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On a class of unitary representations of the braid groups B_3 and B_4

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We describe a class of irreducible non-equivalent unitary representations of the braid group B_3 in every dimension $n \geq 6$ which depends continuously on $n^2/6 + 1$ real parameters. We show that the upper bound on the number of the parameters of which the class of irreducible non-equivalent unitary representations of B_3 depends smoothly is equal to $n^2/4 + 2$. The proof is achieved by a construction of such a class. We also prove that the tensor product of the Burau unitarisable representation of B_4 and the irreducible unitary representation of B_4 that coincide on commuting standard generators always forms *irreducible* unitary representations for the braid group B_4. This gives a new class of unitary representations for the braid group B_4 in $3n$ dimensions.

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1. Introduction

In this paper we shall work with Artin's braid groups B_k, $k \in \mathbb{N}$. B_k has a standard presentation in generators and relations which first appeared in [4]:

$$B_k = \left\langle \sigma_1, \ldots, \sigma_{k-1} \,\middle|\, \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, \ldots, k-2, \\ \sigma_i\sigma_j = \sigma_i\sigma_j, |i-j| > 1 \end{array} \right\rangle$$

There are a lot of representations of B_k in various groups (see, for example, [6]). Our main goal here is to estimate the dimensions of the varieties of non-equivalent unitary matrix representations of B_3 and B_4. The variety of representations of B_3 without involution is directly connected with the known theory of modular groups and quiver representations that allowed to find the dimensions of irreducible components of the variety of irreducible representations of B_3 [2,27]. For unitary representations there does not exist such a nice theory and unitary equivalence complicates the problem of partial classification of unitary representations of B_3. Among the helpful results on the theme let us mention the finding of all representations of B_3 in small dimensions [7,15,26], and the study of representations with a small number of points in the spectrum of the matrix corresponding to σ_1 (see, for example, [11,19,23]) or with a small quotient between the dimension of the representation and n [14]. Our main stimulus in the study was the useful connections between the representations of the braid groups and representations of other objects such as quantum groups and Kac–Moody algebras as well as with R-matrices and solutions of quantum Yang–Baxter equations, see e.g., [5,9]. Let us also mention developments relating representations of braid groups with quantum computations, see e.g. [8,24,28].

Let us now describe the content of our paper. We first observe that the braid group B_k can be generated by two elements $J = \sigma_1\sigma_2 \ldots \sigma_{k-1}$ and $S := \sigma_1 J$. One of the relations between them is $S^{k-1} = J^k$ (for B_3 it is the only one). It follows directly from [13] that B_3 is a $*$-wild group, that is the classification of all its representations up to unitary equivalence is rather complicated. In the present paper, we concentrate our study on the estimation of the dimensions of quotients (by unitary conjugation) of the representation varieties of all irreducible unitary representations of B_3 and B_4, provided the dimension of the representations is a fixed number $n \in \mathbb{N}$. We shall prove in section 2 that the upper bound is $n^2/4 + 2$ for the group B_3.

Also we present in section 3 a class of non-equivalent irreducible *unitary* representations of B_3 by $n \times n$ matrices for every $n \geq 6$. It depends continuously on a number of real parameters for which we give a lower bound close to $n^2/6 + 1$.

Note that B_3 is a factor group of B_4 by the relation $\sigma_3 = \sigma_1$. So every representation of B_3 is also a representation of B_4. However it is interesting to find many irreducible representations π_α of B_4 such that $\pi_\alpha(\sigma_3) \neq \pi_\alpha(\sigma_1)$. We shall show that the tensor product of every irreducible representation π of B_3 with $\pi(\sigma_3) = \pi(\sigma_1)$ and a specialization of the reduced Burau unitary representation of B_4 form an irreducible representation of B_4. This will give us a representation family that depends continuously on $n^2/54$ real parameters. Note that the tensor product of two specializations of the Burau representation is, in general, irreducible for different specializations [1]. The present paper is related to results in a previous unpublished preprint by the authors [3]. In that preprint we did not make any estimation of upper bounds on the dimension of non-equivalent irreducible representation varieties, but gave a simpler construction of a class of unitary non-equivalent representations of B_3 (and of B_4) which depends smoothly on approximately $n^2/9$ real parameters. So here we present a richer class of representations.

In what follows we shall use the notations I_n and 0_n for the $n \times n$ identity and zero matrices respectively, and the notation E_{ij} for the matrix unit with 1 in the $(i\ j)$ position and 0 in the other positions. A diagonal matrix with entries $a_1, a_2, \dots, a_m$ will be denoted by $\operatorname{diag}(a_1, a_2, \dots, a_m)$.

2. The number of non-equivalent irreducible unitary representations of B_3

Let us denote by $\mathrm{Rep}_n B_k$ the variety of all unitary n-dimensional representations of B_k. Since B_k is generated by a finite number of elements, the variety $\mathrm{Rep}_n B_k$ can be viewed as the group of continuous matrix-functions over some algebraic variety $\Omega^2_{n,k}$. For example, putting formally

$$\pi_z(J) = \begin{pmatrix} z_1 & \dots & z_n \\ \vdots & & \vdots \\ z_{n^2+1-n} & \dots & z_{n^2} \end{pmatrix}, \ \pi_z(S) := \begin{pmatrix} z_{n^2+1} & \dots & z_{n^2+n} \\ \vdots & & \vdots \\ z_{2n^2+1-n} & \dots & z_{2n^2} \end{pmatrix}, \tag{1}$$

where $z := (z_1, \dots, z_{2n^2}) \in \mathbb{C}^{2n^2}$, $\|z_i\| \leq 1$ for every i, and applying the corresponding braid relations to the matrices, we define some algebraic variety $\Omega^1_{n,k}$ over $\mathbb{C}$ with the property: π_z *is a representation of* B_k *if and only if* $(z_1, \dots, z_{2n^2}) \in \Omega^1_{n,k}$. Then we write every z_j as a linear combination of two real variables: $z_j = x_j + \mathbf{i}y_j$, $x_j, y_j \in \mathbb{R}$ and verify the equations $\pi(J)_{xy}\pi_{xy}(J)^* = \pi_{xy}(S)\pi_{xy}(S)^* = I_n$. This gives us some algebraic variety $\Omega^2_{n,k}$ over $\mathbb{R}$ which is in one to one correspondence with $\mathrm{Rep}_n B_k$. The set of all irreducible unitary representations of B_k can be defined by the continuous matrix-functions (1), where $(\vec{x}, \vec{y}) = (x_1, \dots, x_{2n^2}, y_1, \dots, y_{2n^2})$ belongs to some open subset $\Omega^3_{n,k} \subset \Omega^2_{n,k}$.

We denote by $\mathrm{Irrep}_n\ B_k$ the set of non-equivalent irreducible unitary representations of B_k in $M_n(\mathbb{C})$. Due to invariant theory this set can also be viewed as a group of continuous matrix-valued functions over some algebraic variety. In fact, the set $\Omega^3_{n,k}$ is stable under the action of $U(n)$. Also $U(n)$ is reductive, whence the quotient set of orbits $\Omega^o_{n,k} := \Omega^3_{n,k}/\sim$ of equivalent representations is an affine variety [17,20].

Let $d(n,k)$ be the *dimension* of $\Omega^o_{n,k}$. In this section we shall find an upper bound for $d(n,3)$. The group B_k is generated by two elements J and S. If $\pi_t \in \mathrm{Irrep}_n\ B_k$, then the entries of $\pi_t(J)$ and $\pi_t(S)$ are defined by $2n^2$ real-valued functions. Therefore $d(n,k) \leq 2n^2$. The following Theorem 4 provides a more strict bound for $d(n,3)$. To prove it we need two Lemmas about local reducing of a continuous matrix-valued function to the triangular form. These lemmas have a close connection with the perturbation theory and we provide the proofs for the sake of completeness. More general results on global holomorphic reducing in this sense can be found in [25].

Lemma 1. *Let $X(t)$, $t \in \mathcal{B}$ be a continuous $n \times n$ matrix-valued function over a unit ball $\mathcal{B} \subset \mathbb{R}^m$. Then there exist an open set $\Omega \subset \mathcal{B}$ and a unitary continuous matrix-valued function $U(t)$, $t \in \Omega$ such that $U^*XU(t)$ is upper triangular. Moreover, if $X(t)$ is normal, then $U^*XU(t)$ is diagonal for $t \in \Omega$.*

Proof. We use induction on n. The case $n = 1$ is obvious. Suppose Lemma 1 is true for all $n-1 \times n-1$ matrix-valued functions.

Let us denote by $N(t)$ the vector space that is isomorphic to $\mathbb{C}^n$ for every $t \in \mathcal{B}$. The action of the matrix $X(t) : N(t) \to N(t)$ in a standard basis defines an operator which we shall also denote by $X(t)$. The number $l(t)$ of different eigenvalues of $X^*(t)$ for every fixed t is less or equal to n. Hence for some $t_0 \in \mathcal{B}$, the value $l(t_0)$ is maximal. The eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ of $X^*(t)$ are continuous functions of t, for t belonging to a small neighborhood Ω_0 of t_0 [10]. Without loss of generality, we may assume that $X(T)$ is nonsingular for $t \in \Omega_0$. Then we can define the polynomial function with continuous coefficients

$$P(z)(t) = (z - \lambda_2(t))(z - \lambda_3(t)) \ldots (z - \lambda_{l(t_0)}(t)) / \prod_{2}^{l(t_0)} (\lambda_1(t) - \lambda_i(t)).$$

If $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$ form an orthonormal basis of $N(t)$, then there exist $k < n$, $t_1 \in \Omega_0$ and $j \in \{1, \ldots, n\}$ such that

$$(X^*(t) - \lambda_1(t) I_n)^{k+1} (P(X^*(t))^n e_j = \vec{0}, \quad t \in \Omega_0$$

and the following vector $\mathbf{f}(t)$ as a function of t,

$$\mathbf{f}(t) = (X^*(t) - \lambda_1(t) I_n)^k (P(X^*(t))^n e_j,$$

is not zero at some point $t_1 \in \Omega_0$: $\mathbf{f}(t_1) \neq \vec{0}$. In virtue of the continuity there exists $\epsilon > 0$ such that $\epsilon < \|\mathbf{f}(t)\|$ for t from a small neighborhood Ω_1 of t_1. Let us denote by $N_1^*(t)$ the vector space of all vectors that are orthogonal to $\mathbf{f}(t)$, $t \in \Omega_1$. By construction, $X(t) : N_1^*(t) \to N_1^*(t)$. Whence concerning the sum

$$N_1^*(t) \oplus < \mathbf{f}(t) >,$$

the matrix of the operator $X(t)$ has an upper block triangular form. If in addition $X(t)$ is normal, then $X(t)\mathbf{f}(t) = \bar{\lambda}_1(t)\mathbf{f}(t)$ and hence the corresponding form is block diagonal. Using the induction assumption, we complete the proof. □

Corollary 2. *Let $X(t)$ be a continuous $m \times n$ matrix-valued function over a unit ball $\mathcal{B}$. Then there exist an open set $\Omega \subset \mathcal{B}$ and two unitary continuous matrix-valued functions $V(t) \in M_m(\mathbb{C})$ and $W(t) \in M_n(\mathbb{C})$ such that $V^*XW(t)$ is a diagonal matrix for every $t \in \Omega$.*

Proof. It runs in a way similar to the singular value decomposition for a complex matrix [10]. For example, if $m = n$ and $X(t)$ is invertible, then there exists a unitary matrix $V(t)$ such that $V^*XX^*V(t)$ is the diagonal matrix $\Lambda(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_m(t))$. In view of Lemma 1, we can assume that $V(t)$ is continuous on some open set Ω. So, putting $W(t) = X^*V\Lambda^{-1}(t)$, we obtain the statement. If $X(t)$ is singular, then the rank of $\Lambda(t)$ is a constant on some small open subset $\Omega_1 \subset \Omega$ and so there exists a projection onto the kernel of Λ, i.e. a diagonal matrix P, with $P\Lambda(t) = 0$ and $\det(P + \Lambda(t)) \neq 0$. Whence $W(t)$ will be the matrix $X^*V(P + \Lambda^{-1})(t)$. If now $m < n$, then adding zero rows, we make $X(t)$ to be a square matrix and then we can apply Corollary 2 to the square matrix-valued function choosing $V(t)$ in the block form $\operatorname{diag}(V_1(t), I_{n-m})$. The case $m > n$ is treated in the same way. □

Lemma 3. *Let $X(t)$ be a continuous matrix-valued function over a unit ball $\mathcal{B} \subset \mathbb{R}^m$. Then there exist an open set $\Omega \subset \mathcal{B}$ and a unitary continuous matrix-valued function $U(t)$, $t \in \Omega$ such that $XU(t)$ is lower triangular.*

Proof. Let $N(t)$ be as in the proof of Lemma 1. If $\vec{u}_1(t), \dots, \vec{u}_n(t)$ are the rows of $X(t)$, then by $r_m(t)$ we denote the rank of the system $< \vec{u}_1(t), \dots, \vec{u}_m(t) >$. Suppose for some $t_0 \in \mathcal{B}$, $s = r_n(t_0) \geq r_n(t)$, $t \in \mathcal{B}$. There exists a closed set $\tilde{\Omega} \subset \mathcal{B}$ in a small neighborhood of t_0 such that for every j, the function $r_j(t)$ is a constant on $\tilde{\Omega}$ and $r_n(t) = s$. If $l_1 < l_2 < \cdots < l_s$ are the indices for which $r_{l_i - 1} = r_{l_i} - 1$, $i = 1, \dots, s$ with $r_0 = 0$, then the vectors $\vec{u}_{l_1}(t), \dots, \vec{u}_{l_s}(t)$ are linearly independent. For a fixed $t = t_1 \in \tilde{\Omega}$ there exist vectors $\vec{h}_{s+1}, \dots, \vec{h}_n \in \mathbb{C}^n$ such that

$$\vec{u}_{l_1}(t), \dots, \vec{u}_{l_s}(t), \vec{h}_{s+1}, \dots, \vec{h}_n \tag{2}$$

is a base in $N(t)$. By continuity, these vectors are linearly independent for every t from some open set $\Omega \ni t_1$. We use Gram–Schmidt orthonormalization for the system (2) from $N(t)$. Since the process is continuous, we obtain a new orthonormal continuous basis for $N(t)$. The vectors of this basis will be the columns of U. □

Remark 1. The matrix $(XU)^*(t) = U^*X^*(t)$ is upper triangular. So we locally can obtain an upper triangular matrix-valued function by multiplying it with a unitary operator from the left.

Theorem 4. *For $n \geq 3$, the following inequality holds $d(n, 3) \leq n^2/4 + 2$.*

Proof. Suppose we have a unitary representation $\pi(t)$ of B_3 by continuous $n\times n$ matrix-valued functions which is irreducible for every t from an open set $\Omega\subset\mathbb{R}^d$. For every element $v\in Z(B_3)$, the matrix $\pi(v)(t)$ is unitary. So

$$\pi(S^2)(t)=\pi(J^3)(t)=u(t)I_n,\quad u(t)\bar{u}(t)=1,\ t\in\Omega.$$

To estimate $d(n,3)$ it suffices to consider the case $u(t)=1$. Note that every representation in this case will be a unitary representation of the unimodular group $PSL_2(\mathbb{Z})$ (see [22]).

Thus let $\pi(S^2)(t)=I_n$ and $d=d(n,3)-1$. We are going to find a closed set $\Omega_8\subset\mathbb{R}^d$ such that the matrix functions $\pi(S)(t)$ and $\pi(J)(t)$ can be reduced continuously in it to a special form. The entries of the obtained matrices will depend on a number of real-valued functions. The value of d will be bounded by this number. We shall suppose in the proof below that all sets Ω_i will be open or closed balls in $\mathbb{R}^d$.

By Lemma 1, there exist an open set $\Omega_1\subset\Omega$ and an orthogonal basis of continuous vectors $e_1(t),e_2(t),\dots,e_n(t)$ in which the matrix $\pi(J)(t)$ has a diagonal form, $\pi(J)(t)=\operatorname{diag}(I_{n_1},\beta I_{n_2},\beta^2 I_{n_3})$, where $t\in\Omega_1$ and $\beta=\sqrt[3]{1}$ is a primitive root. Note that the operator $\pi(S)(t)$ in the basis is a Hermitian unitary matrix. Hence

$$P:=\begin{cases}(\pi(S)+I_n)/2, & \text{if } \operatorname{tr}(S)<0,\\ (\pi(S)-I_n)/2, & \text{otherwise}\end{cases}$$

is an orthogonal projection. Since $\operatorname{tr}P(t)$ is continuous over the connected set Ω_1 and $\operatorname{tr}P(t)\in\mathbb{Z}$, we have that $\operatorname{tr}P(t)=k$ for some $k\in\mathbb{Z}$. By definition, $k\le n/2$.

Let us define the following three matrices.

$$J_1=\operatorname{diag}(I_{n_1},0_{n-n_1}),\quad J_2=\operatorname{diag}(0_{n_1},I_{n_2},0_{n_3}),\quad J_3=\operatorname{diag}(0_{n-n_3},I_{n_3}).$$

Due to symmetry we can suppose that $n_1\ge\max(n_2,n_3)$. For every $i\ge 1$, we have $\operatorname{rank}(PJ_iP)=\operatorname{tr}J_i$, otherwise, P and J_i have a common eigenvector and hence, π is reducible.

Let us consider the block matrix form of $P(t)$:

$$P(t)=\begin{pmatrix}P_{11}(t) & P_{12}(t) & P_{13}(t)\\ P_{21}(t) & P_{22}(t) & P_{23}(t)\\ P_{31}(t) & P_{32}(t) & P_{33}(t)\end{pmatrix}$$

where P_{ij} is an $n_i\times n_j$ matrix. The matrices P_{ii}, $i=1,2,3$ are Hermitian, hence by Lemma 1, there exist an open set $\Omega_2\subset\Omega_1$, unitary matrices U_1, U_2 and U_3 such that every matrix $U_i^*P_{ii}U_i(t)$ is diagonal for $t\in\Omega_2$, $i=1,2,3$.

Changing the order of the base vectors $e_1(t)$, $e_2(t),\dots,e_n(t)$ to

$$e_1(t),e_2(t),\dots,e_{n_1}(t),e_{k_1}(t),e_{k_1+1}(t)\dots,e_{k_{n-n_1}}(t),$$

one can assume that the first k rows of $P(t_1)$ in the new basis are linearly independent for some fixed $t_1 \in \Omega_1$. Whence for a sufficiently small open set $\Omega_2 \subset \Omega_1$, $\Omega_2 \ni t_1$, the first k rows of $P(t)$ in this basis are linearly independent for every t from the closure of Ω_2. Beside this, we can suppose that the operator $\pi(J)(t)$ will have the form

$$\operatorname{diag}(I_{m_1}, \beta I_{m_2}, \beta^2 I_{m_3}, \beta I_{m_4}, \beta^2 I_{m_5}),$$

where $m_1 = n_1$, $m_i + m_{i+2} = n_i$ for $i = 2, 3$ and

$$m_1 + m_2 + m_3 = k \tag{3}$$

and the operator $R(t) = \operatorname{diag}(U_1^*, U_2^*, U_3^*) P \operatorname{diag}(U_1, U_2, U_3)(t)$ will have the following block form

$$R = \begin{pmatrix} Q_{11} & \dots & Q_{15} \\ \vdots & & \vdots \\ Q_{51} & \dots & Q_{55} \end{pmatrix},$$

where Q_{ij} is an $m_i \times m_j$ matrix and Q_{ii} is a diagonal matrix.

Let $h_1(t), \dots, h_{m_2+m_3}(t)$ be the entries of the first rows of Q_{12} and Q_{13} and $h_{m_2+m_3+1}(t), \dots, h_{k-1}(t)$ be the entries of the last column of Q_{13}. Since every $h_i(t)$ is continuous, there exists a closed set $\Omega_3 \subset \Omega_2$ such that for each $i = 1, \dots, k-1$ either $h_i(t) = 0$ for $t \in \Omega_3$ or $h_i(t)$ is invertible on Ω_3.

We define unitary functions for $t \in \Omega_3$: $u_k(t) = 1$,

$$u_i = \begin{cases} h_{i+m_2+m_3}(t)/\|h_{i+m_2+m_3}(t)\|, & \text{if } 0 \le i \le m_1 - 1 \text{ and } h_{i+m_2+m_3}(t) \ne 0, \\ u_0(t) h_i(t)/\|h_i(t)\|, & \text{if } m_1 \le i \le k-1 \text{ and } h_i(t) \ne 0, \\ 1, & \text{otherwise.} \end{cases}$$

Putting $D(t) = \operatorname{diag}(u_1(t), \dots, u_{m_1}(t), \bar{u}_{m_1+1}(t), \dots, \bar{u}_k(t), I_{n-k})$ and considering $D^* R D(t)$ we see that the entries of the first rows of the corresponding blocks Q_{12} and Q_{13} and the entries of the last column of Q_{13} become real-valued. Hence without loss of generality, we can also suppose that the mentioned entries of Q_{12} and Q_{13} are real-valued from the beginning.

To find functional relations between entries of R, it is convenient to write it down in another block form:

$$R(t) = \begin{pmatrix} W_{11}(t) & W_{12}(t) \\ W_{12}^*(t) & W_{22}(t) \end{pmatrix},$$

where $W_{11}(t)$ is a $k \times k$ matrix and $W_{22}(t)$ is an $n - k \times n - k$ matrix.

Note that $W_{11}(t)$ as well as $W_{22}(t)$ are Hermitian. By Lemma 1, there exists an open set $\Omega_4 \subset \Omega_3$ such that $W_{11}(t)$ can be reduced to the diagonal form continuously for

$t \in \Omega_4$. Let the number $t_2 \in \Omega_4$ be such that $\operatorname{rank}(W_{11} - W_{11}^2)(t_2) \geq \operatorname{rank}(W_{11} - W_{11}^2)(t)$ for every $t \in \Omega_4$. Then for a sufficiently small open set $\Omega_5 \subset \Omega_4$, $t_2 \in \Omega_5$, the Moore–Penrose pseudoinverse of the matrix $(W_{11} - W_{11}^2)(t)$ is continuous and uniformly bounded for all $t \in \Omega_4$.

By definition, $R^2(t) = R(t)$, whence

$$W_{11}(t) = W_{11}^2(t) + W_{12}W_{12}^*(t) \text{ and } W_{22}(t) = W_{22}^2(t) + W_{12}^*W_{12}(t). \tag{4}$$

Immediately from this, using the polar decomposition for $W_{12}(t)$, we have that the matrix function

$$V(t) = \left(\sqrt{W_{11} - W_{11}^2}\right)^+ W_{12}(t) \tag{5}$$

is a $k \times n-k$ partial *coisometry*, where $\left(\sqrt{W_{11} - W_{11}^2}\right)^+$ is the Moore–Penrose pseudoinverse. Besides, $W_{22}(t)$ is uniquely determined by $W_{11}(t)$ and $V(t)$. A direct calculation shows that

$$W_{22}(t) = V^*(I_k - W_{11})V(t).$$

If $s = \operatorname{rank} V(t)$ and $s = k$, then $VV^*(t) = I_k$, otherwise there exist an open set $\Lambda \subset \Omega_5$ and a continuous $k \times n-k$ coisometry $\tilde{V}(t)$ such that

$$V(t) = (W_{11} - W_{11}^2)(W_{11} - W_{11}^2)^+\tilde{V}(t) \tag{6}$$

for $t \in \Lambda$. Really, let $S(t)$ be a unitary continuous $k \times k$ matrix that reduces $W_{11}(t)$ to the diagonal form and

$$S^*(W_{11} - W_{11}^2)(W_{11} - W_{11}^2)^+S(t) = \operatorname{diag}(I_s, 0_{k-s}).$$

The set of the first s rows of $S^*V(t)$ is an orthonormal system, say $\vec{v}_1(t), \dots, \vec{v}_s(t)$. For a fixed $t_3 \in \Omega_5$, it can be completed to an orthonormal system of k vectors by some vectors $\vec{v}_{s+1}, \dots \vec{v}_k \in \mathbb{C}^{n-k}$. Gram–Schmidt orthonormalizing process for the vectors

$$\vec{v}_1(t), \dots, \vec{v}_s(t), \vec{v}_{s+1}, \dots, \vec{v}_k$$

is continuous for t from a small neighborhood Λ of t_3. The obtained vectors will be the rows of $S^*\tilde{V}(t)$.

In view of (6), we can restrict ourselves to consider only the case $\operatorname{rank} V(t) = k$. Suppose $l \in \mathbb{N}$ and the l-th column of $W_{12}(t)$ is zero. Then for the $k+l$ base vector $e_{k+l}(t)$, the following equality holds

$$Re_{k+l}(t) = \operatorname{diag}(W_{11}, W_{22})e_{k+l}(t).$$

By (4),

$$\operatorname{diag}(W_{11} - W_{11}^2, W_{22} - W_{22}^2)e_{k+l}(t) = \operatorname{diag}(W_{12}W_{12}^*, W_{12}^*W_{12})e_{k+l}(t) = \vec{0}.$$

Since W_{ii} is Hermitian, we have $Re_{k+l}(t) = \vec{0}$ or $Re_{k+l}(t) = e_{k+l}(t)$. Thus the vector $e_{k+l}(t)$ is invariant under the action of R and J, i.e. π is reducible. Therefore there is no zero column in W_{12} and in $V(t)$. Let us consider the block form of $V(t)$:

$$V(t) = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

with V_{11} being an $m_4 \times m_4$ matrix. The following relation for the rank holds:

$$\operatorname{rank} \begin{pmatrix} V_{11}(t) \\ V_{21}(t) \end{pmatrix} = m_4$$

since otherwise there exists a unitary $m_4 \times m_4$ matrix Z such that

$$\begin{pmatrix} V_{11}(t)Z \\ V_{21}(t)Z \end{pmatrix}$$

is lower triangular. Hence it has a zero column and so does $W_{12}(t) \operatorname{diag}(Z, I_{m_5})$. This contradicts the statement above.

We fix $t_4 \in \Omega_5$. One can suppose that the first k columns of $V(t_0)$ are linearly independent, because we can rearrange the basis vectors $e_{k+m_4+1}, \ldots, e_n$ so that this property becomes valid. There exists a matrix $N \in M_k(\mathbb{C})$ such that $NV(t_4)$ is upper triangular with the corresponding block form

$$NV(t) = \begin{pmatrix} \hat{V}_{11}(t) & \hat{V}_{12}(t) \\ \hat{V}_{21}(t) & \hat{V}_{22}(t) \end{pmatrix}.$$

For a sufficiently small open set $\Omega_6 \subset \Omega_5$, $t_4 \in \Omega_6$, the matrix-valued function $\hat{V}_{11}(t)$ is invertible and $\operatorname{rank}(\hat{V}_{22}(t)) = k - m_4$, $t \in \overline{\Omega}_6$.

By Lemma 3, there exist unitary matrices $V_4(t)$ and $V_5(t)$ such that the matrices

$$\hat{V}_{11}V_4(t) \text{ and } \begin{pmatrix} \hat{V}_{22}V_5(t) \\ \hat{V}_{12}V_5(t) \end{pmatrix}$$

are lower triangular with real functions on the diagonals for t from some open set $\Omega_7 \subset \Omega_6$. Let

$$\tilde{V}(t) = NV(t) \operatorname{diag}(V_4(t), V_5(t)) = \begin{pmatrix} \tilde{V}_{11}(t) & \tilde{V}_{12}(t) \\ \tilde{V}_{21}(t) & \tilde{V}_{22}(t) \end{pmatrix}.$$

Direct calculations show that $\tilde{V}_{11}(t)$ and $\tilde{V}_{22}(t)$ are lower triangular and $\tilde{V}_{11}(t)$ is an invertible $m_4 \times m_4$ matrix-valued function.

We denote by $\tilde{R}(t)$ the matrix

$$\operatorname{diag}(N, V_4^*(t), V_5^*(t))R(t)\operatorname{diag}(N^*, V_4(t), V_5(t))$$

and by $U(t)$ the unitary transformation of the basis that we made to obtain $\tilde{R}(t)$ from $P(t)$. Let $\Omega_8 \subset \Omega_7$ be a closed ball of non-zero radius. Due to (4) and (5), the commutative C^*-algebra $\mathfrak{A}(\Omega_8)$ generated by the entries of $\tilde{R}(t)$ and the identity coincides with the C^*-algebra generated by all entries of the blocks $Q_{ij}(t)$, $i,j = 1,2,3$, $\tilde{V}_{lm}$, $l,m = 1,2$ and the identity function for $t \in \Omega_8$. On the other hand, it separates points of Ω_8. Indeed, the representations $\pi(t)$ and $\pi(z)$ are not equivalent for $t \neq z$. Also we have the equality

$$E_{1i}XE_{j1} + E_{2i}XE_{j2} + \ldots + E_{ni}XE_{jn} = x_{ij}I_n$$

for every matrix $X = (x_{ij})_1^n$. Whence, the center of the C^*-algebra generated by the matrix units E_{ij}, $U^*\pi(S)U$ and $U^*\pi(J)U$ separates the points of Ω_8. Therefore by the Stone–Weierstrass theorem, $\mathfrak{A}(\Omega_8)$ is isomorphic to the matrix algebra of all continuous functions $M_n(C(\Omega_8))$. We recall that $C(\Omega_8)$ cannot be generated by less than $\dim \Omega_8$ real-valued functions in virtue of invariance of the dimension number of $\mathbb{R}^n$ and the theorem about a system of generators of a commutative normed algebras [21, Chapter III, Th. 6].

Let us count the independent entries of $\tilde{R}(t)$. In view of the previous paragraph, this number is at least d. By construction, Q_{ii} is diagonal for $i = 1,2,3$. Taking into account that at least $k-1$ entries of Q_{12} and Q_{13} are real-valued, we have that $W_{11}(t)$ depends on at most

$$k^2 - (m_1^2 + m_2^2 + m_3^2) + 1 \tag{7}$$

real-valued functions. Also

$$\tilde{V}(t)\tilde{V}^*(t) = I_k = \begin{pmatrix} \tilde{V}_{11}\tilde{V}_{11}^*(t) + \tilde{V}_{12}\tilde{V}_{12}^*(t) & \tilde{V}_{11}\tilde{V}_{21}^*(t) + \tilde{V}_{12}\tilde{V}_{22}^*(t) \\ \tilde{V}_{21}\tilde{V}_{11}^*(t) + \tilde{V}_{22}\tilde{V}_{12}^*(t) & \tilde{V}_{21}\tilde{V}_{21}^*(t) + \tilde{V}_{22}\tilde{V}_{22}^*(t) \end{pmatrix}. \tag{8}$$

Whence

$$\tilde{V}_{11}\tilde{V}_{11}^*(t) = I_{m_4} - \tilde{V}_{12}\tilde{V}_{12}^*(t). \tag{9}$$

The matrix $\tilde{V}_{11}$ is lower triangular, hence using the Cholesky decomposition, we obtain that the entries of $\tilde{V}_{11}(t)$ are defined by the entries of $\tilde{V}_{12}(t)$. We are going to show that every entry of $\tilde{V}_{21}(t)$ and $\tilde{V}_{22}(t)$ is completely defined by $\tilde{V}_{12}(t)$ too. It follows from (8),

$$\tilde{V}_{21}\tilde{V}_{11}^*(t) + \tilde{V}_{22}\tilde{V}_{12}^*(t) = 0 \text{ or } \tilde{V}_{21}(t) = -\tilde{V}_{22}\tilde{V}_{12}^*(\tilde{V}_{11}^*)^{-1}(t) \tag{10}$$

and $\tilde{V}_{21}\tilde{V}_{21}^*(t) + \tilde{V}_{22}\tilde{V}_{22}^*(t) = I_{k-m_4}$. Substituting $\tilde{V}_{21}$ by (10), we obtain the equation

$$\tilde{V}_{22}\tilde{V}_{12}^*(\tilde{V}_{11}^*)^{-1}\tilde{V}_{11}^{-1}\tilde{V}_{12}\tilde{V}_{22}^*(t) + \tilde{V}_{22}\tilde{V}_{22}^*(t) = I_{k-m_4}. \tag{11}$$

By construction, $\hat{V}_{22}V_5(t)$ is lower triangular and we can write the block form

$$\tilde{V}_{22}(t) = (D(t)\ 0_{k-m_4\ m_4+m_5-k}), \tag{12}$$

where $D(t)$ is a lower triangular $(k-m_4)\times(k-m_4)$ matrix-valued function with positive functions on the diagonal. Multiplying (11) by $D^{-1}(t)$ from the left and by $(D^*(t))^{-1}$ from the right, we get

$$(I_{k-m_4}\ O)\tilde{V}_{12}^*(\tilde{V}_{11}^*)^{-1}\tilde{V}_{11}^{-1}\tilde{V}_{12}\begin{pmatrix} I_{k-m_4} \\ O^* \end{pmatrix} + I_{k-m_4} = D^{-1}(t)(D^*(t))^{-1},$$

with $O = 0_{k-m_4\ m_4+m_5-k}$. Thus $D^*D(t)$ is a continuous function on $\tilde{V}_{12}(t)$ and $\tilde{V}_{11}(t)$. If $\sigma = E_{n1} + E_{2\ n-1} + E_{3\ n-2}\cdots + E_{1n}$ is the skew diagonal matrix, then $\sigma^*D^*\sigma$ is lower triangular too. So using the Cholesky decomposition for the matrix $(\sigma^*D^*(t)\sigma)(\sigma^*D(t)\sigma)$ yields that the entries of $D^*(t)$ as well as the entries of $\tilde{V}_{22}(t)$ can be calculated by the entries of $\tilde{V}_{12}(t)$ and $\tilde{V}_{11}(t)$. In view of (9) and (10), we conclude that every entry of $\tilde{V}_{ij}(t)$ lies in the C^*-algebra generated by the entries of $\tilde{V}_{12}(t)$.

We count the number of non-zero entries of $\tilde{V}_{12}(t)$. By construction, if one changes the order of the rows in

$$\begin{pmatrix} \tilde{V}_{12}(t) \\ \tilde{V}_{22}(t) \end{pmatrix},$$

one gets a lower triangular $k \times m_5$ matrix-valued function with real functions on the diagonal. Whence $\tilde{V}_{12}(t)$ and $\tilde{V}_{22}(t)$ have $m_5k - m_5(m_5+1)/2$ complex-valued entries and m_5 real-valued entries. It follows from (12), that $\tilde{V}_{22}(t)$ has $(k-m_4)(k-m_4-1)/2$ complex-valued entries and $k - m_4$ real-valued entries. So $\tilde{V}_{12}(t)$ depends on

$$2m_4m_5 - (m_4 + m_5 - k)^2 \tag{13}$$

real-valued functions. Adding (7) with (13), we conclude that

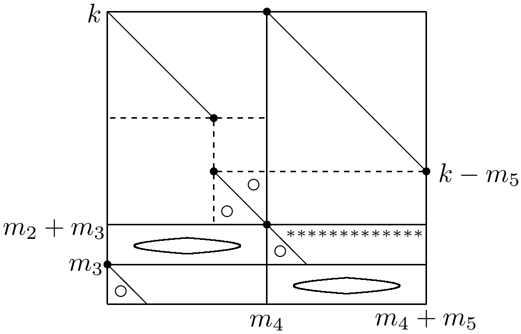
$$d \le k^2 - (m_1^2 + m_2^2 + m_3^2) + 1 + 2m_4m_5 - (m_4 + m_5 - k)^2.$$

Taking into account that $m_1 \ge n/3$, $m_4 + m_5 = n - k$ and the equation (3) holds, one can easily find the maximum on m_i:

$$d \le k^2 + 1 - n^2/9 - 2(k/2 - n/6)^2 + (n-k)^2/2 - (n-2k)^2,$$

or after collecting like terms, $d \le -3k^2 + \frac{10}{3}nk - (\frac{2}{3}n^2 - 1)$. The right part of this inequality as a function of k has a maximum at $k = 5/9n$. But $k \le n/2$. So putting $k = n/2$ we have $d \le -3n^2/4 + \frac{10}{6}n^2 - \frac{2}{3}n^2 + 1 = n^2/4 + 1$. Therefore $d(n,3) \le n^2/4 + 2$. $\square$

If the inclusion $x \in \sigma(W_{11}(t))$ yields to $1 - x \notin \sigma(W_{11}(t))$ for $x \in [0, 1]$, then the matrix-valued function $W_{11}(t)$ is uniquely determine by $W_{12}(t)$. Using Lemmas 1 and 3 and Corollary 2, one can reduce locally $W_{12}(t)$ to a matrix-function of the form



that depends on less than $n^2/6 + 2$ parameters. Here on the picture we mark both ends of each diagonal if the corresponding block matrix is diagonal and only the upper or lower end if it is upper triangular or lower triangular correspondingly. The entry marked by "$*$" is real-valued. In view of the following section, we can prove that in some points the local dimension of $\Omega^o_{n,k}$ is $n^2/6 + 2$.

3. A class of irreducible unitary representations of B_3

Our aim is to show that $d(n, 3)$ is larger than $n^2/6$. We consider at first the case $n = 6m$, $m \in \mathbb{N}$. Let A and D be $3m \times 3m$ matrices, $\|A\|, \|D\| < 1/2$ and assume that the Hermitian matrix A satisfies the equality $A - A^2 = DD^*$. We define the block matrix

$$U := 2 \begin{pmatrix} A & D \\ D^* & D^* A^{-1} D \end{pmatrix} - I_{6m}.$$

Obviously $U^2 = I_{6m}$ and $U^* = U$. So the pair (U, V) of matrices, where

$$V = \operatorname{diag}(I_{2m}, \beta I_{2m}, \beta^2 I_{2m}),$$

defines a representation of B_3. Let D have the following form

$$D = \begin{pmatrix} D_1 & D_2 \\ 0_m & D_3 \end{pmatrix},$$

where D_2 is a $2m \times 2m$ diagonal matrix, D_1, D_2 and D_3 are matrices with real entries on the diagonal, D_1 is a lower triangular matrix, D_3 is an upper triangular matrix with real numbers in the first row.

We assign the set of all non-zero entries of D to a point $\vec{h} \in \mathbb{R}^d$, $d = 6m^2 + 1$ by the following rule: $h_1/2, \ldots, h_{4m}/2$ are the diagonal entries of D_2, D_1 and D_3, the numbers $h_{4m+1}/2, \ldots, h_{6m-1}/2$ are the entries of the upper row of D_3. Calling the other nonzero entries of D_1 and D_3, say, $b_1, \ldots, b_{3m^2-3m}$, we set $h_{6m+2i} = (b_{i-1} + \overline{b}_{i-1})/4$, $h_{6m+2i+1} = (b_{i-1} - \overline{b}_{i-1})/4\mathbf{i}$, $i = 1, \ldots, 3m^2 - 3m + 1$. With such a point $\vec{h}$ we define the representation π_h of B_3 by $\pi_h(S) = U, \pi_h(J) = V$. We shall write D_{ih} for the block matrix D_i to stress that the corresponding block depends on $\vec{h}$.

Theorem 5. *There exists an open set $\Omega \subset \mathbb{R}^d$ such that for every $\vec{h} \in \Omega$, the representation π_h is an irreducible unitary representation of B_3 and for different $\vec{h}, \vec{g} \in \Omega$, the representations π_h and π_g are not equivalent.*

Proof. Let $0 < \epsilon < 1/(100m^4)$ and

$$\Omega = \{\vec{h} \in \mathbb{R}^d \mid (6m^2 - i + 2)\epsilon < h_i < (6m^2 - i + 3)\epsilon, i = 1, \ldots, 6m^2 + 1\}. \tag{14}$$

For $\vec{h} \in \Omega$, we denote by $\mathcal{A}_h$ the $*$-algebra generated by $\pi_h(J)$ and $\pi_h(S)$. Our goal is to show that $E_{ij} \in \mathcal{A}_h$ for every $i, j = 1, \ldots, 6m$. Since $\pi_h(J) = V$ is a block diagonal matrix with different scalar matrices in the blocks, there exists a polynomial P_i such that

$$P_i(V) = J_i$$

where J_i is a diagonal projection matrix from the previous section with $n_i = 2m$. Whence $J_i \in \mathcal{A}_h$ for $i = 1, 2, 3$. Direct calculations show that

$$J_1 \pi_h(S) J_3 \pi_h(S) J_1 = \operatorname{diag}(h_1^2, \ldots, h_{2m}^2, 0, \ldots, 0). \tag{15}$$

Since $h_i \neq h_j$ for $i \neq j$, there exist a polynomial R_i such that $R_i(h_i) = 1$ and $R_i(h_j) = R_i(0) = 0$ for every $i \neq j$. We evaluate it on the diagonal matrix:

$$R_i(\operatorname{diag}(h_1^2, \ldots, h_{2m}^2, 0, \ldots, 0)) = E_{ii} \in \mathcal{A}_h, \quad i = 1, \ldots, 2m.$$

Beside this, we have

$$J_3 \pi_h(S) J_1 \pi_h(S) J_3 = \operatorname{diag}(0, \ldots, 0, h_1^2, \ldots, h_{2m}^2). \tag{16}$$

So

$$R_i(\operatorname{diag}(0, \ldots, 0, h_1^2, \ldots, h_{2m}^2)) = E_{i+4m\ i+4m} \in \mathcal{A}_h.$$

Let us consider

$$J_2 \pi_h(S) J_2 = C_h = \operatorname{diag}(0_{2m}, C_1^h, C_2^h, 0_{2m}),$$

where C_1^h and C_2^h are Hermitian $m \times m$ matrices. By construction, the spectrum of C_1^h is in a small neighborhood of -1 and the spectrum of C_2^h is in a small neighborhood of 1. Therefore there exist two polynomials G_1 and G_2 such that

$$G_1(C_h) = \operatorname{diag}(0_{2m}, I_m, 0_{3m}) \text{ and } G_2(C_h) = \operatorname{diag}(0_{3m}, I_m, 0_{2m}). \tag{17}$$

Multiplying $\pi_h(S)$ with $G_1(C_h)$ from the left and J_3 from the right gives us the block matrix with only nonzero block D_3. We denote this matrix by $\tilde{D}_{3h}$. Thus $\tilde{D}_{3h} \in \mathcal{A}_h$ and

$$\tilde{D}_{3h} E_{4m+1\ 4m+1} \tilde{D}_{3h}^* = h_{3m+1}^2 E_{2m+1\ 2m+1}. \tag{18}$$

Since $h_{3m+1} > 0$, the matrix unit $E_{2m+1\ 2m+1}$ belongs to $\mathcal{A}_h$. Note that

$$\tilde{D}_{3h}^{(1)} = \tilde{D}_{3h} - E_{2m+1\ 2m+1} \tilde{D}_{3h} \tag{19}$$

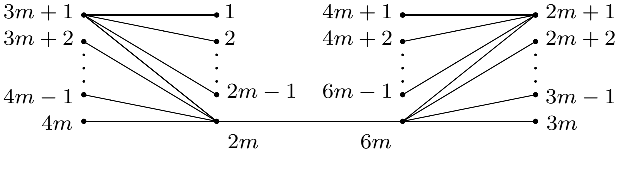
has the $2m+1$ row identically equal to zero, whence

$$\tilde{D}_{3h}^{(1)} E_{4m+2\ 4m+2} \tilde{D}_{3h}^{(1)*} = h_{3m+2}^2 E_{2m+2\ 2m+2} \in \mathcal{A}_h. \tag{20}$$

Continuing this process, we obtain $E_{ii} \in \mathcal{A}_h$ for $i = 2m+1, \ldots, 3m$. A similar argument with $\tilde{D}_{2h} = J_1 \pi_h(S) G_2(C_h)$ gives us $E_{ii} \in \mathcal{A}_h$ for $i = 3m+1, \ldots, 4m$. Therefore, $E_{ii} \in \mathcal{A}_h$ for every $i = 1, \ldots, 6m$. Let $\Gamma = (\gamma_{ij})_1^{6m}$ be a $(0,1)$ matrix:

$$\gamma_{ij} = \begin{cases} 1, & \text{if } E_{ii}\pi_h(S)E_{jj} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix Γ is an adjacency matrix of a *connected* graph on the indices. Indeed, the vertices $1, \ldots, 2m$ are connected with $3m+1$ (the first left column of D_1), the vertex $2m$ is connected with $3m+1, \ldots, 4m$ (the lower row of D_1) and with $6m$ (the diagonal entry of D_2), the vertex $6m$ is connected with $2m+1, \ldots, 3m$ (the last column of D_3) and the vertex $2m+1$ is connected with $4m+1, \ldots, 6m$ (the upper row of D_3):



Since $\pi_h(S)$ is Hermitian, we have the connections in both directions and connectedness of the graph.

Every path from k to l gives us the expression for a scalar multiple of E_{lk} if we substitute every edge $(j\ i)$ of the path by $E_{ii}\pi_h(S)E_{jj}$. Whence $E_{lk} \in \mathcal{A}_h$ and as a result $\mathcal{A}_h$ is the matrix algebra of all $6m \times 6m$ complex matrices. This proves the irreducibility of the representation π_h of the group algebra $\mathbb{C}[B_3]$. Since a reducible representation of a group leads to the reducible representation of the corresponding group algebra, we conclude that π is an irreducible unitary representation of B_3.

Let now $\vec{h}, \vec{g} \in \Omega$ be two different vectors. We are going to show that the corresponding representations are not equivalent. We use reductio ad absurdum. Assume π_h is equivalent to π_g. That means there exists a unitary matrix U such that

$$\pi_h(J) = U^*\pi_g(J)U \text{ and } \pi_h(S) = U^*\pi_g(S)U. \tag{21}$$

By definition,

$$\pi_h(J) = \pi_g(J) = V. \tag{22}$$

Whence

$$U = \operatorname{diag}(U_1, U_2, U_3), \tag{23}$$

where U_i is a $2m \times 2m$ matrix, $i = 1, 2, 3$. As was proved above, $P_i(V) = J_i$. Using (15), (22) and (23), we obtain:

$$\operatorname{diag}(h_1^2, \dots, h_{2m}^2) = U_1^* \operatorname{diag}(g_1^2, \dots, g_{2m}^2)U_1. \tag{24}$$

Since both sequences $h_1, \dots, h_{2m}$ and $g_1, \dots, g_{2m}$ are strictly decreasing as coordinates of vectors from (14), we have that U_1 is a diagonal matrix and

$$h_i = g_i \quad i = 1, \dots, 2m. \tag{25}$$

Using (16), we deduce in a similar way that

$$\operatorname{diag}(h_1^2, \dots, h_{2m}^2) = U_3^* \operatorname{diag}(g_1^2, \dots, g_{2m}^2)U_3 \tag{26}$$

and U_3 is a diagonal matrix too. Another result from (26) and (16) is the existence of polynomials $\tilde{R}_i(x, y)$ such that

$$E_{ii} = \tilde{R}_i(\pi_h(J), \pi_h(S)), \quad i = 1, \dots, 2m, 4m, \dots, 6m. \tag{27}$$

Besides, by (24) and (25), $\tilde{R}_i(\pi_h(J), \pi_h(S)) = \tilde{R}_i(\pi_g(J), \pi_g(S))$. In view of this, we shall use in the formulas below the matrix units E_{ii} and J_s for both representations assuming that they are obtained by the same corresponding formula (27).

Let us consider the action of U_2. By assumption,

$$\operatorname{diag}(C_1^h, C_2^h) = U_2^* \operatorname{diag}(C_1^g, C_2^g)U_2. \tag{28}$$

The spectrum of C_1^h and the spectrum of C_1^g are close to -1 and both $\sigma(C_2^h)$ and $\sigma(C_2^g)$ are close to 1. Therefore there exist two unitary matrices W_1 and W_2 such that

$$C_1^g = W_1^* C_1^h W_1 \text{ and } C_2^g = W_2^* C_2^h W_2 \tag{29}$$

Substituting the expressions for C_1^g and C_2^g from (29) into (28) and evaluating G_1 on both parts, we see that

$$G_1(\operatorname{diag}(C_1^h, C_2^h)) = \operatorname{diag}(I_m, 0_m)$$

and

$$G_1(U_2^*(\operatorname{diag}(W_1^* C_1^h W_1, W_2^* C_2^h W_2))U_2) = \\ U_2^* \operatorname{diag}(W_1^*, W_2^*) G_1(\operatorname{diag}(C_1^h, C_2^h)) \operatorname{diag}(W_1, W_2) U_2.$$

Hence

$$\operatorname{diag}(I_m, 0_m) = U_2^* \operatorname{diag}(I_m, 0_m) U_2$$

and so U_2 is the block diagonal matrix: $U_2 = \operatorname{diag}(U_2^{(1)}, U_2^{(2)})$, where $U_2^{(i)}$ is an $m \times m$ matrix. Whence $G_i(C_h) = G_i(C_g)$, $i = 1, 2$ and we can use the formula (17).

By definition

$$\tilde{D}_{3h} = G_1(C_h)\pi_h(S)J_3 = \operatorname{diag}(I_{2m}, U_2^{(1)*}, I_{3m})\tilde{D}_{3g} \operatorname{diag}(I_{4m}, U_3). \tag{30}$$

Since U_3 is diagonal, the matrix

$$\tilde{D}_{3h} E_{4m+1\ 4m+1} \tilde{D}_{3h}^* = \operatorname{diag}(0_{2m}, h_{3m+1}^2, 0_{4m-1})$$

is equivalent to $\operatorname{diag}(0_{2m}, g_{3m+1}^2, 0_{4m-1})$. So $g_{3m+1} = h_{3m+1}$. On the other hand, D_{3h} and D_{3g} are upper triangular with $D_{3h} = U_2^{(1)*} D_{3g} U_3$ and if $U_3 = \operatorname{diag}(u_1^{(3)}, u_2^{(3)} \dots, u_{2m}^{(3)})$, then the upper left entry of $U_2^{(1)}$ is equal to $u_1^{(3)}$, i.e. $U_2^{(1)}$ is a block diagonal matrix, say, $\operatorname{diag}(u_1^{(3)}, U_{2_2}^{(1)})$, and there exists a polynomial $\tilde{R}_{2m+1}(x, y)$ such that

$$E_{2m+1\ 2m+1} = \tilde{R}_{2m+1}(\pi_h(J), \pi_h(S)) = \tilde{R}_{2m+1}(\pi_g(J), \pi_g(S)).$$

Cutting the $2m + 1$ rows of $\tilde{D}_{3h}$ and $\tilde{D}_{3g}$ (see (19)), we obtain the same case as above but with a smaller number of nonzero rows. Therefore $g_{3m+2} = h_{3m+2}$ and $U_2^{(1)} = \operatorname{diag}(u_1^{(3)}, u_2^{(3)}, U_{2_3}^{(1)})$. Taking these steps inductively, we have that $U_2^{(1)} = \operatorname{diag}(u_1^{(3)}, u_2^{(3)}, \dots, u_m^{(3)})$ and $h_i = g_i$ for $i = 3m + 1, \dots, 4m$.

By definition,

$$\tilde{D}_{2h} = J_2 \pi_h(S) G_2(C_h) = \operatorname{diag}(U_1^*, 0_{4m})\tilde{D}_{2g} \operatorname{diag}(I_{3m}, U_2^{(2)}, I_{2m}). \tag{31}$$

Since U_1 is diagonal, $U_1 = \operatorname{diag}(u_1^{(1)}, u_2^{(1)} \dots, u_m^{(1)})$, we have

$$E_{11}\tilde{D}_{2h}\tilde{D}_{2h}^* E_{11} = \operatorname{diag}(h_{2m+1}^2, 0_{6m-1}) = \operatorname{diag}(g_{2m+1}^2, 0_{6m-1}). \tag{32}$$

Whence $g_{2m+1} = h_{2m+1}$, since $g_{2m+1} > 0$ and $h_{2m+1} > 0$ by definition. The matrices D_{2h} and D_{2g} are lower triangular with $D_{2h} = U_1^* D_{2g} U_2^{(2)}$. This yields that the upper left entry of $U_2^{(2)}$ is equal to $u_1^{(1)}$ and $U_2^{(2)} = \operatorname{diag}(u_1^{(1)}, U_{2_2}^{(2)})$. The product $\tilde{D}_{2h}^* E_{11} \tilde{D}_{2h}$

is a multiple of $E_{3m+1\ 3m+1}$, so we have the existence of a polynomial $\tilde{R}_{3m+1}(x, y)$ such that

$$E_{3m+1\ 3m+1} = \tilde{R}_{3m+1}(\pi_h(J), \pi_h(S)) = \tilde{R}_{3m+1}(\pi_g(J), \pi_g(S)).$$

Cutting the $3m+1$ columns of $\tilde{D}_{2h}$ and $\tilde{D}_{2g}$, we obtain the same case as above but with a smaller number of nonzero columns. Consequently taking these steps, we find $h_i = g_i$ for $i = 2m+1, \ldots, 3m$ and $U_2^{(2)} = \operatorname{diag}(u_1^{(1)}, u_2^{(1)}, \ldots, u_m^{(1)})$. Thus, U is diagonal. Let us show that all $u_i^{(j)}$ are equal. The matrices U and E_{ii} commute, so

$$E_{2m+1\ 2m+1}\pi_h(S)E_{jj}\pi_h(S)^* E_{2m+1\ 2m+1} = h_{j-1}^2 E_{2m+1\ 2m+1}$$

and

$$U^* E_{2m+1\ 2m+1}\pi_g(S)E_{jj}\pi_g(S)^* E_{2m+1\ 2m+1}U = g_{j-1}^2 E_{2m+1\ 2m+1}$$

are equal for $j = 4m+2, \ldots, 6m-1$. Since $h_j > 0$ and $g_j > 0$, we have $h_j = g_j$ for $j = 1, \ldots, 6m-1$. Now (21) yields

$$E_{2m+1\,2m+1}\pi_h(S)E_{4m+i\,4m+i} = h_{4m+i}E_{2m+1\,4m+i} = \overline{u}_1^{(3)}u_i^{(3)}g_{4m+i}E_{2m+1\,4m+i},$$

whence $u_i^{(3)} = u_1^{(3)}$, $i = 2, \ldots, 2m$. Beside this, we have

$$E_{ii}\pi_h(S)E_{4m+i\ 4m+i} = h_i E_{i\ 4m+i} = \overline{u}_i^{(1)}u_i^{(3)}g_i E_{i\ 4m+i}$$

for $i = 1, \ldots, 2m$. Hence $u_i^{(1)} = u_i^{(3)} = u_1^{(3)}$ and U is a scalar matrix. By (21), $\pi_h = \pi_g$ and $\vec{h} = \vec{g}$. A contradiction to the assumption that $\vec{h}$ and $\vec{g}$ are different. So π_h and π_g are not equivalent for different $\vec{h}$ and $\vec{g}$. □

In order to obtain a family of non-equivalent irreducible representations of B_3 in the general case for $n > 6$ when $n \neq 6m$, we shall preserve the triangular and diagonal forms of the matrices D_1, D_3 and D_2 but change slightly their sizes. Let $[x]$ denote the greatest integer that is less or equal to x. The matrix D will be a $[n/2] \times n - [n/2]$ matrix and D_2 will be a $[(n+1)/3] \times [(n+1)/3]$ matrix. The set Ω can be chosen as above for even n and with the following restriction on the norm of the last right column $\tilde{h}$ of D

$$\|\tilde{h}\| < (10n)^{-1} \left\|(DD^*)^{-1}\right\| \tag{33}$$

for odd n. All steps of the proof of Theorem 5 can be repeated without change. We show here only that $\sigma(C_2^h)$ is close to $+1$ for odd n. Note that $2D^*A^{-1}D - I_{n-[n/2]}$ has $[n/2]$ eigenvalues close to $+1$ and one eigenvalue close to -1. Since $A - A^2 = DD^*$, we have

$$\|A^{-1}\| \leq \|(I_{[n/2]} - A)^{-1}A^{-1}\| = \|(DD^*)^{-1}\|.$$

So the norm of the last right column of $D^*A^{-1}D$ is less than $(10n)^{-1}$. Therefore the submatrix $[A^{-1}]$, that is obtained from $D^*A^{-1}D$ by cutting the right last column and the lower last row of it, has all eigenvalues close to +1 and so does C_2^h.

With a so defined D, we again, as at the beginning of the section, form an irreducible unitary representation π_h of B_3 non equivalent for different $\vec{h}$, and to find the lower estimation on $d(n,3)$, we calculate the number of nonzero entries of D (complex and real-valued). The matrix D_2 has $[(n+1)/3]$ real-valued entries, D_1 has

$$\left(n - \left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)\left[\frac{n+1}{3}\right] - \frac{\left(n - \left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)\left(1 + n - \left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)}{2}$$

complex-valued entries and $n - [n/2] - [(n+1)/3]$ real-valued entries and D_3 has

$$\left(\left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)\left[\frac{n+1}{3}\right] - \frac{\left(\left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)\left(1 + \left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)}{2} - \left[\frac{n+1}{3}\right] + 1$$

complex-valued entries and $[n/2] - 1$ real-valued entries. Adding all numbers, we obtain the dependence of D on

$$2\left[\frac{n+1}{3}\right]\left(n - 2\left[\frac{n+1}{3}\right]\right) - \left(\left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)^2 - \left(n - \left[\frac{n}{2}\right] - \left[\frac{n+1}{3}\right]\right)^2 + 1 \quad (34)$$

real parameters.

Corollary 6. *The following inequality holds (see for a comparison [7])*

$$d(n,3) \geq \left(\frac{n}{2} - \left[\frac{n+1}{3}\right]\right)\left(6\left[\frac{n+1}{3}\right] - n\right) + \frac{9 - (-1)^n}{4}.$$

Proof. This is a direct simplification of the formula (34) for odd and even n. □

Remark 2. Let $\mathfrak{A}$ be the $*$-algebra that is generated by three orthogonal projection p_1, p_2 and p_3 with the relation $p_1p_2 = p_2p_1 = 0$. It was proved in [13] that $\mathfrak{A}$ has a representation by 20×20 matrices where the entries are functions of two arbitrary unitary operators. This gives the lower bound $n^2/200 + 1$ on $d(n,3)$. The authors later simplified their construction to 14×14 matrices [12]. Whence $d(n,3) \geq n^2/98 + 1$.

Remark 3. In Theorem 5 we were interested in finding new constructions of irreducible representations of B_3 and by presenting our family of representations, we obtained the estimates on $d(n,3)$. Using ideas from [18] one can also in principle obtain a method of finding the dimension of the representation variety $\mathrm{Irrep}_n\ B_3$ at a fixed point and so get the lower bound on $d(n,3)$. However it is not clear for us how difficult the application of the method might be or what point one should take to achieve good estimations.

4. Irreducible representations of B_4

To construct nontrivial irreducible representations of B_4 we use the notion of tensor products of matrices. Let F and G be $d \times d$ and $l \times l$ matrices respectively. Then the $ld \times ld$ matrix

$$\operatorname{diag}(F, F, \ldots, F) \begin{pmatrix} g_{11} I_d & \ldots & g_{1l} I_d \\ \vdots & \ddots & \vdots \\ g_{l1} I_d & \ldots & g_{ll} I_d \end{pmatrix}$$

is the tensor product $F \otimes G$ of F and G.

Suppose we have two irreducible unitary representations π_1 and π_2 of B_4. Then $\pi_1 \otimes \pi_2$ is unitary representations of B_4 too. It has not to be irreducible in general. But it will be so if we take a special representation π_1, such that

$$\pi_1(\sigma_1) = \pi_1(\sigma_3) \tag{35}$$

and, for the representation π_2, use the reduced Burau representation (see [11]) written in the base where every matrix $\pi_2(\sigma_i)$ is unitary:

$$\pi_2(\sigma_1) = \operatorname{diag}(u, 1, 1), \tag{36}$$

$$\pi_2(\sigma_2) = \begin{pmatrix} (u-1)\alpha_1 + 1 & (u-1)\sqrt{\alpha_1 - \alpha_1^2} & 0 \\ (u-1)\sqrt{\alpha_1 - \alpha_1^2} & (1-u)\alpha_1 + u & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{37}$$

$$\pi_2(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (u-1)\alpha_2 + 1 & (u-1)\sqrt{\alpha_2 - \alpha_2^2} \\ 0 & (u-1)\sqrt{\alpha_2 - \alpha_2^2} & (1-u)\alpha_2 + u \end{pmatrix}, \tag{38}$$

with $u\bar{u} = 1$, $\alpha_1 = -u/(u-1)^2$, $\alpha_2 = \alpha_1/(1-\alpha_1)$. We remark that since both numbers α_1 and α_2 have to be positive and less than 1, we have to assume that the real part of u is less than 0.

Theorem 7. *Let π_1 be an irreducible unitary representation of B_4 and (35) be satisfied. Let also π_2 be the representation of B_4 defined by (36)–(38). Then the representation $\pi_1 \otimes \pi_2$ is an irreducible unitary representation of B_4. If two irreducible representations $\tilde{\pi}_1$, $\hat{\pi}_1$ of B_4 are not equivalent, the corresponding representations $\tilde{\pi}_1 \otimes \pi_2$ and $\hat{\pi}_1 \otimes \pi_2$ of B_4 are not unitary equivalent either.*

Proof. It suffices to prove the theorem for the case $\pi_1((\sigma_1\sigma_2)^3) = I$, where I is an identity matrix. Let $\mathcal{A}$ be the $*$-algebra $\pi_1 \otimes \pi_2(\mathbb{C}\langle B_4 \rangle)$. At first we shall show that $I \otimes E_{ij} \in \mathcal{A}$. We note that

$$[\pi_1(\sigma_1)\otimes\pi_2(\sigma_1)\cdot\pi_1(\sigma_2)\otimes\pi_2(\sigma_2)]^3 = I\otimes\begin{pmatrix} u^3 & 0 & 0\\ 0 & u^3 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

This implies that

$$P = I\otimes\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\in\mathcal{A}.$$

Let us consider the element

$$(\pi_1(\sigma_1)\otimes\pi_2(\sigma_1))^* = \pi_1(\sigma_1^{-1})\otimes\begin{pmatrix} \bar{u} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

It lies in $\mathcal{A}$ and we can multiply it with another element of $\mathcal{A}$, namely with the element $\pi_1(\sigma_3)\otimes\pi_2(\sigma_3) = \pi_1(\sigma_1)\otimes\pi_2(\sigma_3)$ from the right. Using also the projection P constructed above, one has that $P\,(\pi_1(\sigma_1)\otimes\pi_2(\sigma_1))^*\,\pi_1(\sigma_3)\otimes\pi_2(\sigma_3)P$ is equal to

$$I\otimes\begin{pmatrix} \bar{u} & 0 & 0\\ 0 & (u-1)\alpha_2+1 & 0\\ 0 & 0 & 0\end{pmatrix}\in\mathcal{A}.$$

Since $\bar{u}\neq(u-1)\alpha_2+1$ by the definitions of u and α_2, we have $I\otimes E_{ii}\in\mathcal{A}$, $i=1,2,3$. By construction of π_2 every entry of $\pi_2(\sigma_2)$ and $\pi_2(\sigma_3)$ which depends on u, α_1 or α_2 is different from zero. Therefore, $I\otimes E_{12}$ is a scalar multiple of

$$I\otimes E_{11}\cdot\pi_1(\sigma_2)^*\otimes\pi_2(\sigma_2)^*\cdot I\otimes E_{11}\cdot\pi_1(\sigma_2)\otimes\pi_2(\sigma_2)\cdot I\otimes E_{22}$$

and $I\otimes E_{23}$ is a scalar multiple of

$$I\otimes E_{22}\cdot\pi_1(\sigma_3)^*\otimes\pi_2(\sigma_3)^*\cdot I\otimes E_{22}\cdot\pi_1(\sigma_3)\otimes\pi_2(\sigma_3)\cdot I\otimes E_{33}.$$

Since $\mathcal{A}$ is a $*$-algebra, $I\otimes E_{ij}\in\mathcal{A}$ for every i,j. This leads to $I\otimes\pi_2(\sigma_i)^*\in\mathcal{A}$ and

$$I\otimes\pi_2(\sigma_i)^*\cdot\pi_1(\sigma_i)\otimes\pi_2(\sigma_i) = \pi_1(\sigma_i)\otimes I_3\in\mathcal{A} \tag{39}$$

for every i. Using the irreducibility of π_1, we conclude that $E_{mk}\otimes I_3\in\mathcal{A}$ and, hence $E_{mk}\otimes E_{ij}\in\mathcal{A}$. This proves the irreducibility of $\pi_1\otimes\pi_2$.

We deduced (39) without using the actual formulas for $\pi_1(\sigma_1)$ and $\pi_1(\sigma_2)$. Hence, if we have two representations $\tilde{\pi}_1\otimes\pi_2$ and $\hat{\pi}_1\otimes\pi_2$, then they are equivalent if and only if $\tilde{\pi}_1\otimes I_3$ is equivalent to $\hat{\pi}_1\otimes I_3$. $\square$

Remark 4. Obviously $d(n,4)\geq d(n,3)$. Theorem 7 gives us a family of irreducible representations of B_4 that depends on $n^2/54$ parameters with $\pi(\sigma_1)\neq\pi(\sigma_3)$.

Remark 5. In the interesting paper [15] the author classifies complex irreducible representations of B_4 in dimension 4. Although the considered representations studied in [15] in general are not unitary, the ideas and even some formulations of results have similarities with those in our paper.

Remark 6. If we consider a family of n dimensional representations of B_k which depends on d parameters, then Long's construction leads to a new family of kn dimensional representations which depends on $d+1$ parameters [16]. The number of parameters slowly increases with the size of matrices. By Corollary 6 and Remark 4, all non-equivalent finite dimensional representations of B_3 and B_4 cannot be obtained from representations of small dimensions and iterations of Long's construction. It would be very interesting to characterize the representations of a fixed dimension that appear from representations of the smaller dimensions by going through Long's construction.

Conflict of interest statement

The authors confirm that there is no conflict of interest associated with this publication.

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