ON A PARTICULAR COUPLING

AND ITS APPLICATIONS TO RANDOM TREES

(joint works (mainly) with Sasha Iksanov)

Martin Möhle

University of Tübingen

Online workshop on the occasion of Sasha Iksanov's 50th birthday

Kiev, May 24, 2022

A one-sided edge removing procedure

Let T be a rooted deterministic tree.

Meir and Moon (1974) introduce the following procedure to isolate the root of T by successive deletions of edges.

1. Delete (cut) one edge chosen at random.
2. Consider only the subtree containing the root and delete (cut) in it one edge at random.
3. Repeat Step 2 until the root of T is isolated.

$X(T) :=$ (random) number of cuts needed to isolate the root of T

Random recursive trees

Def. (Recursive tree)

A recursive tree is a rooted tree with n vertices $1, \ldots, n$ with the following properties.

- Root has label 1.
- Labels of the vertices on the unique path from the root to any other vertex form an increasing sequence.

Def. (Random recursive tree)

T_n is called random recursive tree with n vertices if T_n is uniformly distributed on the set of all $(n-1)!$ recursive trees with n vertices.

Number of cuts: Basic properties

$X_n := X(T_n) =$ **number of cuts** needed to isolate the root of a random recursive tree with n vertices (Both the tree and the cuts are random, $X_1 = 0$).

Recursion: $\boxed{X_n \stackrel{d}{=} 1 + X_{n-D_n}}$ $\quad n = 2, 3, \ldots$

D_n independent of $X_2, \ldots, X_n$ with $\boxed{\mathbb{P}(D_n = k) \; = \; \frac{n}{(n-1)k(k+1)}}$ $\quad 1 \le k < n.$

Interpretation.

1. $X_n := \inf\{k \ge 0 \,|\, J_k = 1, J_0 = n\} =$ **absorption time** of the non-increasing Markov chain $J := (J_k)_{k \ge 0}$ with state space $\{1, 2, \ldots\}$ and transition probabilities

$$\mathbb{P}(J_{k+1} = j \,|\, J_k = i) \; = \; \frac{i}{(i-1)(i-j)(i-j+1)}, \qquad 1 \le j < i.$$

2. $J =$ jump-chain of the Bolthausen–Sznitman (BS) coalescent; $X_n =$ **number of collisions** in the BS coalescent until there is a single block (Goldschmidt and Martin, 2005).

Number of cuts: Asymptotics

Theorem. (Asymptotics of the moments of X_n, Panholzer, 2004)

$$\mathbb{E}(X_n^j) = \frac{n^j}{\log^j n}\left(1 + \frac{(j+1)h_j - j\gamma}{\log n} + O\Big(\frac{1}{\log^2 n}\Big)\right)$$

($h_j :=$ j-th harmonic number, $\gamma :=$ Euler–Mascheroni constant)

Corollary.

$$\mu_n := \mathbb{E}(X_n) \sim \frac{n}{\log n}, \qquad \mathbb{E}(X_n^2) \sim \frac{n^2}{\log^2 n}, \qquad \sigma_n^2 := \mathrm{Var}(X_n) \sim \frac{n^2}{2\log^3 n}.$$

$$\Rightarrow \boxed{\frac{X_n}{\mu_n} \xrightarrow{P} 1}$$

Number of cuts: Asymptotics

Theorem. (Asymptotics of X_n, Drmota, Iksanov, M., Rösler, 2009)

$$Y_n := \frac{\log^2 n}{n} X_n - \log n - \log\log n \ \xrightarrow{d}\ Y$$

where Y is 1-stable with char. function $\boxed{\mathbb{E}(e^{i\lambda Y}) = (i\lambda)^{i\lambda} = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|}}$, $\lambda \in \mathbb{R}$.

Remarks.

- The same result holds for the total branch length L_n of the BS n-coalescent.
- Limiting behaviour of X_n was unknown for more than 30 years (Meir and Moon, 1974).
- Method of moments not applicable (cf. Panholzer, 2004).
- Contraction methods (Neininger, Rösler, Rüschendorf) not directly applicable.
- Note that $\dfrac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} 0$.

Analytic proof (sketch), Drmota, Iksanov, M., Rösler, 2009

Show that $\mathbb{E}(e^{i\lambda Y_n}) = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|} + O((\log\log n)^2/\log n)$ for $0 \neq \lambda \in \mathbb{R}$.

Need to analyse $\mathbb{E}(s^{X_n})$ for $s = s(n) = e^{i\lambda/b_n}$ with $b_n := n/\log^2 n$.

Recursion. $\Rightarrow$ $\boxed{f(s,t) := \sum_{n\geq 1} \mathbb{E}(s^{X_n}) t^{n-1}}$ satisfies PDE $\dfrac{f_t(s,t)}{f(s,t)} = \dfrac{1}{h(s,t)}$

with $h(s,t) := 1 - t + (1-1/s)t/\log(1-t)$.

For $s = e^{iv}$ (v small), f has singularities at $t = 1$ and $t = t_0(s)$, the (non-real) root of h.

Choose a clever curve γ (Hankel contours) to verify that, for $s = s(n) = e^{i\lambda/b_n}$,

$$\mathbb{E}(s^{X_n}) = \frac{1}{2\pi i}\int_\gamma \frac{f(s,t)}{t^n}\,\mathrm{d}t = (t_0(s))^{-n}\Big(1 + O\Big(\frac{1}{\log n}\Big)\Big).$$

It remains to derive the asymptotics for the root

$$t_0(s(n)) = 1 - \frac{i\lambda \log n}{n} - \frac{i\lambda \log\log n}{n} - \frac{i\lambda \log(i\lambda)}{n} + O\Big(\frac{(\log\log n)^2}{n\log n}\Big). \quad \square$$

Probabilistic proof (sketch), Iksanov and M., 2007

Step 1: (Coupling)

Let $\xi_1, \xi_2, \ldots$ be independent copies of a r. v. ξ with dist. $\boxed{\mathbb{P}(\xi = k) = \frac{1}{k(k+1)}}$ $k \geq 1.$

Let $S := (S_i)_{i \geq 0}$ denote the corresponding random walk ($S_i := \xi_1 + \cdots + \xi_i$).

For $n \geq 1$ define $R^{(n)} := (R_i^{(n)})_{i \geq 0}$ recursively: $R_0^{(n)} := 0$,

$$\boxed{R_i^{(n)} := R_{i-1}^{(n)} + \xi_i \, 1_{\{R_{i-1}^{(n)} + \xi_i < n\}}} \qquad i = 1, 2, \ldots$$

$\Rightarrow$ $\boxed{R_i^{(n)} \leq S_i}$ $i \geq 0$ (**coupling** of $R^{(n)}$ and S)

Define M_n := number of jumps of $R^{(n)}$ $= |\{i \geq 1 \,|\, R_{i-1}^{(n)} \neq R_i^{(n)}\}|$.

A renewal argument yields $\boxed{M_n \overset{d}{=} X_n}$ for all n.

Thus we can consider $(M_n)_n$ instead of $(X_n)_n$.

Probabilistic proof (continued)

Step 2: (Reduction to N_n)

Define $N_n := \inf\{i \geq 1 \,|\, S_i \geq n\}$ $=$ number of steps the random walk S needs to reach a state larger than or equal to n,

$$Y_n := n - S_{\max\{i \,|\, S_i \leq n\}} = n - S_{N_{n+1}-1}.$$

Coupling inequality: $\boxed{0 \leq M_n - N_n + 1 \leq Y_{n-1}}$

Erickson (1970): $\dfrac{\log Y_n}{\log n} \xrightarrow{d} U_{[0,1]}$ (uniform distribution)

$$\Rightarrow \boxed{\frac{\log^2 n}{n}(M_n - N_n) \xrightarrow{P} 0}$$

Thus we can consider $(N_n)_n$ instead of $(M_n)_n$.

Probabilistic proof (continued)

Step 3: (Asymptotics of N_k) Theory of stable distributions $\Rightarrow$

$$\boxed{\frac{S_n}{n} - \log n \xrightarrow{d} Z} \text{ with } \mathbb{E}(e^{i\lambda Z}) = e^{-\frac{\pi}{2}|\lambda| + i\lambda \log|\lambda|}.$$

Fix $x \in \mathbb{R}$. Let the integers n, k be functions of each other such that $\frac{k}{n} - \log n \to x$.

$$\Rightarrow \boxed{\mathbb{P}(N_k \leq n) \;=\; \mathbb{P}(S_n \geq k)} = \mathbb{P}(S_n/n - \log n \geq k/n - \log n) \;\to\; 1 - F(x),$$

($F :=$ distribution function of Z).

$$\Rightarrow 1 - F(x) \;\leftarrow\; \mathbb{P}(N_k \leq n)$$

$$= \;\mathbb{P}\Big(\frac{\log^2 k}{k} N_k - \log k - \log\log k \leq \frac{n}{k}\log^2 k - \log k - \log\log k\Big)$$

$$\sim \;\mathbb{P}\Big(\frac{\log^2 k}{k} N_k - \log k - \log\log k \leq -x\Big).$$

It remains to note that $x \mapsto 1 - F(-x)$ is the distribution function of $-Z \stackrel{d}{=} Y$. $\square$

Absorption times of non-increasing Markov chains

Consider the absorption time $X_n := \inf\{k \geq 1 \,|\, Z_k = 1, Z_0 = n\}$ of some non-increasing Markov chain $Z := (Z_k)_k$ with state space $\{1, 2, \ldots\}$ and transition probabilities $\pi_{ij} > 0$ for $i, j \in \mathbb{N}$ with $j < i$.

Key observation: $\boxed{X_n \stackrel{d}{=} 1 + X_{n-D_n}}$

(D_n independent of $X_2, \ldots, X_n$ with distribution $\mathbb{P}(D_n = k) = \pi_{n,n-k}$, $1 \leq k < n$).

Coupling method works under the assumption that

$$\boxed{\mathbb{P}(D_n = k) \;=\; \frac{p_k}{p_1 + \cdots + p_{n-1}}} \qquad k, n \in \mathbb{N}, k < n$$

where $(p_k)_k$ is some proper probability distribution

$$p_k \;:=\; \mathbb{P}(\xi = k), \quad k \geq 1.$$

Asymptotic results for absorption times

Theorem. (Iksanov, M., 2008) Suppose that $\mathbb{E}(\xi) = \infty$ and that for some function L slowly varying at ∞

$$\mathbb{P}(\xi \geq n) = \sum_{k=n}^{\infty} p_k \sim \frac{L(n)}{n}.$$

Let c be any positive function satisfying $\lim_{x\to\infty} xL(c(x))/c(x) = 1$ and set $\psi(x) := x\int_0^{c(x)} \mathbb{P}(\xi > y)\,\mathrm{d}y$. Let $b(x)$ be any positive function satisfying $b(\psi(x)) \sim \psi(b(x)) \sim x$, and set $a(x) := x^{-1}b(x)c(b(x))$. Then $\boxed{(X_n - a(n))/b(n) \to Y}$ with Y as before.

Remark. For $p_k := \dfrac{1}{k(k+1)}$, $L(n) \equiv 1$, $c(x) := x$, $b(x) := \dfrac{x}{\log x} + \dfrac{x \log\log x}{\log^2 x}$, and $a(x) := \dfrac{b^2(x)}{x} \sim \dfrac{x}{\log^2 x}$ we are back in the special case considered before.

Asymptotic results for absorption times: The case $0 < \alpha < 1$

Theorem. (Iksanov, M., 2008) Suppose that for some $0 < \alpha < 1$ and some function L slowly varying at ∞

$$\mathbb{P}(\xi \geq n) \ = \ \sum_{k=n}^{\infty} p_k \ \sim \ \frac{L(n)}{n^\alpha}, \quad n \to \infty.$$

Then, as $n \to \infty$,

$$\boxed{\frac{L(n)}{n^\alpha} X_n \ \xrightarrow{d} \ \int_0^\infty e^{-U_t}\,\mathrm{d}t}$$

where $(U_t)_{t \geq 0}$ is a drift-free subordinator with Lévy measure

$$\nu(\mathrm{d}t) \ = \ \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}}\,\mathrm{d}t, \quad t > 0.$$

Remark. There is an analog result for $1 < \alpha < 2$. In this case there exist sequences $(a_n)_n$ and $(b_n)_n$ such that $(X_n - a_n)/b_n$ is asymptotically α-stable.

Application: Number of collisions in beta(a,1)-coalescents

Coupling method applicable for $0 < a < 2$ with $p_k = \mathbb{P}(\xi = k) = \dfrac{(2-a)\Gamma(a+k-1)}{\Gamma(a)\Gamma(k+2)}$.

Theorem. (Iksanov, M., 2008)

For the $\beta(a,1)$-coalescent with $0 < a < 1$ ($\Rightarrow\ \mathbb{E}(\xi) = 1/(1-a) < \infty$), the number X_n of collisions satisfies

$$\boxed{\frac{X_n - n(\alpha-1)}{(\alpha-1)n^{1/\alpha}} \xrightarrow{d} Y_\alpha}$$

where Y_α is α-stable with index $\alpha := 2 - a$.

Remarks.

- Characteristic function of Y_α: $\mathbb{E}(e^{itY_\alpha}) = e^{|t|^\alpha(\cos(\pi\alpha/2)+i\sin(\pi\alpha/2)\,\mathrm{sgn}(t))}$, $t \in \mathbb{R}$.

- Gnedin and Yakubovich (2007) use analytic methods to prove the same convergence result for Λ-coalescents satisfying $\Lambda([0,x]) = Ax^a + O(x^{a+\zeta})$, $x \to 0$, $0 < a < 1$, $A > 0$, $\zeta > \max((2-a)^2/(5-5a+a^2), 1-a)$.

Application: Number of collisions in beta(a,1)-coalescents

Theorem. (Iksanov, M., 2008)

For the $\beta(a,1)$-coalescent with $1 < a < 2$ ($\Rightarrow 0 < \alpha := 2 - a < 1$ and $\mathbb{E}(\xi) = \infty$), the number X_n of collisions satisfies

$$\boxed{\frac{X_n}{\Gamma(2-\alpha)n^\alpha} \xrightarrow{d} \int_0^\infty e^{-U_t}\,\mathrm{d}t}$$

where $(U_t)_{t\geq 0}$ is a subordinator with zero drift and Lévy measure

$$\nu(\mathrm{d}t) = \frac{e^{-t/\alpha}}{(1-e^{-t/\alpha})^{\alpha+1}}\,\mathrm{d}t, \quad t > 0.$$

Application: Number of collisions in beta(a,b)-coalescents

Theorem. (Iksanov, Marynych, M., 2009)

For the $\beta(a,b)$-coalescent with $a = 2$ (and arbitrary $b > 0$) the number X_n of collisions satisfies

$$\frac{X_n - \frac{1}{2m_1}\log^2 n}{(\frac{m_2}{3m_1^3}\log^3 n)^{1/2}} \xrightarrow{d} N$$

where N is standard normal distributed and $m_k := \int t^k \mu_b(\mathrm{d}t) = \Gamma(k+1)\zeta(k+1,b)$ is the kth moment of the Lévy measure $\mu_b(\mathrm{d}t) = e^{-bt}/(1-e^{-t})\,\mathrm{d}t$, $t > 0$.

Remark. Proof based on asymptotics of moments and the contraction method with degenerate limit equation (Neininger and Rüschendorf, 2004).

The case $a > 2$ was finally handled in Gnedin, Iksanov, M., 2008 and Gnedin, Iksanov, Marynych, 2011.

Summary: Number of collisions in beta(a,b)-coalescents

Gnedin, Iksanov, Marynych, M., 2014

name of the coalescent	parameter a	number X_n of collisions
Kingman	$a \to 0$	$X_n = n - 1$
	$0 < a < 1$	$\frac{X_n - n(\alpha-1)}{(\alpha-1)n^{1/\alpha}} \xrightarrow{d} Y_\alpha$ (α-stable)
Bolthausen-Sznitman	$a = 1$	$\frac{\log^2 n}{n} X_n - \log(n \log n) \xrightarrow{d} Y$ (1-stable)
	$1 < a < 2$	$\frac{X_n}{\Gamma(a) n^\alpha} \xrightarrow{d} \int_0^\infty e^{-U_t} \mathrm{d}t$ ($U =$ subordinator)
	$a = 2$	$\frac{X_n - (2m_1)^{-1} \log^2 n}{(\frac{m_2}{3m_1^2} \log^3 n)^{1/2}} \xrightarrow{d} N$ (standard normal)
	$2 < a < \infty$	$\frac{X_n - \mu^{-1} \log n}{(\sigma^2 \mu^{-3} \log n)^{1/2}} \xrightarrow{d} N$ (standard normal)
star-shaped	$a \to \infty$	$X_n = 1$

with $\alpha := 2 - a$

Thank you very much for your attention!

Happy birthday Sasha!!!

References I

BASDEVANT, A.-L. AND GOLDSCHMIDT, C. (2008) Asymptotics of the allele frequency spectrum associated with the Bolthausen–Sznitman coalescent. *Electron. J. Probab.* **17**, 586–512.

DELMAS, J.-F., DHERSIN, J.-S., AND SIRI-JEGOUSSE, A. (2008) Asymptotic results on the length of coalescent trees. *Ann. Appl. Probab.* **18**, 997–1025.

DRMOTA, M., IKSANOV, A., MOEHLE, M., AND ROESLER, U. (2007) Asymptotic results concerning the total branch length of the Bolthausen–Sznitman coalescent. *Stochastic Process. Appl.* **117**, 1404–1421.

DRMOTA, M., IKSANOV, A., MOEHLE, M., AND ROESLER, U. (2009) A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree. *Random Struct. Algorithms* **34**, 319–336.

ERICKSON, K.B. (1970) Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151**, 263–291.

GELUK, L. AND DE HAAN, L. (2000) Stable probability distributions and their domain of attraction: a direct approach. *Prob. and Math. Stat.* **20**, 169–188.

References II

GNEDIN, A., IKSANOV, A., MARYNYCH, A. (2011) On Λ-coalescents with dust component. *J. Appl. Probab.* **48**, 1133–1151.

GNEDIN, A., IKSANOV, A., MARYNYCH, A. AND MÖHLE, M. (2014) On asymptotics of the beta coalescents. *Adv. Appl. Probab.* **46**, 496–515.

GNEDIN, A., IKSANOV, A. AND MÖHLE, M. (2008) On asymptotics of exchangeable coalescents with multiple collisions. *J. Appl. Probab.* **45**, 1186–1195.

GNEDIN, A. AND YAKUBOVICH, Y. (2007) On the number of collisions in Λ-coalescents. *Electron. J. Probab.* **12**, 1547–1567.

GOLDSCHMIDT, C. AND MARTIN, J.B. (2005) Random recursive trees and the Bolthausen-Sznitman coalescent. *Electron. J. Probab.* **10**, 718–745.

IKSANOV, A., MARYNYCH, A. AND MÖHLE, M. (2009) On the number of collisions in beta(2,b)-coalescents. *Bernoulli* **15**, 829–845.

References III

IKSANOV, A. AND MÖHLE, M. (2007) A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree. *Electron. Commun. Probab.* **12**, 28–35.

IKSANOV, A. AND MÖHLE, M. (2008) On the number of jumps of random walks with a barrier. *Adv. Appl. Probab.* **40**, 206–228.

MEIR, A. AND MOON, J.W. (1974) Cutting down recursive trees. *Math. Biosci.* **21**, 173–181.

MÖHLE, M. (2006) On the number of segregating sites for populations with large family sizes. *Adv. Appl. Probab.* **38**, 750–767.

NEININGER, R. AND RÜSCHENDORF, L. (2004) On the contraction method with degenerate limit equation. *Ann. Probab.* **32**, 2838–2856.

PANHOLZER, A. (2004) Destruction of recursive trees. In: *Mathematics and Computer Science III*, Birkhäuser, Basel, pp. 267–280.