

We proceed directly to the proof of Theorem 1. For the sequence of random events (A_n) defined by equality (23), we introduce the following notation

$$S_n^{'} = \sum_{j=1}^{n}\mathbf{P}(A_j), \quad S_n^{''} = \sum_{1\leq j<l\leq n}\mathbf{P}(A_j \bigcap A_l).$$

Firstly, we show that under the conditions of Theorem 1 there exists a constant $K < \infty$ such that

$$\limsup_{n\to\infty}\frac{S_n^{''}}{S_n^{'\,2}} \leq K. \tag{38}$$

To this end, we use simple equality

$$S_n^{''} = \sum_{1 \le j < l \le n} \mathbf{P}(A_j)\mathbf{P}(A_l)C_{j,l}, \tag{39}$$

where $C_{j,l} = (\mathbf{P}(\xi_i \le a_l + m, i = 1, \ldots, j))^{-1}$ (see Lemma 3 in [18]).

Let us estimate the value of $C_{j,l}$ from above. Suppose that $a_l = k$. Then $l < \exp(R(k+1))$ and

$$\begin{aligned} C_{j,l} &\le (1 - \exp(-R(k+m+1)))^{-l} \\ &\le (1 - \exp(-R(k+m+1)))^{-\exp(R(k+1))}. \end{aligned}$$

Note that for $x > 1$ the function $(1 - 1/x)^{-x}$ decreases when x increases. Hence and from inequality (32) we obtain

$$C_{j,l} \le C_2 = (1 - \exp(-R(1)))^{-\exp(R(1)+|m|C_0)}.$$

Substituting the last estimate in (39) we have

$$S_n^{''} \le C_2 \sum_{1 \le j < l \le n} \mathbf{P}(A_j)\mathbf{P}(A_l).$$

And finally by Lemma 1 $S_n^{'} \to \infty$ when $n \to \infty$. Then

$$S_n^{'\,2} = 2 \sum_{1 \le j < l \le n} \mathbf{P}(A_j)\mathbf{P}(A_l) + O(S_n^{'}).$$

Together, the latter estimates give inequality (38) at $K = C_2/2$.

It remains to rewrite inequality (38) as follows

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n}\sum_{l=1}^{n} \mathbf{P}(A_j \bigcap A_l)}{(\sum_{j=1}^{n} \mathbf{P}(A_j))^2} \le \limsup_{n\to\infty} \frac{2S_n^{''} + S_n^{'}}{S_n^{'\,2}} \le 2K. \tag{40}$$

The generalized Borel-Cantelli lemma (see [22], Ch. 6, §26) allows us to derive the following inequality from the estimates (24), (40)

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) \ge \frac{1}{2K}.$$

Hence from Hewitt-Savage zero-one law we have

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = 1,$$

that is equality (13) of the Theorem 1 is defined.

The point (ii) of Theorem 1 follows directly from lemma 2 and the Borel-Cantelli lemma.
□

3 Proof of Theorem 2

Definition 1. We say that a function $H : \mathbb{R} \to \mathbb{R}$ satisfies condition $(\mathbb{U}_1)$ if the following holds:

1. $\lim_{x\to+\infty} H(x) = +\infty$;
2. the function H is strictly increasing for $x \in (x_0, \infty)$ where $x_0 := \inf\{x \in \mathbb{R} : H(x) > 0\}$;
3. there exists $\rho \in \mathbb{R}$ such that the function $H'(x)$ is regularly varying at $+\infty$ with index $\rho > -1$.

The conception of regularly varying functions is well-known, see [5], [20].

The main results of [17] about the asymptotic behavior of extreme values of continuous r.v. obtained just in the case where the function $H(x) = -\ln(1 - F(x))$ satisfies condition $(\mathbb{U}_1)$.

If the function $r(x)$ satisfies condition (16), then it is regularly varying at $+\infty$ with index ρ, $r \in RV_\rho$. Then also $R \in RV_{\rho+1}$, $R^{-1} \in RV_{1/(\rho+1)}$ ([5], Proposition 1.5.8, Theorem 1.5.12) and $h \in RV_{-\rho/(\rho+1)}$. Moreover

$$R^{-1}(x) = \int_0^x h(y)dy, \quad h(y) = \frac{1}{r(R^{-1}(y))}. \tag{41}$$

Consider a r.v. ξ^c with a distribution function $F(x) = 1-\exp(-R(x))$, $x > 0$, $F(0) = 0$. Let $(\xi^c_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a r.v. ξ^c ,

$$z^c_n = \max_{1\le i\le n} \xi^c_i.$$

First, we show that under the conditions of the Theorem 2 the following asymptotic equalities hold:
$\forall m \in \mathbb{N}$

$$\limsup_{n\to\infty} \frac{r(a_m(n))(z^c_n - a_m(n))}{L_{m+1}(n)} = 1 \quad \text{a.s.} \tag{42}$$

$$\liminf_{n\to\infty} \frac{L_2(n)r(d(n))(z^c_n - d(n))}{2L_3(n)} = -1 \quad \text{a.s.}, \tag{43}$$

If function $R(x)$ would satisfy condition $(\mathbb{U}_1)$, then equalities (42), (43) would be a simple consequence of theorems 1, 2 in [17]. Unfortunately function $R(x)$ is not differentiated in a countable set of points. Therefore, we will have to slightly modify the corresponding proof from [17].

Establish the equality (42). Let τ^e be a standard exponentially distributed r.v., that is $\mathbf{P}(\tau^e < x) = 1 - \exp(-x)$. Let $(\tau^e_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a r.v. τ^e ,

$$z^e_n = \max_{1\le k\le n} \tau^e_k.$$

Without loss of generality, we can assume that

$$z_n - a_m(n) = R^{-1}(z_n^e) - R^{-1}(\alpha_m(n)). \tag{44}$$

(see proof of Theorem 1 in [17]).

The following equality was obtained in Lema 2 in [17]

$$\limsup_{n\to\infty} \frac{z_n^e - \alpha_m(n)}{L_{m+1}(n)} = 1 \quad \text{a.s.} \tag{45}$$

Furthermore, we assume that

$$z_n^e(n) \geq \alpha_m(n) \tag{46}$$

(since $R^{-1}(x)$ is a monotonically increasing function, taking into account (45), it is sufficient to choose only those n, for which (46)) holds.

We fix an arbitrary sufficiently small $\epsilon > 0$ and introduce the following notation

$$h_n^- = \inf_{\alpha_m(n)\leq t\leq z_n^e} h(t), \quad h_n^+ = \sup_{\alpha_m(n)\leq t\leq z_n^e} h(t),$$

$$\zeta_n^- = \sup(t \leq z_n^e : \quad h(t) \leq h_n^-(1+\epsilon)),$$
$$\zeta_n^+ = \sup(t \leq z_n^e : \quad h(t) \geq h_n^+(1-\epsilon)).$$

Then, respectively (41) we obtain

$$h_n^-(z_n^e - \alpha_m(n)) \leq R^{-1}(z_n^e) - R^{-1}(\alpha_m(n)) \leq h_n^+(z_n^e - \alpha_m(n)).$$

Functions $r(t)$ and $h(t)$ after construction are continuous from the left. Therefore

$$h(\zeta_n^-) \leq h_n^-(1+\epsilon)), \quad h(\zeta_n^+) \geq h_n^+(1-\epsilon)).$$

And therefore

$$\frac{1}{1+\epsilon}h(\zeta_n^-)(z_n^e - \alpha_m(n)) \leq R^{-1}(z_n^e) - R^{-1}(\alpha_m(n)) \leq \frac{1}{1-\epsilon}h(\zeta_n^+)(z_n^e - \alpha_m(n)).$$

Keeping in mind the equality $h(\alpha_m(n)) = 1/r(a_m(n))$ and equality (44) we can rewrite the last inequality as:

$$\frac{1}{1+\epsilon}\frac{h(\zeta_n^-)}{h(\alpha_m(n))}\frac{z_n^e - \alpha_m(n)}{L_{m+1}(n)} \leq \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)} \leq \frac{1}{1-\epsilon}\frac{h(\zeta_n^+)}{h(\alpha_m(n))}\frac{z_n^e - \alpha_m(n)}{L_{m+1}(n)}. \tag{47}$$

It is known, see [7], Chapter 4, Example 4.3.3, that

$$\frac{z_n^e}{\ln n} \to 1 \quad \text{a.s.}$$

and also

$$\frac{z_n^e}{\alpha_m(n)} \to 1 \quad \text{a.s.},$$

as $n \to \infty$.

But $\zeta_n^-, \zeta_n^+ \in (\alpha_m(n), z_n^e)$, therefore

$$\frac{\zeta_n^-}{\alpha_m(n)} \to 1, \quad \frac{\zeta_n^+}{\alpha_m(n)} \to 1.$$

From this we obtain that

$$\frac{h(\zeta_n^-)}{h(\alpha_m(n))} \to 1, \quad \frac{h(\zeta_n^+)}{h(\alpha_m(n))} \to 1 \tag{48}$$

as $n \to \infty$ (see similar conversions in [17]).

Putting together relations (45), (47)(48), we obtain

$$\frac{1}{1+\epsilon} \leq \limsup_{n\to\infty} \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)} \leq \frac{1}{1-\epsilon} \quad \text{a.s.} \tag{49}$$

Estimates (49) are satisfied for any $\epsilon > 0$, therefore, from this we obtain the equality (42).

Similarly, based on Lemma 4 in [17], we can prove the equality (43).

It remains to make the transformation from (42),(43) to equalities (17),(18) in Theorem 2.

Further we notice that for $k \in \mathbb{N}$ random events are

$$\{\xi^c < k\} \quad \text{and} \quad \{[\xi^c] < k\},$$

equivalent, that is

$$\mathbf{P}([\xi^c] < k) = \mathbf{P}(\xi^c < k) = 1 - \exp(R(k)).$$

Thus r.v. $[\xi^c]$ and ξ are identically distributed. The same is true for r.v. $[z_n^c]$ and z_n. If we denote $\theta_n^c = z_n^c - [z_n^c]$, then $z_n^c - \theta_n^c$ and z_n have the same asymptotic behavior at infinity. Hence and (42), (43) we obtain: there exists θ_n, $0 \leq \theta_n \leq 1$, so that for every fixed $m \in \mathbb{N}$

$$\limsup_{n\to\infty} \frac{r(a_m(n))(z_n + \theta_n - a_m(n))}{L_{m+1}(n)} = 1 \quad \text{a.s.}, \tag{50}$$

$$\liminf_{n\to\infty} \frac{L_2(n) r(d(n))(z_n + \theta_n - d(n))}{2L_3(n)} = -1 \quad \text{a.s.} \tag{51}$$

Since the functions $r(x)$ and $R^{-1}(x)$ are regularly varying at infinity, therefore we have an implication:

$$\frac{\alpha_m(n))}{\alpha_1(n))} \to 1 \Rightarrow \frac{r(a_m(n))}{r(a_1(n))} \to 1, \quad n \to \infty,$$

(see [6]).

In the same way we get

$$\frac{r(d(n))}{r(a_1(n))} \to 1, \quad n \to \infty,$$

that together with equalities (50), (51) completes the proof of Theorem 2.

Proof of Corollary 1. Here, only point (iii) needs some explanation. It is simply deduced from Theorem 2. Indeed, let us put

$$\chi_n = \frac{L_2(n) r(a_1(n))}{L_3(n)}.$$

Then in accordance with equality (18), we have: $\forall \epsilon > 0$

$$\mathbf{P}\left(\exists (n_i), \quad z_{n_i} - d(n_i) + \theta_{n_i} \leq \frac{-1+\epsilon}{\chi_{n_i}} \quad i.o. \right) = 1,$$

$$\mathbf{P}\left(\exists n_0, \quad \forall n \geq n_0 \quad z_n - d(n) + \theta_n \geq \frac{-1-\epsilon}{\chi_n} \right) = 1.$$

Given that ϵ is arbitrary positive number and $\chi_n \to \infty$ when $n \to \infty$, from the last relations it follows that

$$\liminf_{n\to\infty} (z_n - d(n)) = \kappa \in [-1, 0] \quad a.s. \tag{52}$$

Moreover, by the Hewitt-Savagee zero-one law, κ is a degenerate r.v., that is (19) holds. □

Remark 1. Since z_n is an integer r.v., then the following relations seem of interest

$$\mathbf{P}(z_n - [d_n] \in \{-1, 0, 1\} \quad \text{i.o.}) = 1,$$

$$\mathbf{P}(z_n - [d_n] < -1 \quad \text{i.o.}) = 0,$$

which we obtain from equality (52) .

4 Proof of proposition 1

From condition (20) we obtain $r(n) = \gamma + o(1)$. Thus the condition (10) is satisfied. Accordingly, by Theorem 1 we have the equality (13). The formula for $a(n)$ simply follows from definition (2):

$$a_n = \max\left(k \geq 0 : \exp(-R(k)) \geq \frac{1}{n}\right) = \max\left(k \geq 0 : k \leq \frac{\ln n - C_1 + o(1)}{\gamma}\right)$$

$$= \left[\frac{\ln n - C_1 + o(1)}{\gamma}\right].$$

Relations (21), (22) are obtained from the theorem 2, since its conditions are also satisfied. It remains only to find the asymptotic behavior of the function $R^{-1}(x)$.

An anonymous reviewer has somewhat refined the interval for κ compared to the original version. Here is his reasoning.

If R denotes the piecewise linear extension of the sequence (R(n)) then

$$R(x) = R([x]) - r([x])([x] - x) =$$

$$= \gamma[x] + C_1 + o(1) - (\gamma + o(1))([x] - x) = \gamma x + C_1 + O(1),$$

as $x \to \infty$. Therefore, denoting $x_u = R^{-1}(u)$, we get, as $u \to \infty$,

$$u = R(x_u) = \gamma x_u + C_1 + o(1),$$

$$x_u = \frac{u - C_1 - o(1)}{\gamma} = \frac{u}{\gamma} - \frac{C_1}{\gamma} + o(1).$$

Hence $d(n) = (L_1(n) - L_3)/\gamma - C_1/\gamma + o(1)$ and (22) holds with $\kappa \in [-1 - C_1/\gamma, -C_1/\gamma]$.
□

5 Examples

Let's consider some examples of application of Theorems 1, 2 and Proposition 1.

Example 1. (Geometric distribution). Let $0 < q < 1$,

$$\mathbf{P}(\xi = k) = p_k = q(1-q)^k, \quad k \geq 0.$$

Then

$$\mathbf{P}(\xi \geq k) = (1-q)^k = \exp(-\gamma k), \quad \gamma = \ln \frac{1}{1-q},$$

that is

$$R(k) = \gamma k, \quad r(k) = \gamma.$$

It is clear that conditions (11), (12) of Theorem 1 hold, and therefore r.v. z_n satisfies equality (13). Moreover, via formula (2) we find $a_n = [(1/\gamma)\ln n]$.

Similarly, the conditions of the Theorem 2 hold. Taking into account Corollary 1, we obtain

$$\limsup_{n\to\infty} \frac{\gamma z_n - \alpha_m(n)}{L_{m+1}(n)} = 1 \quad a.s.,$$

$$\liminf_{n\to\infty} (z_n - (L_1(n) - L_3(n))/\gamma) = \kappa \quad a.s.,$$

where $\kappa \in [-1, 0]$.

In fact, the last equation can be refined. Namely, based on the results of [14] we can prove that for the geometric distribution satisfies the following

$$\mathbf{P}(z_n \leq [(L_1(n) - L_3(n))/\gamma] \quad \text{i.o.}) = 1.$$

From this and Remark 1 we have

$$\mathbf{P}(z_n - [(L_1(n) - L_3(n))/\gamma] \in \{-1, 0\} \quad \text{i.o.}) = 1,$$

$$\mathbf{P}(z_n - [(L_1(n) - L_3(n))/\gamma] < -1 \quad \text{i.o.}) = 0.$$

Example 2. (Queuing system $M/M/m$). Let us now consider a queuing system with m servers, $1 \le m < \infty$, and customers which arrive according to the Poisson process with intensity λ, and service times being independent copies of a random variable η with an exponential distribution

$$\mathbf{P}(\eta \le x) = 1 - \exp(-\mu x), \quad x \ge 0.$$

In the standard notation, this queuing system has the type $M/M/m$, see [9], [15].

We impose the following assumption on the parameters λ and μ ensuring existence of the stationary regime:

$$\rho := \frac{\lambda}{m\mu} < 1. \tag{53}$$

For $t \ge 0$, let $Q(t)$ denote the length of the queue at time t, that is, the total number of customers in service or pending. Set

$$\bar{Q}(t) = \sup_{0 \le s < t} Q(s), \quad t \ge 0.$$

Let us introduce regeneration moments (S_k) for the process Q: $S_0 := 0$ and, for $i \in \mathbb{N}$, S_i is the arrival time of a new customer after the i-th busy period. Let T_i be the duration of the i-th regeneration cycle and $\bar{Q}(T_1)$ be the maximum length of the queue in the first regeneration cycle. It is well-known that $a_T = \mathbf{E}T_1 = 1/(\lambda p_0)$,

$$p_0 = \left(\sum_{k=0}^{m} \frac{(m\rho)^k}{k!} + \frac{\rho^m m^m}{m!(\frac{1}{\rho} - 1)} \right)^{-1}.$$

Put

$$\mathbf{P}(\bar{Q}(T_1) \ge n) = \exp(-R(n)).. \tag{54}$$

In recent paper [10] the authors established that the sequence $(R(n))$ in (54) satisfies conditions (20) with

$$\gamma = \ln \frac{1}{\rho}, \quad C_1 = \ln \frac{\rho\, m!}{m^m(1-\rho)}. \tag{55}$$

Based on these equalities in the article [10] it was found that $\bar{Q}(t)$ satisfies a law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf.

Here we will strengthen this result the following way: $\forall k \ge 1$

$$\limsup_{t\to\infty} \frac{\gamma \bar{Q}(t) - \alpha_k(t)}{L_{k+1}(t)} = 1 \quad \text{a.s.}, \tag{56}$$

and

$$\liminf_{t\to\infty} \left(\bar{Q}(t) - \frac{1}{\gamma}(L_1(t) - L_3(t)) \right) = \kappa \quad \text{a.s.}, \tag{57}$$

where $\kappa \in [-1-(C_1+\ln a_T)/\gamma,\ -(C_1+\ln a_T)/\gamma]$.

Indeed, denote by N the counting process for the sequence (S_k), that is,

$$N(t)=\max\{k\geq 0: S_k\leq t\},\quad t\geq 0.$$

By the strong law of large numbers for N we have

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{a_T}\quad \text{a.s.},$$

whence, as $t\to\infty$,

$$\ln N(t)=\ln\frac{t}{a_T}+o(1)\quad \text{a.s.} \tag{58}$$

Then we put

$$Z_n=\bar{Q}(S_n).$$

From Proposition 1 it follows that for r.v. Z_n the equalities (21) and (22) hold with γ and C_1 , which are defined in (55), that is

$$\limsup_{n\to\infty}\frac{\gamma Z_n-\alpha_k(n)}{L_{k+1}(n)}=1\quad \text{a.s.},$$

$$\liminf_{t\to\infty}(Z_n-\frac{1}{\gamma}(L_1(n)-L_3(n)))=\kappa\quad \text{a.s.},$$

where $\kappa\in[-1-C_1/\gamma,\ -C_1/\gamma]$.

The procedure of the transition from here to (56), (57) is known and is based on the estimate (58) and the following inequalities

$$Z_{N(t)}\leq \bar{Q}(t)\leq Z_{N(t)+1}\quad \text{a.s.}$$

(see, for example, [10]).

Further we consider r.v.

$$\bar{Q}_n=\sup_{0\leq k\leq n}Q(t_k),\quad n\geq 0,$$

where $t_0=0,t_1,t_2\ldots$ moments of receipt of applications in the system.

It is easy to see that

$$\lim_{n\to\infty}\frac{N(t_n)}{n}=\lim_{n\to\infty}\frac{N(t_n)}{t_n}\frac{t_n}{n}=\frac{1}{\lambda a_T}=p_0\quad \text{a.s.}$$

Repeating the observations mentioned from Proposition 1 we obtain

$$\limsup_{n\to\infty}\frac{\gamma\bar{Q}_n-\alpha_k(n)}{L_{k+1}(n)}=1\quad a.s.,$$

$$\liminf_{n\to\infty}(\bar{Q}_n - (L_1(n) - L_3(n))/\gamma) = \kappa \quad a.s.,$$

where $\kappa \in [-1 - (C_1 + \ln p_0)/\gamma,\ -(C_1 + \ln p_0)/\gamma]$.

Example 3. (Birth and death processes). Let $X = (X(t))_{t\geq 0}$ be a birth and death processes with parameters

$$\lambda_n = \lambda v_n + A, \quad \mu_n = \mu v_n + B, \quad n = 1, 2, \dots, \tag{59}$$

$$\lambda_0 = A, \quad \mu_0 = 0, \quad \lambda, \mu, v_n, A, B > 0.$$

That is, $(X(t))_{t\geq 0}$ is a time-homogeneous Markov process such that, for $t \geq 0$, given $\{X(t) = n\}$ the probability of transition to state $n+1$ over a small period of time δ is $(\lambda v_n + A)\delta + o(\delta)$, and the probability of transition to $n-1$ is $(\mu v_n + B)\delta + o(\delta)$, $n = 1, 2, 3, \dots$. The parameter a can be interpreted as the infinitesimal intensity of population growth due to immigration, and B characterizes the intensity of population decline due to emigration.

In case $v_n = n$ the birth–death process X is usually called the process with linear grow (see [15, Ch. 7, §6]).

We assume that $X(0) = 0$, $v_n \uparrow \infty$ as $n \uparrow \infty$,

$$\sum_{n\geq 1} \frac{1}{v_n} < \infty, \tag{60}$$

and

$$\rho := \frac{\lambda}{\mu} < 1. \tag{61}$$

Put

$$\theta_0 = 1, \quad \theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \quad k \in \mathbb{N}.$$

Under conditions (60) and (61), there exists a stationary regime, that is,

$$\lim_{t\to\infty} \mathbf{P}(X(t) = k) = p_k,$$

with

$$p_k = \theta_k p_0, \quad k = 0, 1, 2, 3, \dots, \quad \text{where} \quad p_0 = \left(\sum_{k=0}^{\infty} \theta_k\right)^{-1}. \tag{62}$$

Further, X is a regenerative process with regeneration moments (S_k), where $S_0 = 0$ and S_i, $i \in \mathbb{N}$, is the moment of i-th return to state 0. It is known that

$$a_T = \mathbf{E}T_k = \frac{1}{Ap_0},$$

where $T_k = S_k - S_{k-1}$ is the duration of the k-th regeneration cycle, (see [24]).

Put

$$\bar{X}(t) = \sup_{0 \le s < t} X(s), \quad t \ge 0,$$

and

$$q(n) := \mathbf{P}(\bar{X}(T_1) \ge n) = \exp(-R(n)).$$

It is known, see [3] or Eq. (34) in [24], that

$$q(n) = \frac{1}{\sum_{k=0}^{n-1} \alpha_k}, \tag{63}$$

where $\alpha_0 = 1$ and $\alpha_k = \prod_{i=1}^{k} \frac{\mu_i}{\lambda_i}$ for $k \in \mathbb{N}$.

Further we write α_k in the following way:

$$\alpha_k = \frac{\beta_k}{\rho^k}, \quad \beta_k = \prod_{i=1}^{k}(1+\delta_i), \quad \delta_i = \frac{B/\mu - A/\lambda}{v_i + A/\lambda}. \tag{64}$$

As it is known from the analysis, if the series (60) converges, then exist

$$\lim_{k\to\infty} \beta_k = \beta^* = \prod_{i=1}^{\infty}(1+\delta_i), \tag{65}$$

(see [23], Ch.1, §4).

To estimate the value $q(n)$ we need the following

Lemma 3. *For arbitrary $p > 1$ and β_k, which satisfies the equality (65)*

$$\Lambda_n := \sum_{k=1}^{n} p^k \beta_k = \beta^* \frac{p^{n+1}}{p-1}(1+o(1)), \quad n \to \infty. \tag{66}$$

Proof. By the Stolz-Cesaro theorem we have

$$\lim_{n\to\infty} \frac{\Lambda_n}{p^{n+1}} = \lim_{n\to\infty} \frac{\Lambda_n - \Lambda_{n-1}}{p^{n+1} - p^n} = \lim_{n\to\infty} \frac{p^n \beta_n}{p^{n+1} - p^n} = \frac{\beta^*}{p-1}.$$

The proof is complete.

□

The estimation follows directly from Lemma 3

$$q(n) = \frac{1-\rho}{\rho\beta^*} \rho^n (1+o(1)).$$

That is

$$R(n) = -\ln q(n) = \gamma n + C_1 + o(1),$$

where

$$\gamma = \ln \frac{1}{\rho}, \quad C_1 = \ln \frac{\rho\beta^*}{1-\rho}. \tag{67}$$

It remains to apply Proposition 1 and repeat the reasoning from the previous example. Thus, for birth and death processes with parameters, that are defined in (59), we obtain

$$\limsup_{t\to\infty} \frac{\gamma \bar{X}(t) - \alpha_k(t)}{L_{k+1}(t)} = 1 \quad \text{a.s.},$$

$$\liminf_{t\to\infty} (\bar{X}(t) - \frac{1}{\gamma}(L_1(t) - L_3(t))) = \kappa \quad \text{a.s.},$$

where $\kappa \in [-1 - (C_1 + \ln a_T)/\gamma, \, -(C_1 + \ln a_T)/\gamma]$ and γ, C_1 are given by equalities (67).

Acknowledgments

We thank anonymous referees for a number of useful suggestions and helpful comments.

References

[1] K.S. Akbash and I.K. Matsak, *One improvement of the law of iterated logarithm for the max-scheme*, Ukr. Mat. Zh., **64** 1132–1137 (2012), (in Ukrainian); English translation: Ukr. Mathem. Journ., **64**:1290–1296 (2013).

[2] C.W. Anderson, *Extreme value theory for a class of discrete distribution with application to some stochastic processes,* J. Appl. Prob., **7**:99–113 (1970).

[3] S. Asmussen, *Extreme value theory for queues via cycle maxima*, Extremes, **1**: 137–168 (1998).

[4] O. Barndorff-Nielsen, *On the Limit Behaviour of Extreme Order Statistics*, Ann. Math. Statist., **34**:992–1002 (1963) .

[5] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation.* Cambridge University Press (1989).

[6] V.V. Buldygin, O.I. Klesov and J.G. Steinebach, On some properties of asymptotic quasi-inverse functions and their applications, *Theory Probab. Math. Statist.*, **70**:9–25 (2004).

[7] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley, New York (1978).

[8] P. Glasserman, S.G. Kou, *Limits of first passage times to rare sets in regenerative processes*, Anal.Appl.Probab., **5**:424–445 (1995).

[9] *B.V. Gnedenko, I.N. Kovalenko,* Introduction to Queueing Theory, 2nd ed., Mathematical Modeling, vol.5, Birkhauser, Boston (1989).

[10] B.V. Dovgay, I.K. Matsak, *The asymptotic behavior of extreme values of queue lengths in M/M/m systems*, Cybernetics and systems analysis , **55**(2):171–179 (2019)(In Ukrainian).

[11] L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer-Verlag, Berlin (2006).

[12] L. de Haan and A. Hordijk, *The rate of growth of sample maxima*, Ann. Math. Statist., **43**:1185–1196 (1972).

[13] D.L. Iglehart, *Extreme values in the GI/G/1 gueue* , Ann. Math. Statist., **43**:627–635 (1972).

[14] Klass M.J., *The Robbins-Siegmund criterion for partial maxima*, Ann. Probab., **13**:1369–1370 (1985).

[15] *S. Karlin,* A first course in stochastic processes, Academic Press, New York, (1968).

[16] I. K. Matsak, *Asymptotic behaviour of random variables extreme values. Discrete case*, Ukrainian Math. J , **68**:945–956 (2016) (In Ukrainian).

[17] I. K. Matsak, *Asymptotic behavior of maxima of independent random variables*, Lithuanian Math. J , **59**:185–197 (2019).

[18] I. K. Matsak, *Limit theorem for extreme values of discrete random variables and applications*, Theory Probab. Math. Statist. , **101**:189–202 (2019) (In Ukrainian).

[19] J. Pickands, *An iterated logarithm law for the maximum in a stationary Gaussian sequence*, Zeitschrift für Warhschein. verw. Geb., **12**:344–355 (1969).

[20] E. Seneta. *Regularly Varying Functions.* Springer-Verlag, Berlin, Heidelberg, New York (1976).

[21] R.F. Serfozo *Extreme values of birdh and death processes and queues* , Stochastic processess and their applications, **27**:291–306 (1988).

[22] *F. Spitzer,* Principles of random walk, Princeton, Toronto, New York, London (1964).

[23] *E.C. Titchmarsh,* The Theory of Fuctions, Oxford University Press, Oxford (1939).

[24] O.K. Zakusylo, I.K. Matsak, *On extreme values of some regenerative processes*, Theory Probab. Math. Statist. , **97**:58–71 (2017) (In Ukrainian).