ASYMPTOTIC BEHAVIOR OF EXTREME VALUES OF RANDOM VARIABLES AND SOME STOCHASTIC PROCESSES

Kateryna Akbash and Ivan Matsak *

Volodymyr Vynnychenko Central Ukrainian State Pedagogical University,
Kropyvnytsky, Ukraine (Kateryna Akbash)
Taras Shevchenko National University of Kyiv, Ukraine (Ivan Matsak)

Abstract

This article presents a review of studies of the almost sure asymptotic behavior of extremal values of independent identically distributed random variables and stochastic processes. The central result here is the law of the iterated logarithm for $\limsup$, the law of the triple logarithm for $\liminf$ and some of its refinements. Among random processes, regenerative processes, birth and death processes, and processes in queuing systems are considered.

Key Words: random variables, extreme values, limit theorems almost sure, regenerative process, birth and death processes, queueing system

AMS Subject Classification: 60G70, 60F15, 60K25.

*E-mail address: kateryna.akbash@gmail.com, ivanmatsak@gmail.com

1 Introduction

Together with the scheme of sums of independent random variables, the maximum scheme (the theory of extreme values) occupies a significant place in probability theory.

Problems associated with extreme events arise in various industries. These are, for example, the problems of floods and droughts, extreme meteorological phenomena (pressure, temperature, etc.), a number of problems in aviation (wind gust speed, landing load, overloads acting on the fuselage), analysis of the strength of materials and mechanical structures, etc.

Several historical notes. It is clear that the roots of the theory of extreme values go back centuries: the mortality tables of J. Graunt(1662) [23], C. Huygens (1669) [27], the works of Nicolaus Bernoulli(1709) [6], L. Euler(1760) [12], the Gompertz(1825) [22] distribution for duration of life.

The beginning of the classical theory of extreme values of independent random variables is associated with the works M. Fréchet [15], who for the first time strictly mathematically obtained one of the boundary distributions for the maximum values, as well as the works of R. Fisher and L. Tippett [15], who, in addition to Fréchet boundary distribution, found two other types of extreme value distributions. In 1940-1943 B.V.Gnedenko published several articles on the limit distributions of extreme values, in which he obtained the necessary and sufficient conditions of convergence for all possible limit distributions (see, for example, [18], [19]). The central result of the theory is the theorem about extremal types, which was proved in full also by B.V.Gnedenko [19]. In fact, the works listed above determined the further development of the theory of extreme values. Nowadays, the main results of the classical theory of extremal values of independent identically distributed random variables (i.i.d.r.v.) are well known (see, for example, books by M. Lidbetter, G. Lindgren and H. Rothsen [33], J. Galambos [16], S. Resnik [41], and L. de Haan and A. Ferreira [25]).

It seems that in this area the efforts of mathematicians have been concentrated on the study of the weak convergence of extremal values. At the same time, problems on the asymptotic behavior of extreme values of the i.i.d.r.v. and stochastic processes almost sure, has received much less attention. It should be noted that among a certain part of mathematicians there is an opinion that theorems about the asymptotic behavior of random variables and processes almost sure are not of interest for practical use. We do not share this opinion. Therefore, this study is an attempt to review the results of the asymptotic behavior of extreme values r.v. and random processes almost sure. Of course, these results are less known. But some new theorems will also be given.

In section **??** we study the asymptotic behavior of the extremal values of the i.i.d.r.v. The main result obtained here is the law of the iterated logarithm for the $\limsup$ and the law of the triple logarithm for the $\liminf$. Some non-trivial refinements of this result are given. Note also that asymptotics in continuous and discrete cases can be significantly different. And we present a number of results that are true only for discrete r.v.

Most of the stochastic processes in the works known to us on this topic there are considered regenerative processes, birth and death processes, as well as random processes in queuing systems (QS).

Section 3 focuses on just such processes. Thus, based on the results of section **??**, the law of the iterated logarithm for the lim sup and the law of the triple logarithm for the lim inf, as well as their refinements, are established for regenerative processes. Some related areas of research are also considered.

The list of literature with which the article ends is incomplete and needs to be expanded.

2 Asymptotic behavior of extreme values of random variables

2.1 Some early results

Let (ξ_n) be a sequence of i.i.d.r.v. with the distribution function $F(x) = \mathbf{P}(\xi_i < x)$, and let

$$z_n = \max_{1 \le i \le n} \xi_i. \tag{1}$$

In this subsection we deal with the asymptotic behavior of extreme values z_n as $n \to \infty$.

First, we introduce some notations and concepts.

Let

$$\xi_n \xrightarrow{D} \xi, \quad \xi_n \xrightarrow{P} \xi, \quad \xi_n \xrightarrow{\text{a.s.}} \xi$$

denote, respectively, convergence in distribution, convergence in probability, and almost sure convergence.

As far as we know, the first results on convergence to a degenerate distribution were obtained by B.V. Gnedenko [19] (it seems that the case of a normal distribution was considered earlier).

Definition 1. *A sequence (z_n) of random variables is relatively stable in probability if there exists a sequence of numbers (a_n) such that, as $n \to \infty$,*

$$\frac{z_n}{a_n} \xrightarrow{P} 1. \tag{2}$$

B.V. Gnedenko [19] proved the following criterion.

Theorem 1. *For sequence (z_n) to satisfy relation (2) it is necessary and sufficient that*

$$\forall \varepsilon > 1: \quad \lim_{x\to\infty} \frac{1-F(x\varepsilon)}{1-F(x)} = 0 \quad \text{or} \quad x_F < \infty,$$

where $x_F = \sup(x : F(x) < 1)$.

Definition 2. *A sequence (z_n) is said to be stable in probability if, as $n \to \infty$*

$$z_n - a_n \xrightarrow{P} 0. \tag{3}$$

(see [5], [16], [19]).

B.V. Gnedenko [19] also proved the criterion for the fulfillment of relation (3).

Theorem 2. *For sequence (z_n) to satisfy relation (3) it is necessary and sufficient that*

$$\forall \varepsilon > 0: \quad \lim_{x\to\infty} \frac{1-F(x+\varepsilon)}{1-F(x)} = 0 \quad \text{or} \quad x_F < \infty$$

where $x_F = \sup(x : F(x) < 1)$.

If in relation (2) we change the convergence in probability to the convergence almost sure we get the definition of relative stability almost sure.

Definition 3. *A sequence* (z_n) *of random variables is relatively stable almost sure if there exists a sequence of numbers* (a_n) *such that, as* $n \to \infty$,

$$\frac{z_n}{a_n} \longrightarrow 1 \quad a.s. \tag{4}$$

J. Galambos [16] provides the following criterion for fulfilling the relation (4).

Theorem 3. *For sequence* (z_n) *to satisfy relation (4) it is necessary and sufficient that* $x_F = \infty$ *and*

$$\forall k > 1 \quad \sum_{n=1}^{\infty} [1 - F(ka_n)] < \infty.$$

The last condition is equivalent to the following inequality (see, for example [25])

$$\forall \varepsilon : 0 < \varepsilon < 1 \quad \int_1^{\infty} \frac{dF(x)}{1 - F(\varepsilon x)} < \infty.$$

When replacing convergence in probability in relation (3) with convergence almost sure, then we get the stability almost sure.

A criterion for the almost sure stability of a sequence (z_n) of random variables is proposed by O. Barndorff-Nielsen [5]. It is of the form

Theorem 4. *A sequence* (z_n) *of random variables is stable almost sure if*

$$\forall \varepsilon > 0 \quad \int_{-\infty}^{\infty} \frac{1}{1 - F(x - \varepsilon)} dF(x) < \infty \quad or \quad x_F < \infty. \tag{5}$$

Law of the iterated logarithm for extreme r.v. first appeared in the work of J. Pickands [40]. Here is the result.

Theorem 5. *Let* $X_n, n = 1, 2, \ldots$ *be the successive terms of a discrete coordinate stationary Gaussian stochastic sequence. Assume, without loos of generality, that* $\boldsymbol{E}X_n = 0$ *and* $r_0 = \boldsymbol{E}X_n^2 = 1$ *for all n. Let* $r_n \equiv \boldsymbol{E}X_k X_{k+n}$ *be the covariance function. If*

$$\sum_{n=1}^{\infty} r_n^2 < \infty,$$

then

$$\limsup_{n \to \infty} \frac{\sqrt{2 \log n}(z_n - \sqrt{2 \log n})}{\log \log n} = \frac{1}{2} \quad a.s.$$

The next step in the study of the law of the iterated logarithm for extreme values of r.v. was made by de Haan and A. Hordijk [26].

It is known [26], that the asymptotic behavior of $\{z_n\}$ is closely related to the behavior as $x \to \infty$ of the functions $f(x)$ and $g(x)$ defined by the equalities

$$f(x) = \frac{1 - F(x)}{F'(x)}, \tag{6}$$

$$g(x) = f(x)\ln\ln\left\{\frac{1}{1-F(x)}\right\}. \tag{7}$$

So in the article [26] we have the following theorem.

Theorem 6. *Suppose F ia a distribution function with positive $F'(x)$ for all real x. If some constant $c \quad (0 \le c < \infty)$*

$$\lim_{t\to\infty}\frac{g(t)}{t} = c \tag{8}$$

(with g defined by (7)), then almost surely

$$\liminf_{n\to\infty}\frac{z_n}{a_n} = 1,$$

$$\limsup_{n\to\infty}\frac{z_n}{a_n} = e^c.$$

Here a_n is defined by $F(a_n) = 1 - 1/n$.

If relation (8) holds with $c = \infty$, then almost surely

$$\limsup_{n\to\infty}\frac{z_n}{a_n} = \infty.$$

Also in the article [26] de Haan and A. Hordijk obtained a generalization of the Pickands' law of the iterated logarithm for random variables with an arbitrary distribution.

Theorem 7. *Suppose F is a twice differentiable distribution function and $F'(x)$ is positive for all real x. If*

$$\lim_{t\to\infty} g'(t) = 0$$

(with g defined by (7)), then almost surely

$$\limsup_{n\to\infty}\frac{z_n - a_n}{f(a_n)\log\log n} = 1,$$

$$\liminf_{n\to\infty}\frac{z_n - a_n}{f(a_n)\log\log n} = 0$$

(here f is defined by (6) and a_n is defined by $F(a_n) = 1 - 1/n$).

We give here two more results, which we formulate as a lemma 1 and a lemma 2. In our opinion, these lemmas are independent theoretical results and, in addition, play a central role in the study of asymptotic behavior of extreme values i.i.d.r.v.

In what follows the notation "i.o." is used as a shorthand for "infinitely often".

Lemma 1 (Corollary 4.3.1 in [16]). *Let (ξ_k), $k \ge 1$, be a sequence of independent copies of a random variable ξ with c.d.f. F and let (u_n), $k \ge 1$, be a nondecreasing sequence of real numbers. Then the probability*

$$\boldsymbol{P}\{z_n \ge u_n \quad i.o.\}$$

equals zero or one according to whether the series

$$\sum_{n=1}^{\infty}(1 - F(u_n)) \tag{9}$$

converges or diverges.

The first versions of the following lemma are given in Chapter 4 in [16]. The final version belongs to M.J.Klass [31], [32].

Lemma 2 ([32]). *Let* (ξ_k)*,* $k \geq 1$*, be a sequence of independent copies of a random variable* ξ*. Further, let* (u_n)*,* $k \geq 1$*, be a nondecreasing sequence of real numbers such that*

$$\boldsymbol{P}\{\xi > u_n\} \to 0 \quad \textit{and} \quad n\boldsymbol{P}\{\xi > u_n\} \to \infty, \quad n \to \infty.$$

Then the probability

$$\boldsymbol{P}\{z_n \leq u_n \quad i.o.\}$$

equals zero or one according to whether the series

$$\sum_{n=1}^{\infty} \boldsymbol{P}\{\xi > u_n\} \exp\left(-n\boldsymbol{P}\{\xi > u_n\}\right) \tag{10}$$

converges or diverges.

We also have the following implications:

$$\textit{if} \quad \lim_{n\to\infty} \boldsymbol{P}\{\xi > u_n\} = c > 0, \quad \textit{then} \quad \boldsymbol{P}\{z_n \leq u_n i.o.\} = 0;$$

$$\textit{if} \quad \liminf_{n\to\infty} n\boldsymbol{P}\{\xi > u_n\} < \infty, \quad \textit{then} \quad \boldsymbol{P}\{z_n \leq u_n i.o.\} = 1$$

It should be noted that the asymptotic behavior of z_n for the discrete case has been studied much less. One of these works belongs to Anderson [3]. He obtained an interesting generalization of weak laws for discrete random variables.

Theorem 8. *Let* $X_1, X_2, \ldots$ *be i.i.d.r.v., that take non-negative integer values. Then there exists a sequence of integers* a_n*, that the following equality is true*

$$\lim_{n\to\infty} \boldsymbol{P}(Z_n = I_n \quad \textit{or} \quad I_n + 1) = 1$$

if and only if the total distribution function $F(x)$ *satisfies the condition*

$$\lim_{n\to\infty} \frac{1 - F(n+1)}{1 - F(n)} = 0.$$

In fact we may take $I_n = [a_n + 1/2]$ *is defined by* $F(a_n) = 1 - 1/n$*.*

This result shows that the asymptotics in the continuous and discrete cases can differ significantly (see also [35], [36] and Subsection 2.3).

A more detailed review of early results that were published before 1978 is given in Chapter 4 of J.Galambos' book [16].

2.2 Law of the iterated logarithm and a law of the triple logarithm and its generalization

Let ξ be a random variable with distribution function $F(x)$,

$$R(x) = -\ln(1-F(x)) \quad \text{or} \quad F(x) = 1-\exp(-R(x)).$$

Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a random variable ξ and

$$z_n = \max_{1\le k\le n} \xi_k.$$

It seems that it was first noticed that z_n satisfies a law of the iterated logarithm for the $\limsup$ and a law of the triple logarithm for the $\liminf$ in the work of P.Glasserman et al. [17]. For the case when $R(x) = \gamma x$, ie the random variable ξ has an exponential distribution with the parameter γ (or in some sense close to it), in [17] we have the following theorem.

Theorem 9. *If for all sufficiently large x and for some strictly positive $b_1 \le b_2$ and γ the following condition is satisfied*

$$b_1 \cdot e^{-R(x)} \le P\{Z_1 > x\} \le b_2 \cdot e^{-R(x)},$$

then

$$\limsup \frac{\gamma z_n - \log n}{\log\log n} = 1 \quad a.s. \quad and \quad \liminf \frac{\gamma Z_n - \log n}{\log\log\log n} = -1 \quad a.s.$$

The next step was taken in the work of the authors [1](unfortunately in this work there are no references to the article [17], which drew our attention to Prof. O. Marynych much later).

Before formulating the corresponding results let us recall the concept of regularly varying functions, see [10], [13].

Definition 4. *A positive measurable function U defined in some neighborhood of $+\infty$ is called regularly varying at $+\infty$ with index $\rho \in \mathbb{R}$ if $U(x) = x^{\rho}V(x)$, and the function V is slowly varying at $+\infty$, that is*

$$\lim_{t\to+\infty} \frac{V(tx)}{V(t)} = 1 \quad \text{for all} \quad x > 0.$$

A positive measurable function U defined in some neighborhood of $+\infty$ is called regularly varying at $+\infty$ if it is regularly varying at $+\infty$ with some index $\rho \in \mathbb{R}$.

Given a function $H : \mathbb{R} \to \mathbb{R}$ we denote by H^{-1} its generalized inverse defined by

$$H^{-1}(y) = \inf\{x \in \mathbb{R} : H(x) \ge y\}, \quad y \in \mathbb{R}. \tag{11}$$

We assume that $R(x)$, like $F(x)$, are differentiable functions and

$$R'(x) = \frac{F'(x)}{1-F(x)} = \frac{1}{f(x)}, \quad \forall x > x_0. \tag{12}$$

In article [1], the following result was obtained.

Theorem 10. *Let the function $f(x)$ be defined by equality (6). If one of the following conditions is true:*

(i) $f(x)$ *changes correctly at* $x \to \infty$ *and* $\forall t \in (0,1)$

$$\int_1^{\infty} \frac{dF(x)}{1-F(tx)} < \infty \tag{13}$$

(ii) $h(x) = f(R^{-1}(x))$ *changes correctly at* $x \to \infty$*,*
then

$$\limsup_{n\to\infty} \frac{z_n - a_n}{f(a_n)\ln\ln n} = 1 \tag{14}$$

$$\liminf_{n\to\infty} \frac{z_n - a_n}{f(a_n)\ln\ln\ln n} = -1. \tag{15}$$

In fact, the above conditions (i), (ii) are closely related to the conditions $(\mathbb{U}_1)$ And $(\mathbb{U}_2)$, which are given below.

Definition 5. *We say that a function* $H : \mathbb{R} \to \mathbb{R}$ *satisfies condition* $(\mathbb{U}_1)$ *if the following holds:*

1. $\lim_{x\to+\infty} H(x) = +\infty$*;*
2. *the function* H *is strictly increasing for* $x \in (x_0, \infty)$ *where* $x_0 := \inf\{x \in \mathbb{R} : H(x) > 0\}$*;*
3. *there exists* $\rho \in \mathbb{R}$ *such that the function* $H'(x)$ *is regularly varying at* $+\infty$ *with index* $\rho > -1$.

Definition 6. *We say that condition* $(\mathbb{U}_2)$ *holds for function* $H(x)$ *if conditions (1), (2) of definition 2 are satisfied and for sufficiently large* x *there exists a derivative* $h(x) = H^{'}(x)$ *and there exists* $\rho \in \mathbb{R}$ *such that the function* $\hat{h}(x) = (H^{-1}(x))' = 1/\left[h(H^{-1}(x))\right]$ *is regularly varying at* $+\infty$ *with index* ρ.

Let us define the following functions for sufficiently large $t > 0$:

$$L_0(t) = t, \quad L_m(t) = \ln L_{m-1}(t), \quad m \in \mathbb{N},$$

$$\alpha_m(t) = \sum_{i=1}^{m} L_i(t), \quad a_m(t) = R^{-1}(\alpha_m(t)),$$

$$d(n) = R^{-1}(L_1(n) - L_3(n)).$$

A significant strengthening of the Law of the iterated logarithm and a law of the triple logarithm was obtained in the work of I. Matsak [37]. Here are the results.

Theorem 11. *Let* (ξ_k), $k \geq 1$, *be a sequence of independent copies of a random variable* ξ *with the c.d.f.* F. *Assume that* $F(x) < 1$ *for all* $x \in \mathbb{R}$ *and* F *is strictly increasing on* (x_0, ∞) *where*

$$x_0 := \begin{cases} \inf\{x \in \mathbb{R} : F(x) > 0\}, & \text{if } F(x) = 0 \text{ for some } x \in \mathbb{R}, \\ -\infty, & \text{if } F(x) > 0 \text{ for all } x \in \mathbb{R}. \end{cases}$$

Suppose that one of the conditions (U_1) *or* (U_2) *is fulfilled. Then for every fixed* $m \in \mathbb{N}$ *we have*

$$\limsup_{n\to\infty} \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)} = 1 \quad a.s. \tag{16}$$

Theorem 12. *Under the assumptions of Theorem 11 we have*

$$\liminf_{n\to\infty} \frac{L_2(n)r(d(n))(z_n - d(n))}{2L_3(n)} = -1 \quad a.s., \tag{17}$$

where $\quad d(n) = R^{-1}(L_1(n) - L_3(n)).$

We give two examples of using Theorems 11, 12(see [37]).

Example 2.2.1. (Weibull distribution). Let $F(x) = 1 - \exp(-x^{\beta})$, $\quad x > 0$, $\quad \beta > 0$. Then

$$R(x) = x^{\beta}, \quad r(x) = \beta x^{\beta - 1}, \quad r(R^{-1}(x)) = \beta x^{1-\frac{1}{\beta}}.$$

Obviously, both conditions (U_1), (U_2) are fulfilled.

$$r(a_m(n)) \sim r(d(n)) \sim \beta(\ln n)^{1-\frac{1}{\beta}}, \quad n \to \infty,$$

for every fixed $m \geq 1$. Thus, for every fixed integer $m \geq 1$

$$\limsup_{n\to\infty} \frac{\beta(\ln n)^{1-\frac{1}{\beta}}(z_n - (\alpha_m(n))^{1/\beta})}{L_{m+1}(n)} = 1 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{\beta(\ln n)^{1-\frac{1}{\beta}} L_2(n)(z_n - (L_1(n) - L_3(n))^{1/\beta})}{2L_3(n)} = -1 \quad \text{a.s.}$$

Example 2.2.2. (Standard normal distribution). Let $F(x) = \Phi(x)$ for $x \in \mathbb{R}$, where

$$\Phi(x) = \int_{-\infty}^{x} \varphi(s)ds, \quad \varphi(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right), \quad x, s \in \mathbb{R}.$$

Then for every fixed integer $m \geq 2$ we have

$$\limsup_{n\to\infty} \frac{\sqrt{2\ln n}(z_n - \sqrt{2\ln n} + \frac{\frac{1}{2}L_2(n) - \sum_{i=2}^{m} L_i(n)}{\sqrt{2\ln n}})}{L_{m+1}(n)} = 1 \quad \text{a.s.} \tag{18}$$

and

$$\liminf_{n\to\infty} \frac{\sqrt{2\ln n}L_2(n)(z_n - \sqrt{2\ln n} + \frac{\frac{1}{2}(L_2(n) + \ln(4\pi)) + L_3(n)}{\sqrt{2\ln n}})}{2L_3(n)} = -1 \quad \text{a.s.} \tag{19}$$

These above formulae refine the results obtained earlier in [1], [39] and [40].

Next, we will try to replace conditions (U_1) or (U_2) with weaker conditions.

We will assume that the function $R(x)$ is absolutely continuous and is represented in the form

$$R(x) = R_0 + \int_{x_0}^{x} r(y)dy, \tag{20}$$

where $r(y) \geq 0$ is dimensionally locally integrable function, $R(x) \to \infty$ at $x \to \infty$.

Theorem 13. *Let* ξ *be a random variable with distribution function* $F(x)$*, the function* $R(x)$ *is absolutely continuous and is given by the formula (20),* $m \geq 1$ *some fixed integer. Let the following condition be satisfied:* $\forall x > 0$

$$\lim_{t\to\infty} \frac{r(tx)}{r(t)} = x^{\rho}, \quad \rho > -1, \tag{21}$$

then

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{r(a_1(n))(z_n - a_m(n))}{L_{m+1}(n)} = 1\right) = 1, \tag{22}$$

$$\mathbf{P}\left(\liminf_{n\to\infty} \frac{L_2(n)r(a_1(n))(z_n - d(n))}{2L_3(n)} = -1\right) = 1. \tag{23}$$

Proof of theorem 13. If function $R(x)$ would satisfy conditions (U_1) or (U_2), then equalities (22), (23) would be a simple consequence of theorems 11, 12(see also [37]). Unfortunately function $R(x)$ in the general case is not differentiated in a countable set of points. Therefore, we will have to slightly modify the corresponding proof from [37].

But first let's establish such a lemma.

Lemma 3. *Let the function* $r(t)$ *satisfy condition (21) of Theorem 13,* $A(n)$*,* $B(n)$ *are numerical sequences such that when* $n \to \infty$

$$A(n) \to \infty, \quad B(n) \to \infty, \quad \frac{A(n)}{B(n)} \to 1,$$

$\Delta(n) = B(n) - A(n) > 0$. *Then for an arbitrary small* $\varepsilon > 0$*, for sufficiently large* n

$$\frac{1+o(1)}{1+\varepsilon}\Delta(n) \leq r(R^{-1}(A(n)))(R^{-1}(B(n)) - R^{-1}(A(n))) \leq \frac{1+o(1)}{1-\varepsilon}\Delta(n). \tag{24}$$

Proof of Lemma 3. Let the function $r(x)$ satisfies condition (21), then it is regularly varying at $+\infty$ with index ρ, $r \in RV_{\rho}$. Then also $R \in RV_{\rho+1}$, $R^{-1} \in RV_{1/(\rho+1)}$ ([10], Proposition 1.5.8, Theorem 1.5.12) and $h \in RV_{-\rho/(\rho+1)}$. Moreover

$$R^{-1}(x) = \int_{R_0}^{x} h(y)dy, \quad h(y) = \frac{1}{r(R^{-1}(y))}. \tag{25}$$

We fix an arbitrary sufficiently small $\varepsilon > 0$ and introduce the following notation

$$h_n^- = \inf_{A(n)\leq t\leq B(n)} h(t), \quad h_n^+ = \sup_{A(n)\leq t\leq B(n)} h(t),$$

$$\zeta_n^- = \sup(t \le B(n): \quad h(t) \le h_n^-(1+\varepsilon)),$$
$$\zeta_n^+ = \sup(t \le B(n): \quad h(t) \ge h_n^+(1-\varepsilon)).$$

Then, respectively (25) we obtain

$$h_n^- \Delta(n) \le R^{-1}(B(n)) - R^{-1}(A(n)) \le h_n^+ \Delta(n).$$

Without loss of generality, we can assume that function $r(t)$ is left-continuous, and therefore $h(t)$ are also left-continuous. Therefore

$$h(\zeta_n^-) \le h_n^-(1+\varepsilon), \quad h(\zeta_n^+) \ge h_n^+(1-\varepsilon).$$

And therefore

$$\frac{1}{1+\varepsilon} h(\zeta_n^-)\Delta(n) \le R^{-1}(B(n)) - R^{-1}(A(n)) \le \frac{1}{1-\varepsilon} h(\zeta_n^+)\Delta(n).$$

Keeping in mind the equality $h(A(n)) = 1/r(R^{-1}(A(n)))$ we can rewrite the last inequality as:

$$\frac{1}{1+\varepsilon}\frac{h(\zeta_n^-)}{h(A(n))}\Delta(n) \le r(R^{-1}(A(n)))(R^{-1}(B(n)) - R^{-1}(A(n))) \le \frac{1}{1-\varepsilon}\frac{h(\zeta_n^+)}{h(A(n))}\Delta(n). \quad (26)$$

By constructing $\zeta_n^-, \zeta_n^+ \in (A(n), B(n))$, therefore under the conditions of the Lemma at $n \to \infty$

$$\frac{\zeta_n^-}{A(n)} \to 1, \quad \frac{\zeta_n^+}{A(n)} \to 1.$$

Since $h \in RV_{-\rho/(\rho+1)}$, then

$$\frac{h(\zeta_n^-)}{h(A(n))} \to 1, \quad \frac{h(\zeta_n^+)}{h(A(n))} \to 1 \quad (27)$$

as $n \to \infty$ (see similar conversions in [37]).

The relations (26), (27) together and give us estimate (24) Lemma 3. □

Remark 1. *Lemma 3 remains true in the continuous case when the integer n is replaced by the real number $t \ge 0$.*

Let us proceed directly to the proof of Theorem 13.

Establish the equality (22). Let τ^e be a standard exponentially distributed r.v., that is $\mathbf{P}(\tau^e < x) = 1 - \exp(-x)$. Let $(\tau_k^e)_{k\in\mathbb{N}}$ be a sequence of independent copies of a r.v. τ^e ,

$$z_n^e = \max_{1\le k\le n} \tau_k^e.$$

Without loss of generality, we can assume that

$$z_n - a_m(n) = R^{-1}(z_n^e) - R^{-1}(\alpha_m(n)). \quad (28)$$

(see proof of Theorem 1 in [37]).

The following equality was obtained in Lemma 2 in [37]

$$\limsup_{n\to\infty} \frac{z_n^e - \alpha_m(n)}{L_{m+1}(n)} = 1 \quad \text{a.s.} \tag{29}$$

Furthermore, we assume that

$$z_n^e(n) \geq \alpha_m(n) \tag{30}$$

(since $R^{-1}(x)$ is non-decreasing function, taking into account (29), it is sufficient to choose only those n, for which (30)) holds.

It is known (see [16], Chapter 4, Example 4.3.3), that

$$\frac{z_n^e}{\ln n} \to 1 \quad \text{a.s.}$$

and also

$$\frac{z_n^e}{\alpha_m(n)} \to 1 \quad \text{a.s.},$$

as $n \to \infty$.

We put in Lemma 3 $A(n) = \alpha_m(n)$, $B(n) = z_n^e$.

Putting together relations (29), (24), we obtain

$$\frac{1}{1+\varepsilon} \leq \limsup_{n\to\infty} \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)} \leq \frac{1}{1-\varepsilon} \quad \text{a.s.} \tag{31}$$

Estimates (31) are satisfied for any $\varepsilon > 0$, therefore, from this we obtain the equality (22).

Similarly, based on the following equality (see Lemma 4, in [37])

$$\liminf_{n\to\infty} \frac{L_2(n)(z_n^e - L_1(n) + L_3(n))}{2L_3(n)} = -1 \quad \text{a.s.} \tag{32}$$

we can prove the equality (23). In this case, in Lemma 3, one must choose $A(n) = z_n^e$, $B(n) = L_1(n) - L_3(n)$ and move along n, for which $z_n^e(n) < L_1(n) - L_3(n)$.

□

It should be noted that the above proof of Theorem 13 is actually based on the results and methods of works [2], [37].

In the discrete case, an important theorem was established in the article by K. Akbash et al. [2].

Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i, p_i), i \geq 0$, $F(n) = 1 - \exp(-R(n))$ is a distribution function of r.v. ξ,

$$r(n) = R(n) - R(n-1).$$

To formulate the following result, we introduce some necessary notation. For the sequence $(r(n))$ we define its extension to the function $r : (0,\infty) \to \mathbb{R}$ by setting $r(x) = r(\lceil x \rceil)$, ($\lceil x \rceil$ - the least integer $\geq x$).

Let

$$R(x) = \int_0^x r(y)dy.$$

The function R is a piecewise linear extension of the sequence $R(n)$.

Theorem 14. *([2]) Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution $(i,p_i), i\geq 0$, $m\geq 1$ some fixed integer. Let us assume that condition (21) is satisfied for function $r(x)$.*

Then

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(n))(z_n+\theta_n-a_m(n))}{L_{m+1}(n)}=1\right)=1, \tag{33}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(z_n+\theta_n-d(n))}{2L_3(n)}=-1\right)=1, \tag{34}$$

where θ_n some r.v., $0\leq\theta_n\leq 1$.

2.3 "Confidence intervals" for extrema of discrete random variables

Let z_n be defined by the equality (1). It is well known that in the discrete case the sequence

$$a_n=\max\left(k\geq 0:\sum_{i\geq k}p_i\geq\frac{1}{n}\right) \tag{35}$$

is a good approximation to the r.v. z_n.

It turns out that in some cases it is possible to construct a non-random interval with respect to a_n, in which z_n falls almost sure for large n. This would be the case, for example, if $r(k)\to\infty$ very quickly at $k\to\infty$.

Such studies were carried out in the works of I. Matsak [35], [36]. The following results were obtained.

Theorem 15. *([35]) Let $\xi,\xi_1,\xi_2,\ldots,$ be a sequence of discrete i.i.d.r.v. with distribution (k,p_k), $k\geq 1$, $r(k)$ - monotonic function and $r(k)\to\infty$ at $k\to\infty$. Then the equality*

$$\mathbf{P}(z_n>a_n+1\quad i.o.)=0 \tag{36}$$

takes place if and only if

$$\sum_{k\geq 1}\exp(-r(k))<\infty, \tag{37}$$

where i.o. - infinitely often. Moreover, if (37) is true, then

$$\mathbf{P}(z_n=a_n+1\quad i.o.)=1. \tag{38}$$

Theorem 16. *([35]) Let $\xi,\xi_1,\xi_2,\ldots,$ be a sequence of discrete i.i.d.r.v. with distribution (k,p_k), $k\geq 1$, $r(k)$ - monotonic function and $r(k)\to\infty$ for $k\to\infty$. Then the equality*

$$\mathbf{P}(z_n<a_n-1\quad i.o.)=0 \tag{39}$$

takes place if and only if

$$\sum_{k\geq 1}\exp(-\exp(r(k)))<\infty. \tag{40}$$

Moreover, if (40) is true, then

$$\mathbf{P}(z_n=a_n-1\quad i.o.)=1. \tag{41}$$

This immediately follows

Corollary 1. *If equality (37) holds under the conditions of Theorem 15, then relations (36),(38), (39), (41) are true.*

In the case when the r.v. ξ has a Poisson distribution $p_i = \frac{\lambda^i}{i!}\exp(-\lambda), i \geq 0$, or is in some sense close to a Poisson distribution, the following theorem was proved.

Theorem 17. *([36]) Let ξ be a discrete r.v. with distribution (i, p_i), $i \geq 0$, $\beta > 0$ is an arbitrary number, a_n is given by the equality (35). If the function $r(n)$ defined above satisfies the condition: when $n \to \infty$*

$$r(n) = \beta \ln n + o(L_2(n)), \tag{42}$$

then

$$\mathbf{P}(\exists n_0, \forall n \geq n_0 \quad z_n \in J_n = \{a_n + m, \quad m \in I_\beta\}) = 1, \tag{43}$$

$$\forall m \in I_\beta \quad \mathbf{P}(z_n = a_n + m \quad i.o.) = 1 \tag{44}$$

and

$$a_n = \frac{\ln n}{\beta L_2(n)}(1 + o(1)), \tag{45}$$

where $I_\beta = \{-1, 0, 1, \ldots, [1 + 1/\beta]\}$.

For a Poisson distribution with parameter $\lambda > 0$ (where $r(n) = \ln n + o(1), \beta = 1$) the equalities (43), (44) are fulfilled at $I_\beta = I_1 = \{-1, 0, 1, 2\}$ and

$$a_n = \frac{\ln n}{L_2(n)}\left(1 + \frac{L_3(n) + \ln\lambda + 1 + o(1)}{L_2(n)}\right).$$

When function $r(n)$ grows, it increases more slowly than at (42), for example, if the following conditions are satisfied

$$r(n) = o(\ln n), \tag{46}$$

$$\sum_{n \geq 1} \exp(-e^{r(n)}) < \infty, \tag{47}$$

then the equalities (43), (44) will remain true for $\beta = 0$ [36].

Note that under condition

$$r(n) = v_n \ln n, \quad v_n \to \infty, \quad n \to \infty, \tag{48}$$

equalities (16), (17) are also true for $\beta = \infty$.

In the article K.Akbash et al. [2] a similar problem was considered for a geometric distribution and random variables, the distribution tails of which fall off more slowly than the tails of the geometric distribution.

Theorem 18. *([2]) Let* $(\xi_k)_{k\in\mathbb{N}}$ *be a sequence of independent copies of a discrete random variable* ξ *with distribution* (i, p_i), $i \geq 0$, a_n *is given by the equality (35), and for any fixed* m

$$\lim_{n\to\infty} \frac{r(n+m)}{r(n)} = 1, \tag{49}$$

$$\exists C_0 < \infty, \quad \forall n \geq 1 \quad r(n) \leq C_0. \tag{50}$$

(i) if

$$\sum_{n\geq 1} |r(n)|^2 = \infty, \tag{51}$$

then for any integer m

$$\mathbf{P}(z_n = a_n + m \quad i.o.) = \mathbf{P}(z_n = \xi_n = a_n + m \quad i.o.) = 1. \tag{52}$$

(ii) If condition (51) is not satisfied, then for any integer $m > 0$

$$\mathbf{P}(z_n = \xi_n = (a_n - m, a_n + m) \quad i.o.) = \mathbf{P}(\xi_n = (a_n - m, a_n + m) \quad i.o.) = 0. \tag{53}$$

3 Asymptotic behavior of extreme values of regenerative processes

The problem of finding the asymptotics of the extremal values of regenerative processes is of considerable practical interest and, therefore, has been considered by many authors. So, for example, extrema in queuing systems, and also extrema of birth and death processes (see [3], [4], [9]), [28], [42], [43]). In this case, the classical theory of extreme values of the i.i.d.r.v. was usually used. These results are well known (see review [4]).

3.1 General regenerative processes

Let's start from the definition of the regenerative process (see, for example, [44], p. II, ch.2).

Definition 7. *By a cycle of duration* T *we mean an ordered pair* $£ = (T, \xi(t))$ *, in which* T *is a non-negative r.v., and* $\xi(t)$ *is a random process defined on [0, T),*

$$\mathbf{P}(T = 0) < 1, \quad \mathbf{P}(T < \infty) = 1.$$

In the general case, r.v. T *and process* $\xi(t)$ *are dependent.*

Suppose $£_i = (T_i, \xi_i(t))$, $i \geq 1$, *is an infinite sequence of independent cycles equally distributed with* $£$ *. Let's define a random process* $X(t)$ *,* $t \geq 0$, *by the formula*

$$X(t) = \xi_i(t - S_{i-1}), \quad at \quad t \in [S_{i-1}, S_i),$$

where $S_i = T_1 + \ldots + T_i$, $i \geq 1$, $S_0 = 0$ *.*

Then we will call process $X(t)$ *regenerative, points* S_i *- moments of regeneration, and the interval* $[S_{i-1}, S_i)$ *- the* i^{th} *regeneration period.*

Let's put

$$\overline{X}(t) = \sup_{0\le s<t} X(s), \quad \overline{X}_k = \sup_{S_{k-1}\le s<S_k} X(s). \tag{54}$$

To avoid questions related to the dimensions of $\overline{X}(t)$ and $\overline{X}_k$, on random processes $\xi_i(t)$ we will establish a separability condition.

It is clear that then $\overline{X}_k$ are i.i.d.r.v. We will assume that for all $u \in \mathbf{R}$

$$q(u) = \mathbf{P}(\overline{X}_k \ge u) > 0 \quad \text{and}$$

$$q(u) \downarrow 0 \quad \text{at} \quad u \uparrow \infty$$

(in fact, the last condition means that $\overline{X}_k$ is almost sure a finite r.v.).

It seems that for the first time The laws of iterated and triple logarithms for extreme values of regenerative processes appeared in the work P. Glasserman et al. [17]. In this case, a random process with discrete time was considered and the following condition was established:

For all sufficiently large x

$$b_1 \exp(-\gamma u) \le q(u) \le b_2 \exp(-\gamma u) \tag{55}$$

Theorem 19. *([17]) Let* $(X(t), t = 1, 2, \ldots)$ *be a regenerative random process with discrete time. Assume that* (55) *holds . Then*

$$\limsup_{t\to\infty} \frac{\gamma\overline{X}(t) - \ln t}{L_2(t)} = 1 \quad a.s., \tag{56}$$

and

$$\liminf_{t\to\infty} \frac{\gamma\overline{X}(t) - \ln t}{L_3(t)} = 1 \quad a.s., \tag{57}$$

The next two important results about the asymptotics of almost probably extreme values of regenerative processes are obtained in the paper by A.Marynych et al. [34]. It also gives examples of the use of the obtained results for birth and death processes, which describe the length of the queue and the waiting time in queuing systems.

Throughout this section, we denote by F the distribution function of $\overline{X}_1$, that is,

$$F(x) := \mathbf{P}(\overline{X}_1 < x).$$

Put

$$R(x) := -\log(1 - F(x)), \quad x \in \mathbb{R},$$

and

$$\alpha_T = \mathbf{E}T_1 = \mathbf{E}T.$$

Note also that it is always possible to write a decomposition

$$R(x) = R_0(x) + R_1(x), \quad x \in \mathbb{R}, \tag{58}$$

where

$$|R_1(x)| \le C_1 < \infty, \quad x \in \mathbb{R}. \tag{59}$$

Here and hereafter we denote by C, C_1, C_2 etc. some positive constants which may vary from place to place and may depend on parameters of the process $X(\cdot)$.

We are ready to formulate first result.

Theorem 20. *([34]) Let $(X(t))_{t\ge 0}$ be a regenerative random process. Assume that there exists a decomposition* (58) *such that* (59) *holds and the function R_0 satisfies condition $(\mathbb{U}_2)$. Suppose further that $\alpha_T < \infty$. For large enough $x \in \mathbb{R}$, let r_0 be the derivative of R_0. Then*

$$\limsup_{t\to\infty} \frac{r_0(A_0(t))(\overline{X}(t) - A_0(t))}{L_2(t)} = 1 \quad a.s., \tag{60}$$

and

$$\liminf_{t\to\infty} \frac{r_0(A_0(t))(\overline{X}(t) - A_0(t))}{L_3(t)} = -1 \quad a.s., \tag{61}$$

where

$$A_0(t) = R_0^{-1}\left(\ln \frac{t}{\alpha_T}\right).$$

Next, we consider the discrete case, that is, when the process $X(t)$ takes the values in some lattice in $\mathbb{R}$. Such processes are important in applied areas. Assume that

$$\mathbf{P}(X(t) \in \{0,1,2,3,\dots\}) = 1, \quad t \ge 0, \tag{62}$$

and, for $k = 0,1,2,3,\dots$, put

$$R(k) := -\log \mathbf{P}(\overline{X}(T_1) \ge k).$$

Similarly to (58) and (59) we can write a decomposition:

$$R(k) = R_0(k) + R_1(k), \quad k = 0,1,2,3,\dots, \tag{63}$$

where $R_0 : \mathbb{R} \to \mathbb{R}$ and $R_1 : \mathbb{R} \to \mathbb{R}$ are real-valued functions and R_1 is such that

$$|R_1(k)| \le C_1 < \infty, \quad k = 0,1,2,3,\dots. \tag{64}$$

Theorem 21. *([34]) Let $(X(t))_{t\ge 0}$ be a regenerative random process such that* (62) *holds. Assume that there exists a decomposition* (63) *such that* (64) *is fulfilled and the function R_0 satisfies condition $(\mathbb{U}_2)$. Suppose also that $\alpha_T < \infty$.*

(i) The asymptotic relation

$$r_0(R_0^{-1}(x)) = o(\log x), \quad x \to \infty, \tag{65}$$

entails

$$\limsup_{t\to\infty} \frac{r_0(A_0(t))(\overline{X}(t) - A_0(t))}{L_2(t)} = 1 \quad a.s. \tag{66}$$

(ii) The asymptotic relation

$$r_0(R_0^{-1}(x)) = o(\log\log x), \quad x \to \infty, \tag{67}$$

entails

$$\liminf_{t\to\infty} \frac{r_0(A_0(t))(\overline{X}(t) - A_0(t))}{L_3(t)} = -1 \quad a.s. \tag{68}$$

The functions A_0 and r_0 were defined in Theorem 20.

Note that when function $r_0(x))$ satisfies condition (21) when $-1 < \rho < 0$, then conditions (65), (67) are satisfied, and are not satisfied when $\rho > 0$.

Next, we will obtain boundary theorems for the regenerative random processes, in which we will try to explore these cases in more detail. In addition, conditions (58), (59) and (63), (64) will be reduced. We will also replace conditions like $(\mathbb{U}_2)$ with the condition that $R(x)$ is absolutely continuous and is a regularly varying function.

Theorem 22. *Let $(X(t))_{t\geq 0}$ be a regenerative random process. Suppose that function $R(x)$ is absolutely continuous and is given by formula* (20)*, and function $r(x)$ satisfies condition* (21)*. And let $\alpha_T < \infty$, and m a fixed integer, $m \geq 1$. Then*

$$\mathbf{P}\left(\limsup_{t\to\infty} \frac{r(a_1(t))(\overline{X}(t) - a_m(\frac{t}{\alpha_T}))}{L_{m+1}(t)} = 1\right) = 1. \tag{69}$$

If, in addition, for some $\gamma > 1$

$$\mathbf{E}T^{\gamma} < \infty, \tag{70}$$

then

$$\mathbf{P}\left(\liminf_{t\to\infty} \frac{L_2(t) r(a_1(t))(\overline{X}(t) - d(\frac{t}{\alpha_T}))}{2L_3(t)} = -1\right) = 1, \tag{71}$$

where $L_m(t)$,$a_m(t)$, $d(t)$ are defined in Subsection 2.2 *before Theorem 11 .*

Proof of theorem 22. Let's start from equality (69). Put

$$\overline{X}_k = \sup_{S_{k-1}\leq t<S_k} X(t), \quad Z_n = \max_{1\leq k\leq n} \overline{X}_k.$$

Since (S_k) is the regeneration moment of process $X(t)$, then $(\overline{X}_k)$ is a sequence of i.i.d.r.v. Moreover, it is clear from the formulation of Theorem 22 that the sequence $(\overline{X}_k)$ satisfies the conditions of Theorem 13. Therefore,

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{r(a_1(n))(Z_n - a_m(n))}{L_{m+1}(n)} = 1\right) = 1. \tag{72}$$

Denote by $N(t)$ the counting process for sequence (S_k),

$$N(t) = \max(k \geq 0 : S_k < t), \quad t \geq 0.$$

It is clear that when t changes from 0 to ∞, then the process $N(t)$ runs through all natural numbers almost sure. And accordingly, it is possible to substitute in the equality (72) instead of n the process $N(t)$. Get

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(N(t)))(Z_{N(t)}-a_m(N(t)))}{L_{m+1}(N(t))}=1\right)=1. \tag{73}$$

From the renewal theory it is known [24], that

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\alpha_T}\quad a.s., \tag{74}$$

thus for $t\to\infty$

$$\ln N(t)=\ln\frac{t}{\alpha_T}+o(1)\quad a.s.,$$

From here and equalities (73) we have

$$\mathbf{P}\left(\limsup_{t\to\infty}\frac{r(R^{-1}(\ln\frac{t}{\alpha_T}+o(1)))(Z_{N(t)}-a_{m-1}(\ln\frac{t}{\alpha_T}+o(1)))}{L_m(\ln\frac{t}{\alpha_T}+o(1)))}=1\right)=1. \tag{75}$$

Obviously, in relation (75), the expression in the denominator is equivalent to $L_{m+1}(t)$.

Further, we recall that under the conditions of Theorem 22 he function $r(R^{-1}(x))$ regularly varying at infinity. That is, the functions $r(R^{-1}(\ln\frac{t}{\alpha_T}+o(1)))$ and $r(a_1(t))$ are asymptotically equivalent. Therefore, equality (75) can be rewritten in the following form

$$\mathbf{P}\left(\limsup_{t\to\infty}\frac{r(a_1(t))(Z_{N(t)}-a_{m-1}(\ln\frac{t}{\alpha_T}+o(1)))}{L_{m+1}(t)}=1\right)=1. \tag{76}$$

At the next step, we will show that for $t\to\infty$

$$r(a_1(t))\left|a_{m-1}(\ln\frac{t}{\alpha_T}+o(1))-a_{m-1}(\ln\frac{t}{\alpha_T})\right|=\frac{|o(1)|}{L_1(t)L_2(t)\ldots L_{m-1}(t)}. \tag{77}$$

Indeed, it is not difficult to check by induction that

$$L_m(\ln\frac{t}{\alpha_T}+o(1))-L_m(\ln\frac{t}{\alpha_T})=\frac{o(1)}{L_1(t)L_2(t)\ldots L_m(t)}$$

Next, let's put

$$B(t)=\max(L_{m-1}(\ln\frac{t}{\alpha_T}+o(1)),L_{m-1}(\ln\frac{t}{\alpha_T}),$$

$$A(t)=\min(L_{m-1}(\ln\frac{t}{\alpha_T}+o(1)),L_{m-1}(\ln\frac{t}{\alpha_T}),$$

$$\Delta(t)=B(t)-A(t)=\frac{|o(1)|}{L_1(t)L_2(t)\ldots L_{m-1}(t)}.$$

And once again we use Lemma 3 (see also Remark 1 after it) and get: for sufficiently small $\varepsilon>0$ for $t\to\infty$

$$\frac{1+o(1)}{1+\varepsilon}\Delta(t)\le$$

$$\le r(a_1(t))|R^{-1}(L_{m-1}(\ln\frac{t}{\alpha_T}+o(1)))-R^{-1}(L_{m-1}(\ln\frac{t}{\alpha_T}))|\le$$

$$\le\frac{1+o(1)}{1-\varepsilon}\Delta(t).$$

The last inequalities give the relation (77).

Given (77), relation (76) will be rewritten as follows

$$\mathbf{P}\left(\limsup_{t\to\infty}\frac{r(a_1(t))(Z_{N(t)}-a_m(\frac{t}{\alpha_T}))}{L_{m+1}(t)}=1\right)=1. \tag{78}$$

Next, we take the function $N(t)+1$ instead of $N(t)$. Repeating the above considerations, we find that $Z_{N(t)+1}$ satisfies the equality (78).

It remains to be seen that

$$Z_{N(t)}\le\bar{X}(t)\le Z_{N(t)+1}\quad a.s.$$

All this together gives equality (69).

Let's move on to the proof of equality (71). From Theorem 13 we have

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(Z_n-d(n))}{2L_3(n)}=-1\right)=1.$$

Hence, repeating the above considerations regarding equality (73), we have

$$\mathbf{P}\left(\liminf_{t\to\infty}\frac{L_2(N(t))r(a_1(N(t)))(Z_{N(t)}-d(N(t)))}{2L_3(N(t))}=-1\right)=1. \tag{79}$$

Next, we use the Marcinkiewicz-Zygmund strong law of large numbers (see [8], ch.1, Theoreme 1.3.2):
under the condition (70)

$$\mathbf{P}\left(\lim_{t\to\infty}\frac{N(t)-\frac{t}{\alpha_T}}{t^{1/\gamma}}=0\right)=1,$$

that is, for $t\to\infty$

$$\ln N(t)=\ln\frac{t}{\alpha_T}+o(t^{1/\gamma-1})\quad a.s.$$

Thus, equality (79) is equivalent to the following relation

$$\mathbf{P}\left(\liminf_{t\to\infty}\frac{L_1(\hat{L}(t))\,r(R^{-1}(\hat{L}(t)))(Z_{N(t)}-R^{-1}(\hat{L}(t)-L_2(\hat{L}(t)))}{2L_2(\hat{L}(t))}=-1\right)=1, \tag{80}$$

where $\hat{L}(t)=\ln\frac{t}{\alpha_T}+o(t^{1/\gamma-1})$.

The transition from equality (80) to (71) basically repeats the considerations used above. We will not bring them up again. But there exist one point that probably requires explanation. Namely, the following asymptotic relation:
at $t\to\infty$

$$\frac{L_2(t)\,r(a_1(t))}{L_3(t)}\left|R^{-1}(\hat{L}(t)-L_2(\hat{L}(t)))-R^{-1}(L_1(\frac{t}{\alpha_T})-L_3(\frac{t}{\alpha_T}))\right|=o(1). \tag{81}$$

To prove (81) we again use Lemma 3. Let's put

$$B(t) = \max\left(\hat{L}(t) - L_2(\hat{L}(t)), L_1(\frac{t}{\alpha_T}) - L_3(\frac{t}{\alpha_T})\right),$$

$$A(t) = \min\left(\hat{L}(t) - L_2(\hat{L}(t)), L_1(\frac{t}{\alpha_T}) - L_3(\frac{t}{\alpha_T})\right).$$

Then, using elementary calculations, we find

$$\Delta(t) = B(t) - A(t) = \left|\hat{L}(t) - L_2(\hat{L}(t)) - (L_1(\frac{t}{\alpha_T}) - L_3(\frac{t}{\alpha_T})\right| = o(t^{1/\gamma - 1}). \tag{82}$$

And according to Lemma 3 for $t \to \infty$

$$r(a_1(t))\left|R^{-1}(\hat{L}(t) - L_2(\hat{L}(t))) - R^{-1}(L_1(\frac{t}{\alpha_T}) - L_3(\frac{t}{\alpha_T}))\right| \leq O(\Delta(t)).$$

This inequality, together with relation (82), gives equality (81). □

Next, consider the case of a regenerative process $X(t)$ that takes discrete values, or rather, satisfy condition (62).

Let $r(n) = R(n) - R(n-1)$. As in theorem 14 for the sequence $(r(n))$ we define its extension to the function

$$r(x) = r(\lceil x \rceil). \tag{83}$$

Let

$$R(x) = \int_0^x r(y)dy. \tag{84}$$

Theorem 23. *Let $(X(t))_{t\geq 0}$ be a regenerative random process with discrete values, i.e. satisfies condition (62), $\alpha_T < \infty$, m is a fixed integer, $m \geq 1$. And let functions $r(x)$ and $R(x)$ be given by formulas (83) and (84) , respectively, and condition (21) is satisfied.*

Then

$$\mathbf{P}\left(\limsup_{t\to\infty} \frac{r(a_1(t))(\overline{X}(t) + \theta(t) - a_m(\frac{t}{\alpha_T}))}{L_{m+1}(t)} = 1\right) = 1, \tag{85}$$

If, in addition, for some $\gamma > 1$, condition (70) from Theorem 22 is satisfied, then

$$\mathbf{P}\left(\liminf_{t\to\infty} \frac{L_2(t)r(a_1(t))(\overline{X}(t) + \theta(t) - d(\frac{t}{\alpha_T}))}{2L_3(t)} = -1\right) = 1, \tag{86}$$

where $\theta(t)$ some r.v., $0 \leq \theta(t) \leq 1$.

Proof of theorem 23. Let $(\overline{X}_k)$ and (Z_n) be sequences of r.v. constructed by the process $(X(t))_{t\geq 0}$, which we introduced in the proof of Theorem 22. Consider their continuous analogues $(\overline{X}_k^c)$ and (Z_n^c).

Let $(\overline{X}^c)$ r.v. with distribution function $F^c(x) = 1 - \exp(-R(x))$, where $R(x)$ is given by formula (84). Denote by $(\overline{X}_k^c)$ the sequence of independent copies of $(\overline{X}^c)$,

$$Z_n^c = \max_{1\leq k\leq n} \overline{X}_k^c.$$

Under our conditions, the function $F^c(x)$ is absolutely continuous, that is, we can apply Theorem 13 to the sequence Z_n^c.

We get

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(n))(Z_n^c-a_m(n))}{L_{m+1}(n)}=1\right)=1, \tag{87}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(Z_n^c-d(n))}{2L_3(n)}=-1\right)=1. \tag{88}$$

Next, we define the following equalities

$$\mathbf{P}([\overline{X}^c]<k)=\mathbf{P}(\overline{X}^c<k)=\mathbf{P}(\overline{X}_i<k)=1-\exp(R(k)).$$

Thus r.v. $[\overline{X}_i^c]$ and $\overline{X}_i$ are identically distributed. The same is true for r.v. $[Z_n^c]$ and Z_n. (see [2]).

Thus, the relation (87), (88) can be rewritten as

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(n))(Z_n+\theta_n-a_m(n))}{L_{m+1}(n)}=1\right)=1, \tag{89}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(Z_n+\theta_n-d(n))}{2L_3(n)}=-1\right)=1. \tag{90}$$

where θ_n some r.v., $0\le\theta_n\le 1$.

The transition from the last equalities to (85) and (86) basically repeats the considerations from the proof of the previous theorem 22. Therefore, we skip them. □

Corollary 2. *(i) If under the conditions of Theorem 23*

$$\frac{r(a_1(t))}{L_{m+1}(t)}\to 0,\quad t\to\infty,$$

then value $\theta(t)$ in formula (85) can be omitted.
(ii) The same way can be omitted $\theta(t)$ in formula (86), if

$$\frac{L_2(t)r(a_1(t))}{L_3(t)}\to 0,\quad t\to\infty.$$

(iii) If

$$\frac{r(a_1(t))}{L_{m+1}(t)}\to\infty,\quad t\to\infty,$$

then

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(\overline{X}(t)-a_m(\frac{t}{\alpha_T})\right)=\chi\right)=1. \tag{91}$$

(iiii) If

$$\frac{L_2(t)r(a_1(t))}{L_3(t)}\to\infty,\quad t\to\infty,$$

then

$$\mathbf{P}\left(\liminf_{t\to\infty}\left(\bar{X}(t)-d(\frac{t}{\alpha_T})\right)=\chi\right)=1, \tag{92}$$

where $\chi \in [-1,0]$. *Here and further by* χ *we denote nonrandom constant, not necessarily the same in different parts of the article.*

Items (i) and (ii) of Corollary 2 are obvious. It is not important to check that the conditions of these points are satisfied at $-1<\rho<0$ and are not satisfied at $\rho>0$, where the parameter ρ is defined in inequality (21). On the contrary, points (iii) and (iiii) are satisfied at $\rho>0$, and are not satisfied at $-1<\rho<0$.

For $\rho=0$ we have a boundary case that requires additional verification.

Regarding the transition from Theorem 23 to points (iii) and (iiii) of Corollary 2, some explanations in a similar situation can be found in article [2].

Remark 2. *An analysis of the proofs of Theorems 22 and 23 showed that if condition (70)is omitted in them, then instead of relations (71) and (86), the following equalities hold:*

$$\mathbf{P}\left(\liminf_{t\to\infty} r(a_1(t))(\bar{X}(t)-d(\frac{t}{\alpha_T}))=0\right)=1, \tag{93}$$

$$\mathbf{P}\left(\liminf_{t\to\infty} r(a_1(t))(\bar{X}(t)+\theta(t)-d(\frac{t}{\alpha_T}))=0\right)=1, \tag{94}$$

3.2 Birth and death processes

Suppose that $X(t)$ is a Markov process with states $0,1,2,\ldots$, its transition probabilities $p_{i,j}(t)$ are stationary and satisfy the conditions: for $h\to 0$

1. $p_{i.i+1}(h)=\lambda_i h+o(h), \quad i\geq 0,$
2. $p_{i,i-1}(h)=\mu_i h+o(h), \quad i\geq 1,$
3. $p_{i,i}(h)=1-(\lambda_i+\mu_i)h+o(h), \quad i\geq 0,$
4. $\mu_0=0,\lambda_0>0,\mu_i>0,\lambda_i>0, \quad i=1,2,\ldots.$ (95)

Then $X(t)$ is called the process of the process of birth and death. Such processes are widely used in biology, queuing theory, reliability theory, etc. ([20], §1.4,, [21], §6.3, [29], §7.4).

Let's put

$$\theta_0=1, \quad \theta_k=\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i}, \quad k\geq 1.$$

Further in this subsection, we indicate that the process of the process of birth and death satisfies the condition(95), $X(0)=0$ a.s., as well as

$$\sum_{k\geq 1}\theta_k<\infty, \tag{96}$$

$$\sum_{k\geq 1}\frac{1}{\lambda_k\theta_k}=\infty. \tag{97}$$

It is known ([29], [30]) that then there are stationary state probabilities

$$\lim_{t\to\infty} \mathbf{P}(X(t)=k) = \lim_{t\to\infty} p_k(t) = p_k, \tag{98}$$

and

$$p_k = \theta_k p_0, \quad p_0 = (\sum_{k=0}^{\infty} \theta_k)^{-1}. \tag{99}$$

In addition, $X(t)$ will be a special type of regenerative process with regeneration moments $S_0 = 0, S_1, S_2, \dots$, where S_k is the first moment of entering state 0 after the k^{th} exit from it. And

$$\alpha_T = \mathbf{E}T_k = \frac{1}{\lambda_0 p_0},$$

where $T_k = S_k - S_{k-1}$ is the duration of the k^{th} regeneration cycle [21], [45].

Here we are interested in the asymptotic behavior of the almost sure extreme values of the populations:

$$\overline{X}(t) = \sup_{0\le s<t} X(s), \quad t \ge 0.$$

Example 3.2.1. Let $X(t)$ be the birth and death process with parameters

$$\lambda_n = \lambda n + a, \quad \mu_n = \mu n, \quad \lambda > 0, \mu > 0, a > 0, \quad n = 0, 1, 2, \dots \tag{100}$$

(see [29, ch.7 , §6]).

Such a process is called a process with linear growth and immigration. If state n describes the size of the population at some point in time, then the probability of transition to state $n+1$ in a small time interval δ is equal to $(\lambda n + a)\delta + o(\delta)$, and the transition probability $n \to n-1$ is given by equality $\mu n\delta + o(\delta)$. Coefficient a can be interpreted as the infinitesimal intensity of population growth due to migration.

Assume that the following condition is satisfied

$$\rho = \frac{\lambda}{\mu} < 1. \tag{101}$$

It is easy to check that, under condition (101) are satisfied under condition (96),(97).

For such a process, the following law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf was established in the article by A.Marynych et al. [34].

Theorem 24. *([34]) Let $X(t)$ be the process of birth and death with parameters $\lambda_n = \lambda n + a$, $\mu_n = \mu n$, $\lambda > 0$, $\mu > 0$, $a > 0$, $n = 0, 1, 2, \dots$ and the following condition is satisfied (101). Then a.s.*

$$\limsup_{t\to\infty} \frac{\overline{X}(t) \ln \frac{1}{\rho} - \ln t}{L_2(t)} = 1 + \frac{a}{\lambda}, \tag{102}$$

$$\liminf_{t\to\infty} \frac{\overline{X}(t) \ln \frac{1}{\rho} - \ln t - \frac{a}{\lambda} L_2(t)}{L_3(t)} = -1. \tag{103}$$

Here, for this process, we slightly strengthen the above statement. In this case, to find the asymptotics of $\bar{X}(t)$, we use the results of Theorem 23.

To use Theorem 23, we first need to find the value

$$q(n) = \mathbf{P}(\bar{X}(T_1) \geq n) = \exp(-R(n)).$$

As is known (see [4], [29]), for the general case of the process of birth and death, the formula is true:

$$q(n) = \frac{1}{\sum_{k=0}^{n-1} \alpha_k}, \tag{104}$$

where $\alpha_0 = 1, \quad \alpha_k = \prod_{i=1}^{k} \frac{\mu_i}{\lambda_i}, \quad k \geq 1.$

When process $X(t)$ satisfies conditions (100), (101), then the following equality is derived from (104)

$$q(n) = \frac{1/\rho - 1}{C} \rho^n n^{a/\lambda} (1 + o(1)), \tag{105}$$

where

$$C = \lim_{n \to \infty} n^{a/\lambda} \prod_{i=1}^{n} \left(1 - \frac{1}{1 + i\lambda/a}\right)$$

(see [34]).

Hence it follows that

$$R(n) = -\ln q(n) = n \ln \frac{1}{\rho} - \frac{a}{\lambda} \ln n - \ln \frac{1/\rho - 1}{C} + o(1) \tag{106}$$

and

$$r(n) = \ln \frac{1}{\rho} + o(1). \tag{107}$$

It is clear that the conditions of Theorem 23 and item (i) of Corollary 2 are satisfied for the process $X(t)$ (of course, condition (70) equires additional non-trivial checks, but using Remark 2, we miss it).

It remains to find asymptotic formulas for the functions $a_m(t)$ and $d(t)$ from Theorem 23.

Let's start with the function $a_m(t)$. After we construct the function $R(x)$ is absolutely continuous and increasing. Therefore

$$R(a_m(t)) = R(R^{-1}(L_m(t))) = L_m(t).$$

Therefore

$$L_m(t) \geq \ln \frac{1}{\rho} (a_m(t) - \theta') - \frac{a}{\lambda} \ln(a_m(t) - \theta') - \ln \frac{1/\rho - 1}{C} + o(1)$$

and

$$L_m(t) \leq \ln \frac{1}{\rho} (a_m(t) + \theta'') - \frac{a}{\lambda} \ln(a_m(t) + \theta'') - \ln \frac{1/\rho - 1}{C} + o(1),$$

where $\theta' = a_m(t) - [a_m(t)], \quad \theta'' = \lceil a_m(t) \rceil - a_m(t)$.

Since the function $z \ln \frac{1}{\rho} - \frac{a}{\lambda} \ln z$ increases monotonically and continuously for sufficiently large z, there exists $\theta = \theta_t$, $|\theta| \le 1$ such that

$$L_m(t) = \ln \frac{1}{\rho} (a_m(t) + \theta) - \frac{a}{\lambda} \ln(a_m(t) + \theta) - \ln \frac{1/\rho - 1}{C} + o(1). \tag{108}$$

After taking the logarithm of the last equality, we get

$$L_{m+1}(t) = \ln a_m(t) + L_2(\frac{1}{\rho}) + o(1).$$

From here we find $\ln a_m(t)$ and substitute it into equality (108). We get

$$a_m(t) = (\ln \frac{1}{\rho})^{-1} (L_m(t) + \frac{a}{\lambda}(L_{m+1}(t) - L_2(\frac{1}{\rho})) + \ln \frac{1/\rho - 1}{C}) + \theta + o(1). \tag{109}$$

To find the asymptotics of the function $d(t)$, we start from the equality

$$L_1(t) - L_3(t) = \ln \frac{1}{\rho} (d(t) + \theta) - \frac{a}{\lambda} \ln(d(t) + \theta) - \ln \frac{1/\rho - 1}{C} + o(1),$$

where $|\theta| \le 1$.

Next, we repeat the considerations close to those given above. We get

$$d(t) = (\ln \frac{1}{\rho})^{-1} (L_1(t) - L_3(t) + \frac{a}{\lambda}(L_2(t) - L_2(\frac{1}{\rho})) + \ln \frac{1/\rho - 1}{C} + \theta \ln \frac{1}{\rho}) + o(1). \tag{110}$$

Thus, according to Theorem 23, according to equality (94) of Remark 2 and (109), (110), we have

Corollary 3. *Let $X(t)$ be a birth and death process with parameters $\lambda_n = \lambda n + a$, $\mu_n = \mu n$, $\lambda > 0$, $\mu > 0$, $a > 0$, $n = 0, 1, 2, \dots$ and condition (101) is fulfilled.*

Then

$$\mathbf{P}\left(\limsup_{t\to\infty} \frac{\bar{X}(t) \ln \frac{1}{\rho} - L_m(t)}{L_{m+1}(t)} = 1 + \frac{a}{\lambda} \right) = 1, \tag{111}$$

$$\mathbf{P}\left(\liminf_{t\to\infty} (\bar{X}(t) \ln \frac{1}{\rho} - L^*(\frac{t}{\alpha_T})) \in [-2\ln(\frac{1}{\rho}), \ln(\frac{1}{\rho})] \right) = 1, \tag{112}$$

where

$$L^*(t) = L_1(t) - L_3(t) + \frac{a}{\lambda}(L_2(t) - L_2(\frac{1}{\rho})) + \ln \frac{1/\rho - 1}{C},$$

C is given by the equality after formula (105), $\alpha_T = 1/(ap_0)$, p_0 is defined in equality (99).

Example 3.2.2. Let $X(t)$ be the birth and death process with parameters

$$\lambda_n = \lambda, \quad \mu_n = \mu n^{\beta}, \quad \lambda > 0, \mu > 0, \beta > 0, \quad n = 0, 1, 2, \dots. \tag{113}$$

It is natural to call such a process a process with a power-law increase in mortality and immigration (see the previous example 2.1). If state n describes the size of the population at some point in time, then the probability of transition to state $n+1$ in a small period of time δ is equal to $\lambda\delta+o(\delta)$, , and the transition probability $n \to n-1$ is given by $\mu n^{\beta}\delta+o(\delta)$. Coefficient λ is interpreted as the infinitesimal rate of population growth due to immigration.

Process $X(t)$ with a gradual increase in mortality and immigration was studied in the article by I. Matsak [36]. Among other things, this model is of interest, for example, when studying the queue length in the Queuing system $M/M/\infty$.

Although at first glance this example is close to the previous one, the asymptotic behavior of the corresponding processes differs significantly.

Let

$$\rho = \frac{\lambda}{\mu}, \quad \theta_0 = 1, \quad \theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \frac{\rho^k}{(k!)^{\beta}}, \quad k \geq 1.$$

It is easy to check that relations (96),(97) hold. Therefore, there exists stationary probabilities of states, and

$$p_k = \theta_k p_0, \quad p_0 = \left(\sum_{k=0}^{\infty} \frac{\rho^k}{(k!)^{\beta}} \right)^{-1}. \tag{114}$$

To formulate the corresponding result, we introduce the following notation:

$$a(t) = \max(k \geq 0 : R(k) \leq \ln t), \quad t > 1, \quad \alpha_T = 1/(\lambda p_0), \tag{115}$$

where

$$R(n) = \ln \left(\sum_{k=0}^{n-1} \frac{(k!)^{\beta}}{\rho^k} \right).$$

Theorem 25. *([36]) Let $X(t)$ be a birth and death process with parameters that are given by equalities (113). Then*

$$\mathbf{P}(\exists t_0, \forall t \geq t_0 \quad \bar{X}(t) \in J_t = \{a(\frac{t}{\alpha_T}) + m, \quad m \in I_{\beta}^{+}\}) = 1, \tag{116}$$

and $\forall m \in I_{\beta}$

$$\mathbf{P}(\bar{X}(t) \in J_{t,m} = \{a(\frac{t}{\alpha_T}) + l, \quad l = m-1, m, m+1\} \quad i.o.) = 1 \tag{117}$$

where $I_{\beta}^{+} = I_{\beta} \bigcup \{-2, [2+1/\beta]\}$, $I_{\beta} = \{-1, 0, 1, \ldots, [1+1/\beta]\}$, $a(t)$ *is given by equality (115) and satisfies the asymptotic relation*

$$a(t) = \frac{\ln t}{\beta L_2(t)} \left(1 + \frac{L_3(t) + 1 + \frac{\ln \rho}{\beta} + \ln \beta + o(1)}{L_2(t)} \right), \quad t \to \infty. \tag{118}$$

Remark 3. *When formulating the results, in order to simplify the notation, we imposed the condition $X(0) = 0$ a.s. on the process $X(t)$. From general considerations, it is clear that the obtained asymptotic relations remain valid even in the case when the process $X(t)$ starts from any point $i \geq 0$.*

3.3 Processes in queuing systems

Example 2.3. $M/M/\kappa$ Queuing system ($1 \le \kappa < \infty$).

Let us now consider a queuing system with κ servers and customers which arrive according to the Poisson process with intensity λ, and service times being independent copies of a random variable η with an exponential distribution

$$\mathbf{P}(\eta \le x) = 1 - \exp(-\mu x), \quad x \ge 0.$$

In the standard notation, this queuing system has the type $M/M/\kappa$, (see [20], [29]).

We impose the following assumption on the parameters λ and μ ensuring existence of the stationary regime:

$$\rho := \frac{\lambda}{\kappa\mu} < 1. \tag{119}$$

Let us assume that in the QS the customers arriving at $0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots$. Let $0 = W_0, W_1, W_2, \ldots, W_i, \ldots$ be the actual waiting times of the customers. Thus, at time $t = 0$ a first customer arrives and the service starts immediately.

Let $W(t)$ be the waiting time of the last customer in the queue at time $t \ge 0$, that is,

$$W(t) = W_{\nu(t)}, \quad \text{where} \quad \nu(t) = \max(k \ge 0 : t_k \le t),$$

and

$$W(t_n) = W(t_n+) = W_n.$$

Set

$$\bar{W}(t) = \sup_{0 \le s \le t} W(s) = \max_{0 \le t_k \le t} W_k,$$

then

$$\bar{W}_n = \max_{1 \le i \le n} W_i = \bar{W}(t_n).$$

For $t \ge 0$, let $Q(t)$ denote the length of the queue at time t, that is, the total number of customers in service or pending. Set

$$\bar{Q}(t) = \sup_{0 \le s < t} Q(s), \quad t \ge 0.$$

In this subsection, we are interested in the asymptotic behavior of a.s. processes $\bar{Q}(t)$, $\bar{W}(t)$, $\bar{W}_n$.

It is clear that such studies usually use results on the asymptotic behavior of regenerative processes and birth and death processes.

But sometimes the corresponding results are obtained by direct miscalculations. For example, in this way in paper [11](B.Dovhay et al) it was found that $\bar{Q}(t)$ satisfies a law of the iterated logarithm for the $\limsup$ and a law of the triple logarithm for the $\liminf$.

Theorem 26. *([11]) Assume that for a queuing system $M/M/\kappa$, $1 \le \kappa < \infty$, the condition* (119) *is fulfilled. Then*

$$\limsup_{t\to\infty} \frac{\bar{Q}(t)\log\frac{1}{\rho} - \log t}{L_2(t)} = 1 \quad a.s., \tag{120}$$

and

$$\liminf_{t\to\infty} \frac{\bar{Q}(t)\log\frac{1}{\rho} - \log t}{L_3(t)} = -1, \quad a.s. \tag{121}$$

The next step in the study of the asymptotics of the process $\bar{Q}(t)$ was made in the work [2](K.Akbash et al).

Let

$$\gamma = \ln\frac{1}{\rho}, \quad C_1 = \ln\frac{\rho\,\kappa!}{\kappa^{\kappa}(1-\rho)}.$$

Theorem 27. *([2]) Under the conditions of Theorem 26 the following relations hold:* $\forall m \ge 1$

$$\limsup_{t\to\infty} \frac{\gamma\bar{Q}(t) - \alpha_m(t)}{L_{m+1}(t)} = 1 \quad a.s., \tag{122}$$

and

$$\liminf_{t\to\infty} \left(\bar{Q}(t) - \frac{1}{\gamma}(L_1(t) - L_3(t))\right) = \chi \quad a.s., \tag{123}$$

where $\chi \in [-1-(C_1+\ln a_T)/\gamma, -(C_1+\ln a_T)/\gamma]$.

Next, we turn to the problem of the asymptotic behavior a.s. of processes $\bar{W}(t)$, and $\bar{W}_n$ in system $M/M/\kappa$.

A law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf for $\bar{W}(t)$ and $\bar{W}_n$ was obtained in the work I.Matsak [38].

Theorem 28. *([38]) Let the QS $M/M/\kappa$, $1 \le \kappa < \infty$, satisfy condition (119). Then*

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{m\mu(1-\rho)\bar{W}_n - \ln n}{L_2(n)} = \limsup_{t\to\infty} \frac{m\mu(1-\rho)\bar{W}(t) - \ln t}{L_2(t)} = 1\right) = 1, \tag{124}$$

$$\mathbf{P}\left(\liminf_{n\to\infty} \frac{m\mu(1-\rho)\bar{W}_n - \ln n}{L_3(n)} = \liminf_{t\to\infty} \frac{m\mu(1-\rho)\bar{W}(t) - \ln t}{L_3(t)} = -1\right) = 1. \tag{125}$$

Note that in the case of QS $GI/G/1$, processes $\bar{W}(t)$ and $\bar{W}_n$ also satisfy a law of the iterated logarithm for the lim sup and a law of the triple logarithm for the lim inf . In this case, it is necessary to impose sufficiently strong conditions on the input flow and service time (see article A.Marynych et al [34]).

In the next corollary, Theorem 28 will be slightly strengthened. Here we will rely on Theorem 22 and the results of article [38].

Corollary 4. *Let the QS $M/M/\kappa$, $1 \leq \kappa < \infty$, satisfy the condition (119). Then for an arbitrary integer $m \geq 1$*

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{\kappa\mu(1-\rho)\overline{W}_n - L_m(n)}{L_{m+1}(n)} = \limsup_{t\to\infty}\frac{\kappa\mu(1-\rho)\overline{W}(t) - L_m(t)}{L_{m+1}(t)} = 1\right) = 1, \quad (126)$$

$$\mathbf{P}(\liminf_{n\to\infty}(\kappa\mu(1-\rho)\overline{W}(n) - L_1(n) + L_3(n)) = -\ln(\lambda\alpha_T) - \rho(1-\rho)) = 1, \qquad (127)$$

$$\mathbf{P}(\liminf_{t\to\infty}(\kappa\mu(1-\rho)\overline{W}(t) - L_1(t) + L_3(t)) = -\ln\alpha_T - \rho(1-\rho)) = 1, \qquad (128)$$

where $L_m(t)$, are defined in Section 1 before Theorem 13.

Proof of Corollary 4. For the $M/M/\kappa$ QS, we introduce regenerative moments (S_k) for the process $Q(t)$ and $W(t)$ as follows. Let

$$S_1 = \inf(t > 0 : Q(t) = \kappa / Q(0+) = 1), \quad S_1^{'} = \inf(t > S_1 : Q(t) < \kappa),$$

$$T^z = S_1^{'} - S_1, \quad \overline{W}^{(z)} = \max_{S_1 \leq t_k < S_1^{'}} W_k,$$

thus S_1 is the first moment in time when all service channels are busy, $S_1^{'}$ is the first after S_1 moment of release of one of the channels, $\overline{W}^{(z)}$ is the maximum waiting time for customers that arrived during the period when all channels are busy.

Let $[S_1, S_1^{'}), [S_2, S_2^{'}), \ldots, [S_i, S_i^{'}), \ldots$ be the successive periods on which all channels are busy. Since the service time η has an exponential distribution, the points $S_1, S_2, \ldots, S_k, \ldots$ create moments of regeneration of the processes $Q(t)$ and $W(t)$, $T_i = S_{i+1} - S_i, i \geq 1$.

In what follows, we will use the following equalities established in article [38]:

$$\alpha_T = \mathbf{E}T_i = \frac{(\kappa-1)!}{\mu(\kappa\rho)^\kappa p_0}, \qquad (129)$$

$$p_0 = \left(\sum_{i=0}^{\kappa}\frac{(\kappa\rho)^i}{i!} + \frac{\rho^\kappa\kappa^\kappa}{\kappa!(\frac{1}{\rho}-1)}\right)^{-1},$$

$$F_W(x) = \mathbf{P}(\overline{W}^{(z)} \leq x) = \frac{1-\rho\exp(-\kappa\mu(1-\rho)x)}{1-\rho^2\exp(-\kappa\mu(1-\rho)x)}, \quad x \geq 0, \qquad (130)$$

where $F_W(x)$ is actually a distribution function of the maximum waiting time on one regeneration cycle.

From here we get

$$\begin{aligned} R_W(x) &= -\ln(1-F_W(x)) \\ &= -(\rho(1-\rho) - \kappa\mu(1-\rho)x - \ln(1-\rho^2\exp(-\kappa\mu(1-\rho)x))) \\ &= -\rho(1-\rho) + \kappa\mu(1-\rho)x + o(1) \quad \text{for} \quad x\to\infty, \end{aligned} \qquad (131)$$

and accordingly

$$r_W(x) = (R_W(x))^{'} = \kappa\mu(1-\rho) + \frac{\rho^2\kappa\mu(1-\rho)\exp(-\kappa\mu(1-\rho)x)}{1-\rho^2\exp(-\kappa\mu(1-\rho)x)},$$

$$r_W(x) \to \kappa\mu(1-\rho) \quad \text{for} \quad x \to \infty, \tag{132}$$

thus function $r_W(x)$ is regularly varying at the infinity.

Thus, the conditions of Theorem 22 are satisfied, except for condition (70). But here we again circumvent this condition by using Remark 2.

Under our conditions, at $t \to \infty$

$$L_m(t) = R_W(a_m(t)) = -\rho(1-\rho) + \kappa\mu(1-\rho)a_m(t) + o(1)$$

or

$$a_m(t) = \frac{L_m(t) + \rho(1-\rho) + o(1)}{\kappa\mu(1-\rho)}. \tag{133}$$

Therefore, the second equality in (126) immediately follows from Theorem 22 and relations (132), (133).

To obtain the first equality in (126), one must also take into account the following asymptotic relation: at $n \to \infty$

$$\frac{N(t_n)}{n} = \frac{N(t_n)}{t_n} \cdot \frac{t_n}{n} \to \frac{1}{\alpha_T} \cdot \frac{1}{\lambda} \quad a.s. \tag{134}$$

and inequality

$$Z_{N(t_n)} \le \overline{W}_n \le Z_{N(t_n)+1} \quad a.s. \tag{135}$$

Further, just as in (133) we establish that

$$d(t) = \frac{L_1(t) - L_3(t) + \rho(1-\rho) + o(1)}{\kappa\mu(1-\rho)}.$$

Hence, based on equality (93) rom Remark 2, we obtain (128).

If we add relations (134) and (135)to the above considerations, then we will have the equality (127).

□

Example 2.4. $M/M/\infty$ queuing system.

Consider a QS with an unlimited number of service channels, which receives a Poisson flow of customers with intensity λ, and service time η has an exponential distribution with the parameter μ.

By the queue length here we mean the total number of customers that are being serviced. The designations $Q(t)$ and $\overline{Q}(t)$ are similar to those introduced in the previous example.

For $M/M/\infty$ QS for arbitrary $\lambda > 0$, $\mu > 0$, $Q(t)$ will be a birth and death process with parameters $\lambda_n = \lambda$, $\mu_n = \mu n$, $n \ge 0$, so it satisfies conditions (113) from example 2.2 for $\beta = 1$.

In addition, for $M/M/\infty$ QS there exists stationary probabilities (p_k) and the following formulas are known :

$$p_0 = \exp(-\rho), \quad \rho = \frac{\lambda}{\mu},$$

$$ET_k = \alpha_T = \frac{1}{\lambda p_0} = \frac{\exp(\rho)}{\lambda}, \tag{136}$$

(see [20], [29], [45]).

From here and Theorem 25 we have

Corollary 5. *Let $Q(t)$ be the queue length at time t in $M/M/\infty$ QS. Then*

$$\mathbf{P}(\exists t_0, \forall t \geq t_0 \quad \bar{Q}(t) \in \{a(\frac{t}{\alpha_T}) + m, \quad m \in (-2,-1,0,1,2,3)\}) = 1, \tag{137}$$

and $\forall m \in (-1,0,1,2)$

$$\mathbf{P}(\bar{Q}(t) \in \{a(\frac{t}{\alpha_T}) + l, \quad l = m-1, m, m+1\} \quad i.o.\ ) = 1 \tag{138}$$

where α_T is defined in formula (136), $a(t)$ is given by equality(115) and satisfies the asymptotic relatio

$$a(t) = \frac{\ln t}{L_2(t)}(1 + \frac{L_3(t) + \ln\rho + 1 + o(1)}{L_2(t)}), \quad t \to \infty$$

.

4 Conclusion

We can say that there is already a fairly wide range of results about the asymptotic behavior of almost sure extremal values of independent random variables. For the continuous case, the law of the iterated logarithm for the $\limsup$ and a law of the triple logarithm for the $\liminf$ are established, which, together with their generalizations, basically ends the theory, which began in the 60-70s with the work of J.Pickands, L.de Haan and others.

The class of discrete random variables has not been studied in full. For example, for such an important distribution as the geometric one, at the moment we do not know the asymptotics for $\liminf(z_n - d_n)$. But even here a more or less complete theory has been constructed, which gives answers to the main questions about the asymptotic behavior of the extrema of discrete random variables.

A number of interesting applications of the results on the asymptotics of extrema of random variables to regenerative processes, birth and death processes, as well as processes that describe processes in queuing systems are obtained.

References

[1] K.S.Akbash, I.K.Matsak, *One improvement of the law of the iterated logarithm for the maximum scheme*, Ukrainian Mathematical Journal, vol. 64, no. 8, pp. 1290–1296, 2013.

[2] K.Akbash, N.Doronina, I.Matsak, *Asymptotic behavior of maxima of independent random variables. Discrete case*, Lithuanian Mathematical Journal, vol. 61, no. 2, pp. 145–160, 2021.

[3] C.W. Anderson, *Extreme value theory for a class of discrete distribution with application to some stochastic processes*, Journal of Applied Probability, vol. 7, pp. 99–113, 1970.

[4] S. Asmussen, *Extreme values theory for queues via cycle maxima*, Extremes, vol. 1, pp. 137–168, 1998.

[5] O.Barndorff-Nielsen, *On the limit behaviour of extreme order statistics* , Annals of Mathematical Statistics, vol. 34, no. 3, pp. 992-1002, 1963.

[6] N.Bernoulli, *Dissertatio Inauguralis Mathematico-Juridica de Usu Artis Conjectandi in Jure*, Juris doctor Universitat Basel, 56 p., 1709. Available at: https://gallica.bnf.fr/ark:/12148/bpt6k1303890

[7] V.V.Buldygin, O.I.Klesov, J.G.Steinebach, *Pseudo Regularly Varying Functions and Generalized Renewal Processes*, Theory of Probability and Mathematical Statistics, vol. 87, pp. 1–441, 2012.

[8] V.V. Buldygin, O.I. Klesov, J.G. Steinebach, *On some properties of asymptotic quasi-inverse functions and their applications*, Theory of Probability and Mathematical Statistics, vol. 70, pp. 9–25, 2004.

[9] J.W. Cohen, *Extreme values distribution for the M/G/1 and GI/M/1queueing systems*, Annales de l'Institut Henri Poincare (B) Probability and Statistics, vol. 4, pp. 83–98, 1968.

[10] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, New York : Cambridge University Press, 491 p., 1987.

[11] B.V. Dovgay, I.K. Matsak, *The asymptotic behavior of extreme values of queue lengths in (M/M/m) systems*, Cybernetics and systems analysis, vol. 55, no. 2. pp. 171–179, 2019.

[12] L.Euler, *Recherches generales sur la mortalite et la multiplication du genre humain*, Mem. De l'Acad. d. Se. de Berlin, vol. 16, p. 144-164, 1760.

[13] W. Feller, *An introduction to probability theory and its applications*, Vol.II. Wiley Series in Probability and Mathematical Statistics. New York, London, Sydney, J. Wiley & Sons, Inc., 626 p., 1968.

[14] R.A.Fisher, L.H.C.Tippett, *Limiting forms of the frequency distribution of the largest or smallest member of a sample*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 24, no. 2, pp. 180-190, 1928.

[15] M.Frechet, *Sur la loi de probabilite de lecart maximum*, Annales de la SociÃ©tÃ© Polonaise de MathÃ©matique, vol. 6, pp. 93-116, 1927.

[16] J.Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 352 p., 1978.

[17] P. Glasserman, S.G. Kou, *Limits of first passage times to rare sets in regenerative processes*, Annals of Applied Probability, vol. 5, pp. 424–445, 1995.

[18] B.V. Gnedenko, *On the theory of domains of attraction of stable laws*, Uchenye zapiski MGU, vol. 30, pp. 61–81, 1939.

[19] B.V. Gnedenko, *Sur la distribution limit du terme maximum d'une serie aleatoire*, Annals of Mathematics, vol. 44, no. 3, pp. 423–453, 1943.

[20] B.V. Gnedenko, I.N. Kovalenko, *Introduction to Queueing Theory*, Birkhauser, Boston, 315 p., 1989. doi:10.1007/978-1-4615-9826-8

[21] B. V. Gnedenko, Yu. K. Belyayev, A. D. Solovyev, *Mathematical Methods of Reliability Theory*, Academic Press, 471 p., 1969.

[22] B.Gompertz, *On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies*, Philosophical Transactions of the Royal Society of London, vol. 115, pp. 513–585, 1825. doi:10.1098/rstl.1825.0026.

[23] J.Graunt, *Natural and Political Observations Made Upon the Bills of Mortality*, Journal of Actuaries, vol. 90, 1662. Available at: http://www.edstephan.org/Graunt/bills.html

[24] A. Gut, *Stopped Random Walks*, Springer, New York, 263 p., 2009. doi:10.1007/978-0-387-87835-5

[25] L.de Haan, A.Ferreira, *Extreme Values Theory: An Introduction*, Berlin: Springer, 417 p., 2006.

[26] L.de Haan, *The rate of growth of sample maxima*, Annals of Mathematical Statistics, vol. 43, pp. 1185-1196, 1972.

[27] C.Huygens *The Correspondence of Huygens Concerning the Bills of Mortality of John Graunt*, Extracted from Volume V of the Oeuvres Completes of Christiaan Huygens, 1669.

[28] D.L. Iglehart, *Extreme values in the GI/G/1 gueue*, Annals of Mathematical Statistics, vol. 43, pp. 627–635., 1972.

[29] S. Karlin, *A first course in stochastic processes*, Academic Press, New York, 1968.

[30] S. Karlin, J. McGregor, *The clasification of birth and death processes*, Transactions of the American Mathematical Society, vol. 86, pp. 366–400, 1957.

[31] M.J. Klass, *The minimal growth rate of partial maxima*, Annals of Probability, vol. 12, pp. 380–389, 1984.

[32] M.J. Klass, *The Robbins-Siegmund criterion for partial maxima*, Annals of Probability, vol. 13, pp. 1369–1370, 1985.

[33] M.R.Leadbetter, G.Lindgren and H.Rootzen, *Extremes and related properties of random sequences and processes*, Springer, New York, 336 p., 1983. doi:10.1007/978-1-4612-5449-2

[34] A.Marynych, I.Matsak, *The laws of iterated and triple logarithms for extreme values of regenerative processes*, Modern Stochastics: Theory and Applications, vol. 7, pp. 61–78, 2020.

[35] I.K.Matsak, *Asymptotic behaviour of random variables extreme values. Discrete case*, Ukrainian Mathematical Journal, vol. 68, pp. 945–956, 2016.

[36] I.K.Matsak, *Limit theorem for extreme values of discrete random variables and its application*, Theory of Probability and Mathematical Statistics, vol. 101, pp. 189–202, 2019.

[37] I.K.Matsak, *Asymptotic behavior of maxima of independent randomvariables*, Lithuanian Mathematical Journal, vol. 59, pp. 185–197, 2019.

[38] I.K.Matsak, *On the of extreme values of the M/M/m queueing systems*, Georgian Mathematical Journal, vol. 28, pp. 917–924, 2021.

[39] J.Pickands, *Sample sequences of maxima*, Annals of Mathematical Statistics, vol. 38, no. 5, pp. 1570-1574, 1967.

[40] J.Pickands, *An iterated logarithm law for the maximum in a stationary Gaussian sequence*, Zeitschrift fur Warhschein. verw. Geb., vol. 12, no. 3, pp. 344-355, 1969.

[41] S.I. Resnick *Extreme Values, Regular Variation and Point Processes*, Berlin: Springer, 320 p., 1987.

[42] J. Sadowsky, W. Szpankowski, *Maximum queue length and waiting time revisied: GI/G/c queue*, Probability in the Engineering and Informational Sciences, vol. 6, pp. 157–170, 1995.

[43] R.F. Serfozo, *Extreme values of birdh and death processes and queues*, Stochastic processess and their applications, vol. 27, pp. 291–306, 1988.

[44] W.L. Smith, *Renewal theory and its ramifications*, Journal of the Royal Statistical Society, vol. 20, no. 2, pp. 243–302, 1958.

[45] O.K. Zakusylo, I.K. Matsak, *On extreme values of some regenerative processes*, Theory of Probability and Mathematical Statistics, vol. 97, pp. 58–71, 2017.