Tail behavior of the derivative martingale in a branching random walk

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Joint work with Dariusz Buraczewski (University of Wrocław) and Alexander Iksanov (Taras Shevchenko National University of Kyiv)

LAGA – Université Sorbonne Paris Nord

Prof. Alexander Iksanov's 50th birthday

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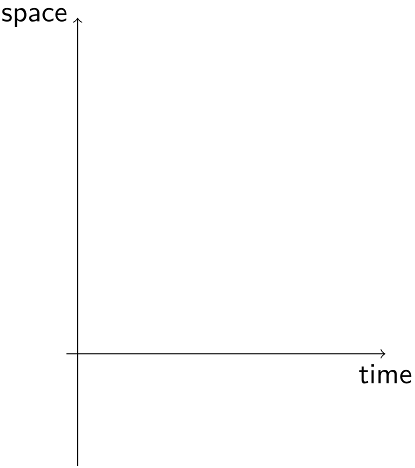


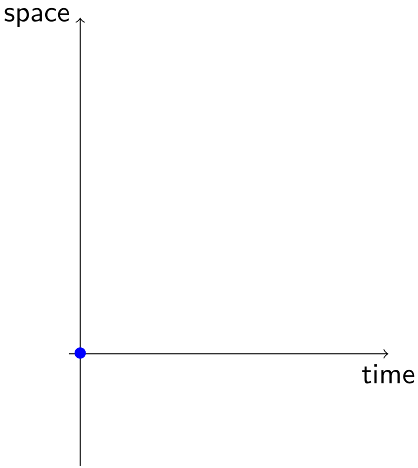
Definition

The branching random walk

- starts with one particle,
- that creates children according to a point process,
- who then reproduce independently according to copies of the same point process centred around their position.
- The process keeps branching forever.

The branching random walk

Definition of the process



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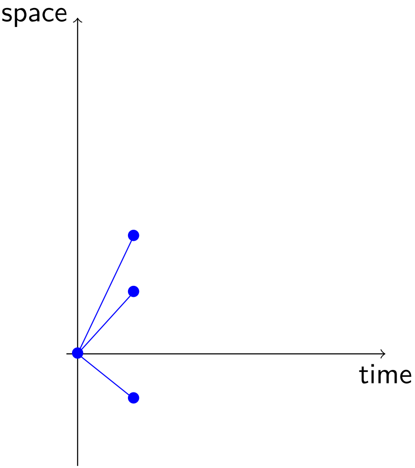

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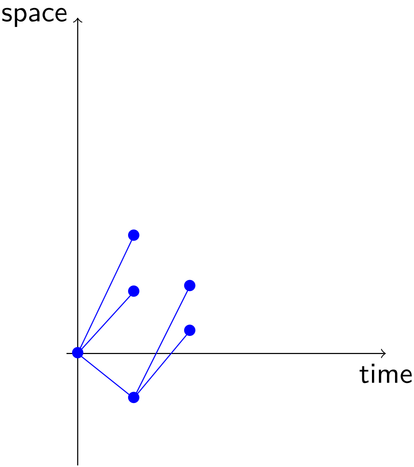

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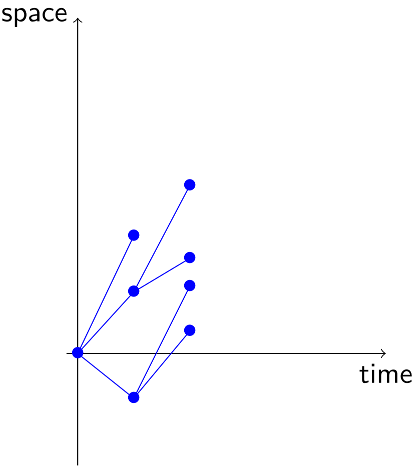

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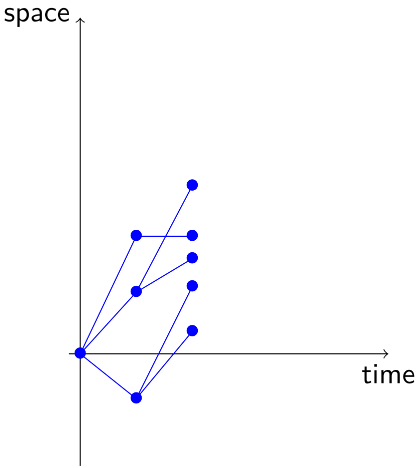

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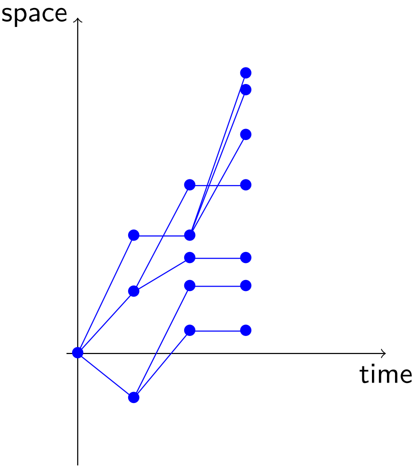

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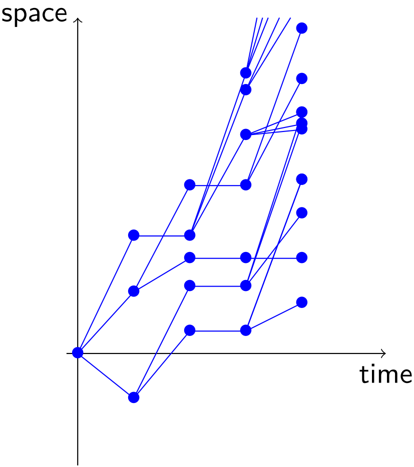

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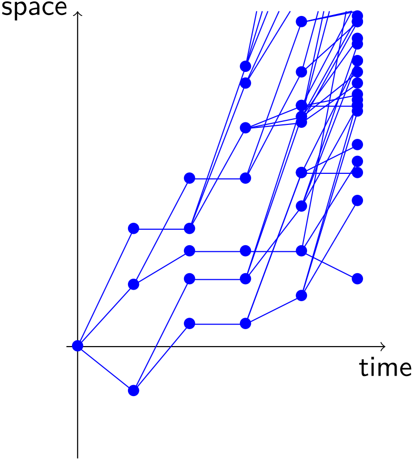

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A sample path of the branching random walk

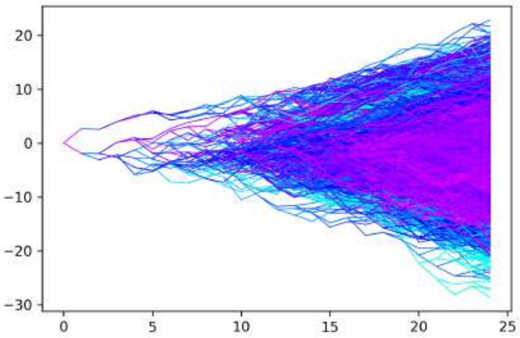

Figure: Trajectories of particles in a branching random walk. Colours are inherited from parents with random mutations in order to visualize families.

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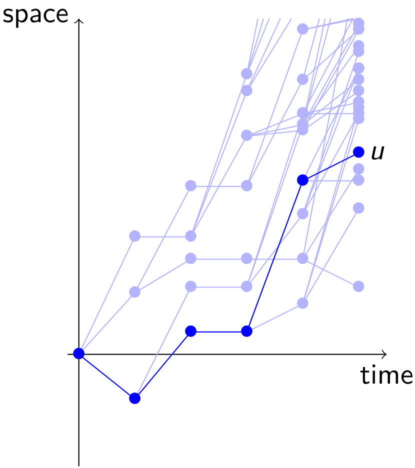
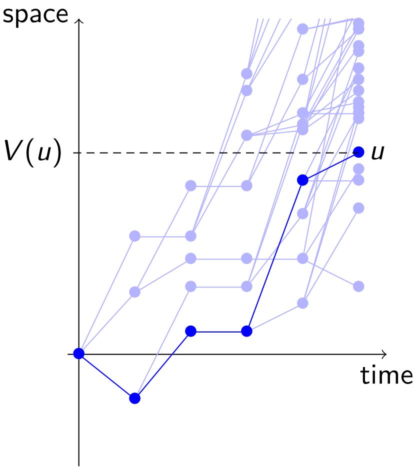


Notation

For u an individual of the branching random walk, we write

- $V(u)$ the position of u;
- $|u|$ the generation to which u belongs,
- u_k the ancestor at generation k of u.

The branching random walk



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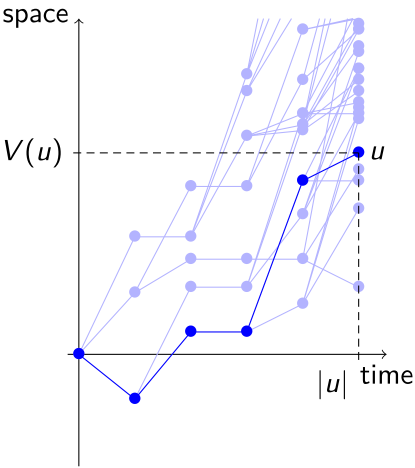

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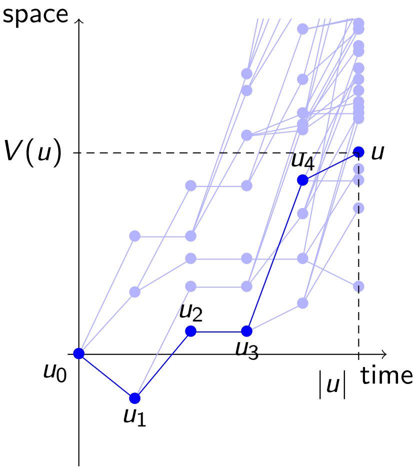

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The branching random walk

We only consider supercritical branching random walks, i.e. $\mathbf{E}(\#\{|u| = 1\}) > 1$.

Branching random walk in the boundary case

Up to a space-time linear transform, we assume the branching random walk to be in the boundary case, i.e.

$$\mathbf{E}\left(\sum_{|u|=1} e^{-V(u)}\right) = 1 \quad \text{and} \quad \mathbf{E}\left(\sum_{|u|=1} V(u)e^{-V(u)}\right) = 0.$$

Minimal displacement (Hammersley '74, Kingman '75, Biggins '76)

For all $n \in \mathbb{N}$, we set $M_n = \min_{|u|=n} V(u)$. We have

$$\lim_{n\to\infty} \frac{M_n}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} M_n = \infty \quad \text{a.s.}$$

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Bringing a branching random walk to the boundary case

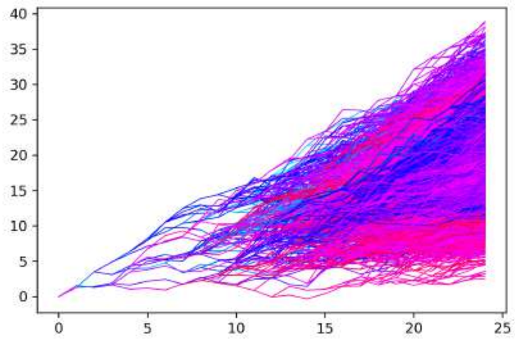

Figure: Trajectories of particles of a branching random walk in the boundary case. Colors are inherited from parents with random mutations in order to visualize families.

Additive martingales

For all $n \in \mathbb{N}$, we write

$$W_n = \sum_{|u|=n} e^{-V(u)} \quad \text{and} \quad Z_n = \sum_{|u|=n} V(u) e^{-V(u)}.$$

Lemma

The process $(W_n, n \geq 1)$ is a non-negative martingale.
The process $(Z_n, n \geq 1)$ is a (signed) martingale.

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Additive martingales

We assume that $\mathbf{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) \in (0, \infty)$.

Convergence of the additive martingale (Biggins '77, Lyons '95)

We have $\lim_{n\to\infty} W_n = 0$ a.s.

Convergence of the derivative martingale (Aïdékon '13, Chen '15)

We have $\lim_{n\to\infty} Z_n = Z_\infty$ a.s., where Z_∞ is a non-negative random variable, positive on the survival set of the branching random walk, if and only if

$$\mathbf{E}\left(W_1 \log_+(W_1)^2\right) + \mathbf{E}\left(\widetilde{W}_1 \log_+(\widetilde{W}_1)\right) < \infty,$$

where $\widetilde{W}_1 = \sum_{|u|=1} V(u)_+ e^{-V(u)}$.

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The condition $\mathcal{S}$

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We refer to the condition $\mathcal{S}$ as the joint conditions

$$\mathbf{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) \in (0,\infty)$$

$$\mathbf{E}\left(W_1 \log_+(W_1)^2\right) + \mathbf{E}\left(\widetilde{W}_1 \log_+(\widetilde{W}_1)\right) < \infty.$$

When adding the non-lattice condition

$$\sup_{a,b>0} \mathbf{P}(\forall |u| = 1, V(u) \in a\mathbb{Z} + b) < 1,$$

the condition is called $\mathcal{S}_{nl}$.

The branching random walk

Asymptotic behavior of extremal particles

The derivative martingale appears in the asymptotic behavior of extreme particles.

Convergence of the minimal displacement (Aïdékon '13)

Under assumption $\mathcal{S}_{nl}$, setting $m_n = \frac{3}{2}\log n$, there exists $c_\star > 0$ such that

$$\lim_{n\to\infty} \mathbf{P}(M_n \geq m_n + z) = \mathbf{E}\left(e^{-c_\star Z_\infty e^{z}}\right).$$

Convergence of the extremal process (Madaule '15)

Under assumption $\mathcal{S}_{nl}$, we have

$$\lim_{n\to\infty} \sum_{|u|=n} \delta_{V(u)-m_n} = \mathrm{DPPP}(c_\star Z_\infty e^{-z}\mathrm{d}z, \mathcal{D}).$$

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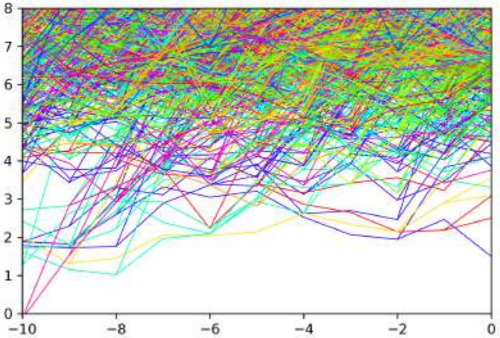

Figure: The extremal process of the branching random walk.

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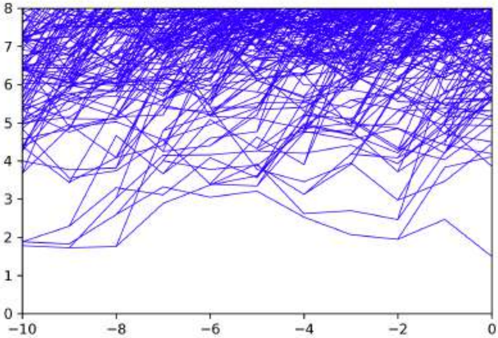

Figure: The extremal process of the branching random walk, can be decomposed into families of close relatives.

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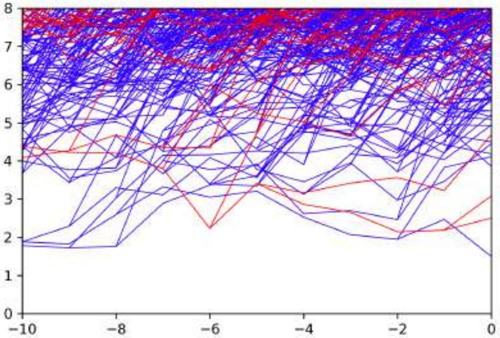

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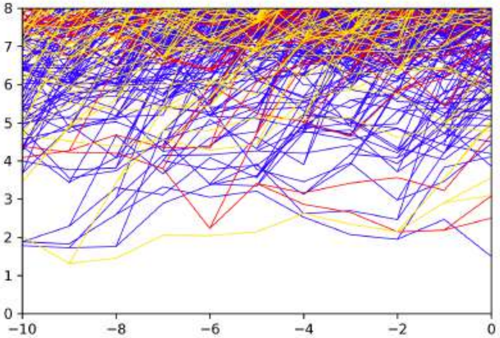

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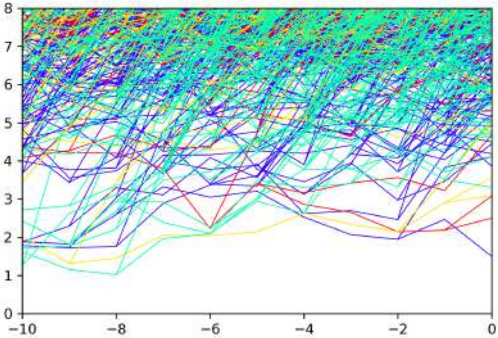

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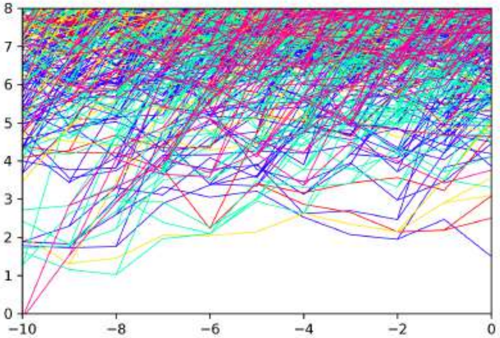

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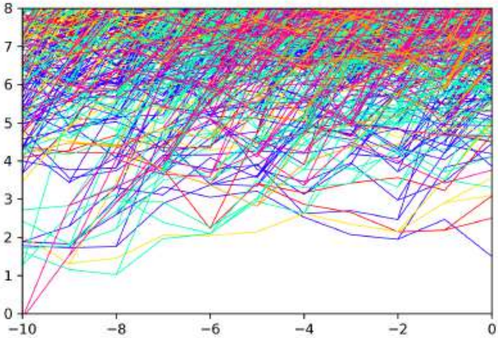

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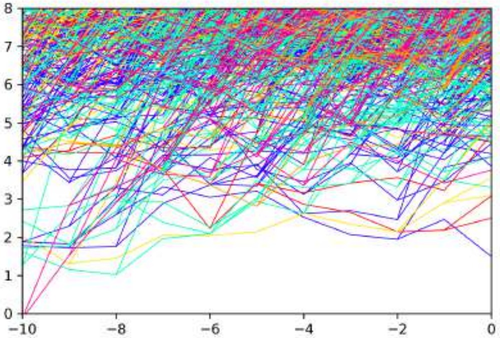

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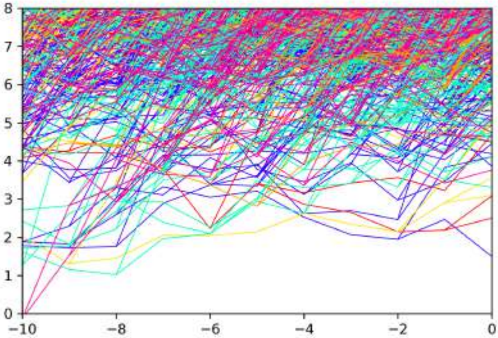

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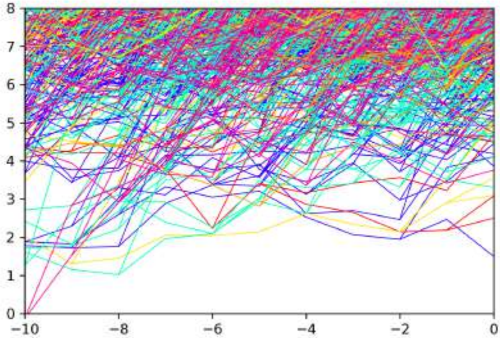

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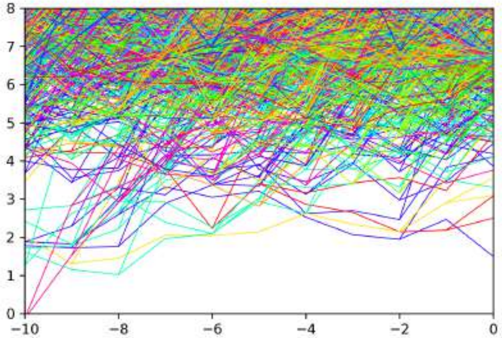

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The branching random walk

Counting particles going at critical speed

Particles going at critical speed (Madaule '16)

Under assumption $\mathcal{S}_{nl}$, for any continuous bounded function f, we have

$$\lim_{n\to\infty} \sum_{|u|=n} f(V(u)n^{-1/2})e^{-V(u)} = Z_\infty \mathbf{E}\left(f(M_1)\right),$$

in probability, where M is a Brownian meander.

Trajectory of particles at critical speed (Pain '18)

Under assumption $\mathcal{S}_{nl}$, for any continuous bounded function f, we have

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Tail of the derivative martingale

Our objective is to obtain a tight estimate of the left tail of the derivative martingale.

Theorem (Buraczewski, Iksanov, M.)

Under assumption $\mathcal{S}$, we have $\mathbf{E}\left(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}}\right) \sim \log x$ *as* $x \to \infty$.

Theorem (Buraczewski, Iksanov, M.)

We assume that $\mathcal{S}_{nl}$ *holds. There exists* $c \in \mathbb{R}$ *such that*

$$\mathbf{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}}) = \log x + c + o(1) \text{ as } x \to \infty$$

if and only if condition $\mathcal{S}^*$ *(to be disclosed) holds. In this case, we have* $\mathbf{P}(Z_\infty > x) \sim 1/x$ *as* $x \to \infty$.

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Branching random walk estimates

Laplace transform of the derivative martingale

Notation

For all $\lambda > 0$, we set $\phi(\lambda) = \mathbf{E}\left(e^{-\lambda Z_\infty}\right)$.

Remark

By a Tauberian-type theorem, the asymptotic behavior of $\mathbf{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}})$ as $x \to \infty$ is related to the asymptotic behavior of $\phi(\lambda)$ as $\lambda \to 0$.

Lemma

Let X be a non-negative random variable. Let $b > 0$, $c \in \mathbb{R}$, we have

$$\mathbf{E}(X\mathbf{1}_{\{X \leq x\}}) = b \log x + (c - b) + o(1) \text{ as } x \to \infty$$
$$\iff \mathbf{E}(e^{-\lambda X}) = 1 + b\lambda \log \lambda + (c - \gamma)\lambda + o(1) \text{ as } \lambda \to 0.$$

Branching random walk estimates

Laplace transform of the derivative martingale

Notation

For all $\lambda > 0$, we set $\phi(\lambda) = \mathbf{E}\left(e^{-\lambda Z_\infty}\right)$.

Remark

By a Tauberian-type theorem, the asymptotic behavior of $\mathbf{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}})$ as $x \to \infty$ is related to the asymptotic behavior of $\phi(\lambda)$ as $\lambda \to 0$.

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Branching random walk estimates

The branching property

Branching property

Using that $Z_\infty \overset{(d)}{=} \sum_{|u|=1} e^{-V(u)} Z_\infty^{(u)}$, the Laplace transform of Z_∞ satisfies

$$\phi(\lambda) = \mathbf{E}\left(\prod_{|u|=1} \phi(\lambda e^{-V(u)}) \right).$$

Branching random walk estimates

Asymptotic of the Laplace transform

Notation

For all $x \in \mathbb{R}$, we write $D(x) = e^x(1 - \phi(e^{-x}))$.

We have $\mathbf{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}}) = \log x + (c-1) + o(1)$ if and only if $D(x) = x + c - \gamma + o(1)$ as $x \to \infty$.

Tameness lemma (Alsmeyer, M. '20)

Under assumption $\mathcal{S}_{nl}$, there exists $C > 0$ such that $D(x) \leq C(x+1)$ for all $x > 0$.

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Branching random walk estimates

The many-to-one lemma

Many-to-one lemma (Kahane, Peyrière '74)

There exists a random walk $(S_n, n \geq 0)$ such that

$$\mathbf{E}(f(S_j, j \leq n)) = \mathbf{E}\left(\sum_{|u|=n} e^{-V(u)} f(V(u_j), j \leq n) \right).$$

$$\mathbf{E}(D(x+S_1)) - D(x) = \mathbf{E}\left(\sum_{|u|=1} e^{-V(u)} D(x+V(u)) \right) - D(x)$$
$$= e^x \mathbf{E}\left(\prod_{|u|=1} \phi(e^{-x-V(u)}) - 1 + \sum_{|u|=1} (1 - \phi(e^{-x-V(u)})) \right) \geq 0.$$

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Corollary

Under the assumption $\mathcal{S}_{nl}$, for all $x \in \mathbb{R}$, we have

$$\mathbf{E}(D(x + S_1)) \geq D(x) \quad \text{with} \quad \sup_{x \in \mathbb{R}_+} \frac{D(x)}{1 + x} < \infty.$$

In particular, D can be described as a subharmonic function of the random walk S with at most linear growth at ∞.

Subharmonic functions of the killed random walk

The Poisson equation with (at most) linear growth

Poisson equation with linear growth

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ a càdlàg function and $h : \mathbb{R}_- \to \mathbb{R}$ a right-continuous bounded function. A function f is called a solution to the Poisson equation for the killed random walk if

$$\begin{cases} \mathbf{E}(f(x+S_1)) = f(x) + g(x) & x \geq 0 \\ f(x) = h(x) & x < 0 \\ \sup_{x \in \mathbb{R}_+} |f(x)|/(1+x) < \infty. \end{cases} \tag{P}$$

Property

If f is a solution of (P) for all $n \in \mathbb{N}$, then

- $(f(S_n), n \geq 0)$ *is a submartingale.*
- $(f(S_{n\wedge\tau}) - \sum_{k=1}^{n\wedge\tau-1} g(S_k))$ *is a martingale.*

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Linearity of the Poisson equation

Assume that f_1 and f_2 are two solutions of (P). Then $f = f_1 - f_2$ satisfies

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Theorem (Alsmeyer, M. '20)

If $(S_n, n \geq 0)$ is a non-lattice centred random walk with finite variance, then the set of solutions of (P0) is given by $\{cU, c \in \mathbb{R}\}$, with U the renewal function of the random walk.

Subharmonic functions of the killed random walk

Some remarks on the Poisson equation

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The renewal function of the random walk

Renewal function of the random walk

Let $(S_n, n \geq 0)$ be a non-lattice centered random walk with finite variance. We set

$$U(x) = \begin{cases} \sum_{k \geq 0} \mathbf{P}_x \left(S_k \geq 0, S_k < \min_{0 \leq j \leq k-1} S_j \right) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Property

We have $U(x) = \mathbf{E}\left(U(x + S_1)\right)$ for all $x \geq 0$, and $U(x) \sim \mu x$ as $x \to \infty$.

Remark

Similar estimates also hold for lattice random walks, up to taking into account a periodicity due to the presence of the lattice.

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Subharmonic functions of the killed random walk

Characterization of solutions to (P)

Theorem (Buraczewski, Iksanov, M.)

Let $(S_n, n \geq 0)$ *be a non-lattice centered random walk with finite variance. If there exists a solution to* (P)*, then for each* $x > 0$

$$\mathbf{E}_x\left(\sum_{k=0}^{\tau-1} g(S_k)\right) < \infty, \ \textit{where } \tau = \inf\{n \geq 0 : S_n < 0\}.$$

Conversely, if this integrability condition is satisfied and g *is dRi on* $\mathbb{R}_+$ *then any solution to* (P) *can be written as*

$$f(x) = cU(x) + \mathbf{E}_x(h(S_\tau)) - \mathbf{E}_x\left(\sum_{k=0}^{\tau-1} g(S_k)\right)$$

for some $c > 0$.

Subharmonic functions of the killed random walk

The Brownian motion case

Lemma

Let g be a continuous positive function and $a > 0$. If f is a function such that $f''(x) = g(x)$ for all $x > 0$ with $f(0) = a$, we have

$$f(x) = a + cx + \int_0^x (x-y)g(y)\mathrm{d}y.$$

Corollary

In particular, we observe that

$$f(x) \sim px \quad \text{as } x \to \infty \iff \int_0^\infty g(y)\mathrm{d}y < \infty$$

$$f(x) = px + q + o(1) \quad \text{as } x \to \infty \iff \int_0^\infty yg(y)\mathrm{d}y < \infty.$$

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Subharmonic function of the killed random walk

Asymptotic behavior of subharmonic functions of linear growth

Let $(S_n, n \geq 1)$ be a centered random walk with finite variance.

Corollary

If f is a solution of (P) with g being dRi on $\mathbb{R}_+$, then there exists $c > 0$ such that $f(x) \sim cx$ as $x \to \infty$.

Corollary

If in addition $\lim_{x\to\infty} \mathbf{E}_x \left(\sum_{k=0}^{\tau-1} g(S_k) \right)$ exists and is finite, then $f(x) = cx + d + o(1)$ as $x \to \infty$.

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Proof of the tail estimates

Application to the Laplace transform of the derivative martingale

Proof.

Under condition $\mathcal{S}$, function D is a solution to

$$\mathbf{E}(D(x+S_1)) = D(x) + G(x), \text{ where } G(x) = \mathbf{E}(D(x+S_1)) - D(x),$$

therefore $D(x) \sim x$ as $x \to \infty$.

Therefore, $\mathbf{E}(e^{-\lambda Z_\infty}) = \phi(\lambda) - 1 \sim \lambda \log \lambda$ as $\lambda \to 0$.

As a result, $\mathbf{E}(Z_\infty \mathbf{1}_{\{Z_\infty \leq x\}}) \sim \log x$ as $x \to \infty$. □

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Proof of the tail estimates

The condition $\mathcal{S}^*$

Condition $\mathcal{S}^*$

We refer to the condition $\mathcal{S}^*$ as the joint conditions

$$\mathbf{E}\left(\sum_{|u|=1} e^{-V(u)} V(u)_-^3\right) < \infty.$$

$$\mathbf{E}(W_1^+ \log_+(W_1^+)^3) + \mathbf{E}(\widetilde{W}_1(\log_+ \widetilde{W}_1)^2) < \infty$$

$$\text{and } \mathbf{E}\left(W_1^- \log_+(W_1^-)^3 \mathbf{1}_{\left\{\sum_{|u|=1, V(u)<0}(1+V(u)-M_1)e^{M_1 V(u)} > C_0\right\}}\right) < \infty,$$

for some $C_0 > 0$, where we have set

$$W_1^+ = \sum_{|u|=1, V(u)\geq 0} e^{-V(u)} \quad \text{and} \quad W_1^- = \sum_{|u|=1, V(u)<0} e^{-V(u)}$$

Proof of the tail estimates

An intermediate lemma

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Under assumption $\mathcal{S}$, writing $G(x) = \mathbf{E}(D(x + S_1)) - D(x)$, we have

$$\mathcal{S}^* \iff \sup_{x \in \mathbb{R}} \mathbf{E}_x \left(\sum_{j=0}^{\tau-1} G(S_k) \right) < \infty$$

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Under assumption $\mathcal{S}$, we have

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Proof of the tail estimates

Case of i.i.d. branching

Branching random walk with i.i.d. increments

Consider a branching random walk in which individuals create independently N children, whose positions are given by i.i.d. copies of a random variable X.

Condition $\mathcal{S}$

$$\mathbf{E}\left(N(\log N)^2\right) < \infty \quad \text{and} \quad \mathbf{E}\left(X^2 e^{-X}\right) < \infty.$$

Condition $\mathcal{S}^*$

$$\mathbf{E}\left(N(\log N)^3\right) < \infty \quad \text{and} \quad \mathbf{E}\left((X_-)^3 e^{-X}\right) < \infty.$$

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Central limit theorem for the derivative martingale

Last corollary

Theorem (Buraczewski, Iksanov, M.)

Under assumptions $\mathcal{S}_{nl}$ and $\mathcal{S}^*$, we have

$$\lim_{n\to\infty} n^{1/2} \left(Z_\infty - Z_n + (\log n/2) W_n\right) = Z_\infty S \text{ in law,}$$

with S a spectrally positive 1-stable Lévy process.

Thank you for your attention.