ZEROS AND EXPONENTIAL PROFILES OF POLYNOMIALS II: EXAMPLES

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Abstract. In [Jalowy, Kabluchko, Marynych, arXiv:2504.11593v1, 2025], the authors discuss a user-friendly approach to determine the limiting empirical zero distribution of a sequence of real-rooted polynomials, as the degree goes to ∞. In this note, we aim to apply it to a vast range of examples of polynomials providing a unifying source for limiting empirical zero distributions.

We cover Touchard, Fubini, Eulerian, Narayana and little q-Laguerre polynomials as well as hypergeometric polynomials including the classical Hermite, Laguerre and Jacobi polynomials. We construct polynomials whose empirical zero distributions converge to the free multiplicative normal and Poisson distributions. Furthermore, we study polynomials generated by some differential operators. As one inverse result, we derive coefficient asymptotics of the characteristic polynomial of random covariance matrices.

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1. Introduction

We consider a sequence of polynomials $P_n(x) = \sum_{k=0}^{n} a_{k:n} x^k$ of degree at most n, with real coefficients and nonpositive real zeros. This paper is a continuation of the recent study [31], where the authors discuss the method of exponential profiles for establishing the existence of a limiting empirical zero distribution of P_n as $n \to \infty$. While [31] focused on the theoretical foundations and examined how limiting empirical zero distributions are affected by certain polynomial operations, such as finite free convolution and repeated differentiation, the present paper demonstrates the versatility of the method by applying it to various classical polynomials. Although we shall briefly recall the necessary definitions and tools below, we recommend that readers first familiarize themselves with [31], where they will find detailed bibliographic remarks, motivation, and further applications.

2. A quick recap on zeros and exponential profiles

2.1. **Definitions.** We shall briefly recall necessary definitions and concepts from [31] and refer to that source for detailed explanations.

Definition 2.1. The *empirical distribution of zeros* of a polynomial $P \not\equiv 0$ of degree at most n is the probability measure $[\![P]\!]_n$ on $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ which assigns equal weight $1/n$ to each zero (counting multiplicities) and places the remaining mass at ∞. Thus, letting δ_x denote the Dirac measure at $x \in \overline{\mathbb{C}}$, we define

$$[\![P]\!]_n := \frac{1}{n} \sum_{z \in \mathbb{C} \colon P(z)=0} \delta_z + \frac{n - \deg P}{n} \delta_\infty. \tag{1}$$

We shall work with sequences of polynomials rather than with a single polynomial P, and the variable n in Definition 2.1 should be understood as the variable indexing some sequence $(P_n)_{n \in \mathbb{N}}$. In many cases, n coincides with the degree of P_n, but in general, we shall assume only $\deg P_n \le n$.

If a polynomial P has only real zeros all of the same sign, we identify the point at infinity, ∞, with either $+\infty$ (when the roots are nonnegative) or $-\infty$ (when the roots are nonpositive). Accordingly, $[\![P]\!]_n$ is supported on $[0,+\infty]$ in the former case, and on $[-\infty,0]$ in the latter.

An important characteristic of a sequence of polynomials (P_n) with *nonnegative real coefficients* is its exponential profile.

Definition 2.2. For every $n \in \mathbb{N}$ let $P_n(x) = \sum_{k=0}^{n} a_{k:n} x^k$ be a polynomial of degree at most n with nonnegative coefficients $a_{k:n} \geq 0$. We say that the sequence $(P_n)_{n\in\mathbb{N}}$ has an *exponential profile* g if $g : (\underline{m}, \overline{m}) \to \mathbb{R}$ is a function defined on a nonempty interval $(\underline{m}, \overline{m}) \subseteq [0,1]$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log a_{\lfloor \alpha n \rfloor : n} = \begin{cases} g(\alpha), & \text{if } \alpha \in (\underline{m}, \overline{m}), \\ -\infty, & \text{if } \alpha \in [0, \underline{m}) \cup (\overline{m}, 1]. \end{cases} \tag{2}$$

Here, and in the following, we stipulate that $\log 0 = -\infty$.

The basic principle lying behind the method of exponential profiles discussed in [31] is as follows.

> A sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials with only nonpositive roots possesses an exponential profile if and only if the sequence of probability measures $([\![P_n]\!]_n)_{n\in\mathbb{N}}$ converges weakly on $[-\infty, 0]$ to a limiting probability measure which assigns a positive mass to $(-\infty, 0)$.

Theorems 2.4 and 2.5, established in [31] and restated below for the reader's convenience, provide a rigorous foundation for the aforementioned principle. For the purposes of the present paper, the direct implication stating that existence of an exponential profile implies convergence of the empirical distribution is of greater relevance. Indeed, given a sequence of polynomials with only nonpositive roots, a direct proof of the convergence of their empirical distributions is typically a technically involved task. By contrast, establishing the existence of an exponential profile usually follows straightforwardly from classical asymptotic relations.

The main principle outlined above suggests that it should be possible to recover the limiting measure of $[\![P_n]\!]_n$ as $n \to \infty$ from the exponential profile g, and conversely. The most convenient framework for this purpose is provided by Cauchy transforms.

Definition 2.3. For any probability measure μ on $\mathbb{R}$ its *Cauchy transform* $G : \mathbb{C}\backslash \operatorname{supp}(\mu) \to \mathbb{C}$ is defined by

$$G(t) = \int_{\mathbb{R}} \frac{\mu(\mathrm{d}z)}{t - z}. \tag{3}$$

The negative of the Cauchy transform is known as the Stieltjes transform. By the Stieltjes–Perron inversion formula, see [51, Proposition 2.1.2 on p. 35] and [1, pp. 124–125], the Cauchy transform uniquely determines the measure μ via

$$\mu(I) = -\lim_{y\to 0+} \int_I \frac{1}{\pi} \operatorname{Im} G(x + iy) \mathrm{d}x, \tag{4}$$

where I is any interval such that μ is continuous at its endpoints. Also, locally uniform convergence on $\mathbb{C}\backslash\mathbb{R}$ of the sequence of Cauchy transforms is equivalent to weak convergence of the corresponding sequence of probability measures.

Observe that the definition of Cauchy transforms extends naturally to probability measures on $(-\infty,+\infty]$ or $[-\infty,+\infty)$. Moreover, if μ is supported by $[-\infty,A]$ (respectively, $[A,+\infty]$) for some $A\in\mathbb{R}$, then

$$1-\mu(\{-\infty\})=\lim_{t\to+\infty} tG(t) \qquad \Big(\text{respectively, } 1-\mu(\{+\infty\})=\lim_{t\to-\infty} tG(t)\Big). \tag{5}$$

2.2. **Theorems on zeros and profiles.** Having recalled the necessary concepts, we are now ready to formulate two theorems, proved in [31], that support the method of exponential profiles.

Theorem 2.4 (Profile for coefficients implies limiting distribution of zeros, Theorem 2.2 in [31])**.** *For every $n\in\mathbb{N}$ let $P_n(x)=\sum_{k=0}^n a_{k:n}x^k$ be a polynomial of degree at most n. Suppose that the sequence $(P_n)_{n\in\mathbb{N}}$ possesses an exponential profile $g:(\underline{m},\overline{m})\to\mathbb{R}$ in the sense of Definition 2.2. If, additionally, all roots of P_n are nonpositive for all sufficiently large n, then the following hold.*

(a) We have weak convergence

$$[\![P_n]\!]_n \xrightarrow[n\to\infty]{w} \mu$$

of probability measures on $[-\infty,0]$ for some μ.

(b) The measure μ is uniquely determined by the profile g as follows. If G denotes the Cauchy transform of μ, then $t\mapsto tG(t)$, $t>0$, is the inverse of the function[1] $\alpha\mapsto \mathrm{e}^{-g'(\alpha)}$, $\alpha\in(\underline{m},\overline{m})$. The function $t\mapsto tG(t)$ is a strictly increasing continuous bijection between $(0,+\infty)$ and $(\underline{m},\overline{m})$.

(c) It holds $\mu(\{0\})=\underline{m}$ and $\mu(\{-\infty\})=1-\overline{m}$.

The converse of Theorem 2.4 is as follows.

Theorem 2.5 (Distribution of zeros implies exponential profile, Theorem 2.4 in [31])**.** *For every $n\in\mathbb{N}$ let $P_n(x)=\sum_{k=0}^n a_{k:n}x^k$ be a polynomial of degree at most n with real, nonpositive roots. If $[\![P_n]\!]_n$ converges weakly to some probability measure μ on $[-\infty,0]$, then the following hold.*

(a) Define $\underline{m}:=\mu(\{0\})$ and $\overline{m}:=1-\mu(\{-\infty\})$ and observe that $0\le\underline{m}\le\overline{m}\le 1$. There exists a strictly concave and infinitely differentiable function $g:(\underline{m},\overline{m})\to(-\infty,0]$ such that

$$\sup_{(\underline{m}+\varepsilon)n\le k\le(\overline{m}-\varepsilon)n}\left|\frac{1}{n}\log\frac{a_{k:n}}{P_n(1)}-g\Big(\frac{k}{n}\Big)\right|\xrightarrow[n\to\infty]{}0, \tag{6}$$

for all sufficiently small $\varepsilon>0$ as well as

$$\sup_{0\le k\le(\underline{m}-\varepsilon)n}\frac{1}{n}\log\frac{a_{k:n}}{P_n(1)}\xrightarrow[n\to\infty]{}-\infty, \qquad \sup_{(\overline{m}+\varepsilon)n\le k\le n}\frac{1}{n}\log\frac{a_{k:n}}{P_n(1)}\xrightarrow[n\to\infty]{}-\infty. \tag{7}$$

[1]The derivative g' exists on $(\underline{m},\overline{m})$. In fact, g is automatically infinitely differentiable and strictly concave on $(\underline{m},\overline{m})$.

(b) If $\underline{m} < \overline{m}$, then the Cauchy transform G of μ is such that $(0, +\infty) \ni t \mapsto tG(t)$ is the inverse of $(\underline{m}, \overline{m}) \ni \alpha \mapsto \mathrm{e}^{-g'(\alpha)}$.

(c) If $\underline{m} = \overline{m}$, then (6) is a void statement but (7) holds true.

A somewhat weaker version of Theorem 2.5 appeared in the paper of Fano, Ortolani, Van Assche [23]. We refer to [31] for a discussion of further related references.

Remark 2.6. Theorem 2.4 is not directly applicable if the real-rooted polynomials (P_n) are allowed to have positive roots. However, if the positive roots of the polynomials (P_n) are uniformly bounded by some $A \geq 0$, Theorem 2.4 can be applied to the shifted polynomials $\hat{P}_n(x) = P_n(x + A)$. If this shifted sequence possesses an exponential profile, we can find its limiting empirical zero distribution $\hat{\mu}$. The limiting empirical zero distribution μ of the original sequence is then $\mu(\cdot) = \hat{\mu}(\cdot - A)$.

2.3. **Logarithmic and normalized logarithmic potentials.** The exponential profile g is also connected to the normalized logarithmic potential of μ. It is proved in [31] that under the assumptions of Theorem 2.4,

$$\lim_{n\to\infty} \frac{\log P_n(1)}{n} = \sup_{\alpha\in(\underline{m},\overline{m})} g(\alpha) =: M_g \tag{8}$$

and the function $-g(\cdot) + M_g$ is the Legendre transform of $u \mapsto \Psi(\mathrm{e}^u)$, where

$$\Psi(t) = \int_{(-\infty,0]} \log\left(\frac{t-z}{1-z}\right)\mu(\mathrm{d}z), \quad t > 0, \tag{9}$$

This means that $g(\alpha) = M_g + \inf_{u\in\mathbb{R}}(\Psi(\mathrm{e}^u) - \alpha u)$ for all $\alpha \in (\underline{m}, \overline{m})$. The function Ψ can be analytically continued to the domain $\mathbb{C}\backslash(-\infty, 0]$. Recall that in this domain the usual logarithmic potential U of μ is defined by

$$U(t) := \int_{(-\infty,0]} \log|t-z|\mu(\mathrm{d}z) = \mathrm{Re}\left(\int_{(-\infty,0]} \log(t-z)\mu(\mathrm{d}z)\right), \quad t \in \mathbb{C}\backslash(-\infty, 0],$$

and is infinite if $\mu(\{-\infty\}) > 0$ (and can also be infinite even if $\mu(\{-\infty\}) = 0$). On the other hand, Ψ is well-defined for any measure μ on $[-\infty, 0]$. If U is finite, then of course

$$\Psi(t) = U(t) - U(1), \quad t \in \mathbb{C}\backslash(-\infty, 0].$$

The (normalized) logarithmic potential and the Cauchy transform are related via $2\frac{\mathrm{d}}{\mathrm{d}t}U = 2\frac{\mathrm{d}}{\mathrm{d}t}\Psi = G$, where t is a complex variable and $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\partial}{\partial\,\mathrm{Re}(t)} - \mathrm{i}\frac{\partial}{\partial\,\mathrm{Im}(t)}\right)$ is the complex (Wirtinger) derivative. Restricted to the real-valued argument $t > 0$, both logarithmic potentials Ψ and U are antiderivatives of G, satisfying $\Psi'(t) = U'(t) = G(t)$ for $t > 0$.

3. Overview of the main results

The main contribution of this article is to provide a unified reference for the limiting empirical zero distributions of a broad class of real-rooted polynomials. While many of the individual results are known, this work brings them together under a common framework using a universal proof method based on exponential profiles.

Polynomial $P_n(z)$	Coefficient $a_{k:n}$	profile $\rightsquigarrow \mathrm{e}^{-g'(\alpha)}$	distribution μ
Stirling	$\left[{n \atop k}\right] n^k$	$\frac{\alpha}{w_S(\alpha)}$	$\mathrm{Unif}_{[-1,0]}$
Touchard $T_n(nz)$	$\left\{{n \atop k}\right\} n^k$	$\frac{1}{w_T(\alpha)\mathrm{e}^{w_T(\alpha)}}$	$\int x^k d\mu = \frac{(-k)^k}{(k+1)!}$
Fubini $F_n(z)$	$\left\{{n \atop k}\right\} k!$	$\frac{1}{\mathrm{e}^{w_T(\alpha)}-1}$	$\frac{dx}{\lvert x\rvert(x+1)\left(\pi^2+\log^2\lvert 1+\frac{1}{x}\rvert\right)}$
Eulerian $E_n(z)$	$\left\langle {n \atop k}\right\rangle$	$\mathrm{e}^{w_E(\alpha)}$	– log-Cauchy
gen. Narayana	$\binom{n}{k}^\gamma$	$\left(\frac{\alpha}{1-\alpha}\right)^\gamma$	$G_{B,\gamma}(z) = \frac{1}{z(z^{-1/\gamma}+1)}$
Laguerre $L_n^{(\gamma n)}(nz)$	$(-1)^k\binom{n+\gamma}{n-k}\frac{n^k}{k!}$	$\frac{\alpha(\alpha+\gamma)}{1-\alpha}$	$\mathsf{MP}_{\gamma+1,\frac{1}{\gamma+1}}$
Hermite $\mathrm{He}_n(\sqrt{n}z)$	$\frac{(-1)^k}{k!2^k}\cdot\frac{n^{n/2-k}}{(n-2k)!}$	$\leftrightarrow \frac{\alpha^2}{1-\alpha}$	$\mathsf{sc}_1 = \frac{dx}{2\pi}\sqrt{4-x^2}$
Legendre	$\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}$	$\leftrightarrow \frac{\alpha^2}{1-\alpha^2}$	$\mathrm{arcsine}_{[-1,1]}$
Gegenbauer	$(-1)^{\frac{n-k}{2}}\frac{\Gamma(\gamma+\frac{n+k}{2})}{k!(\frac{n-k}{2})!}2^k$	$\leftrightarrow \frac{\alpha(\gamma+\alpha-1/2)}{(1-\alpha)(2\gamma+\alpha)}$	$c\frac{(b-x)(x-a)}{1-x^2}dx$
little q-Laguerre	$\prod_{j=1}^k \frac{q^{j-n}-q}{(1-aq^j)(1-q^j)}$	$\frac{(1-a\mathrm{e}^{-\lambda\alpha})(1-\mathrm{e}^{-\lambda\alpha})}{\mathrm{e}^{\lambda}\mathrm{e}^{-\lambda\alpha}-1}$	$p_{a\vert\lambda}(x)\mathrm{d}x$, (47)

TABLE 1. An overview of some explicit examples of polynomials P_n in the first column and our findings on their limiting empirical zero distributions. Further details on notation used can be found in Section 4.

A brief summary of our findings is presented in Table 1. The second column lists the coefficients of P_n, while the third column displays the (typically simpler) function $\mathrm{e}^{-g'}$, where g is the exponential profile associated with the sequence (P_n).[2] The symbol $\leftrightarrow$ indicates that the profile corresponds not directly to (P_n), but to related polynomials obtained through simple transformations (like a scaling, shift or reflection), as the Hermite, Legendre and Gegenbauer polynomials do not have nonnegative coefficients. The final column presents several ways of characterizing the limiting empirical zero distribution μ, such as by name, moments, Lebesgue density, or Cauchy transform. Additional details regarding the notation used, further properties and references to earlier discoveries can be found in the respective subsections.

Our general approach to filling Table 1, which can also be applied to any sequence of real-rooted polynomials (P_n) with nonpositive roots, can be briefly summarized as follows. Given such a sequence (P_n), the first step is to pass to a normalized polynomial $\widetilde{P}_n(x) = b_n P_n(a_n x)$ using appropriate scaling factors a_n and b_n. These sequences (a_n) and (b_n) are chosen so that $\widetilde{P}_n$ admits an exponential profile, meaning that (2) holds for some function $g = g_P$. If such a normalization is possible, then Theorem 2.4 guarantees that the empirical measures of the roots of $\widetilde{P}_n$ converge to a limiting measure $\mu = \mu_P$. The second step is to characterize μ via its Cauchy transform G, which can be recovered from the profile g by inverting the function $\alpha \mapsto \mathrm{e}^{-g'(\alpha)}$. On the third step, knowing the Cauchy transform G, we calculate the moments, the logarithmic potential of μ

[2]This function is closely related to the S-transform of μ, see [31, Lemma 6.1], [3] and Equation (52) below.

and the Lebesgue density using the Stieltjes–Perron inversion formula (4). In some situations not only $t \mapsto tG(t)$ but also the Cauchy transform G itself can be written as an inverse of some simple function f. In this case, a classic formula for the antiderivative of an inverse function

$$\int f^{\leftarrow}(t)\mathrm{d}t = tf^{\leftarrow}(t) - F(f^{\leftarrow}(t)) + C, \tag{10}$$

can be used to calculate the logarithmic potential of μ. In this formula F is the antiderivative of f and $f^{\leftarrow}$ denotes the inverse function to f.

4. Classical polynomials

4.1. **Stirling numbers of the first kind.** Consider the polynomials

$$S_n(x) := x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{11}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the unsigned Stirling number of the first kind. Obviously, the sequence of probability measures $[\![S_n(nx)]\!]_n$ converges weakly to the uniform distribution $\mu_S = \mathrm{Unif}_{[-1,0]}$ on $[-1,0]$ with the Cauchy transform

$$G_S(t) = \log\left(1 + \frac{1}{t}\right), \qquad t \in \mathbb{C}\backslash[-1,0].$$

As a warm up example, let us check this trivial result by applying Theorem 2.4 to the polynomials $S_n(nz)/n^n$. By the classic logarithmic asymptotic of the Stirling numbers of the first kind, see, for example, Eq. (5.7) in [49], we have

$$g_S(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log\left(\begin{bmatrix} n \\ \lfloor \alpha n \rfloor \end{bmatrix} n^{\lfloor \alpha n \rfloor - n}\right) = -1 + \alpha + (1-\alpha)\log\alpha + w_S + (\alpha - 1)\log w_S, \quad \alpha \in (0,1),$$

where $w_S = w_S(\alpha)$ is a unique solution in $(0,\infty)$ of the equation

$$\frac{w_S}{\mathrm{e}^{w_S} - 1} = \alpha. \tag{12}$$

Taking the derivative g_S' and simplifying the expression using (12) we obtain $\mathrm{e}^{-g_S'(\alpha)} = \alpha/w_S(\alpha) = (\mathrm{e}^{w_S(\alpha)} - 1)^{-1} =: L_S(w_S(\alpha))$, where $L_S(t) := (\mathrm{e}^t - 1)^{-1}$. Recalling that $t \mapsto tG_S(t)$ is the inverse of $\alpha \mapsto \mathrm{e}^{-g_S'(\alpha)}$ we conclude that

$$G_S(t) = t^{-1} w_S^{\leftarrow}(L_S^{\leftarrow}(t)) = \log\left(1 + \frac{1}{t}\right), \quad t > 0.$$

4.2. **Touchard polynomials and Stirling numbers of the second kind.** These polynomials, also called the *one-variable Bell* or the *exponential polynomials* are defined by

$$T_n(x) = \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^k,$$

where $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}$ is the Stirling number of the second kind. There are several equivalent definitions of Touchard polynomials including the following ones:

$$T_n(x) = \mathrm{e}^{-x} \sum_{\ell=0}^{\infty} \frac{\ell^n x^\ell}{\ell!} = n![t^n]\mathrm{e}^{x(\mathrm{e}^t - 1)} = \mathrm{e}^{-x}\,(xD)^n\,\mathrm{e}^x = B_n(x,\ldots,x), \tag{13}$$

where $B_n(z_1,\dots,z_n)$ is the n-th Bell polynomial and we abbreviated the derivative with $D = \frac{\mathrm{d}}{\mathrm{d}x}$. We refer to the books [29, Section 6.1] and [47] for an introduction to Stirling numbers of both kinds. Also, there is a Rodrigues-type formula

$$T_n(\mathrm{e}^x) = \mathrm{e}^{-\mathrm{e}^x} D^n \mathrm{e}^{\mathrm{e}^x}.$$

Harper [30] showed that all roots of $T_n(x)$ are real and nonpositive and used this fact to prove a CLT for the Stirling numbers of the second kind. As for the Stirling numbers of the first kind, it is natural to consider the scaled Touchard polynomials $T_n(nx)/n^n$.

The following result is originally due to Elbert [20, 21], see also Proposition 2.14 in [34].

Theorem 4.1. *The sequence of probability measures $[\![T_n(nx)]\!]_n$ converges weakly to a probability measure μ_T which is concentrated on the interval $(-\mathrm{e}, 0)$ and has the following*

$$\textit{Cauchy transform:} \quad G_T(t) = \frac{1}{tW_0(1/t)} - 1 = \mathrm{e}^{W_0(1/t)} - 1, \qquad t \in \mathbb{C}\backslash[-\mathrm{e}, 0], \tag{14}$$

$$\textit{Lebesgue density:} \quad p_T(x) = \frac{1}{\pi} \operatorname{Im} \mathrm{e}^{W_0(\frac{1}{x}+\mathrm{i}0)}, \qquad x \in (-\mathrm{e}, 0), \tag{15}$$

$$\textit{moments:} \quad \int_{-\mathrm{e}}^{0} x^k \mu_T(\mathrm{d}x) = \frac{(-k)^k}{(k+1)!}, \qquad k \in \mathbb{N}, \tag{16}$$

$$\textit{log-potential:} \quad U_T(t) = \operatorname{Re}\left(\frac{1}{W_0(1/t)} + W_0(1/t) - t - 1 + \log t \right), \qquad t \in \mathbb{C}\backslash[-\mathrm{e}, 0]. \tag{17}$$

Here, W_0 denotes the principal branch of the Lambert W-function.

Remark 4.2. In the Appendix A we collected some properties of the Lambert function which are relevant for us. For further information on this function we refer to the excellent sources [15, 46].

Proof. By the well-known logarithmic asymptotic of the Stirling numbers of the second kind, see, for example, Eq. (5.9) in [7], we have

$$g_T(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log\left(\left\{ {n \atop \lfloor \alpha n \rfloor} \right\} n^{\lfloor \alpha n \rfloor - n} \right) = f(w_T; \alpha) + \alpha - 1 - \alpha \log \alpha, \quad \alpha \in (0,1),$$

where $f(x;\alpha) = \alpha \log(\mathrm{e}^x - 1) - \log x$ and $w_T = w_T(\alpha) \in (0,\infty)$ is a unique solution to the equation $1 - \mathrm{e}^{-w_T} = \alpha \cdot w_T$. This solution can be explicitly written via the Lambert function as follows

$$w_T(\alpha) = \alpha^{-1} + W_0(-\alpha^{-1}\mathrm{e}^{-\alpha^{-1}}).$$

Taking the derivative, a short calculation yields

$$g_T'(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}(f(w_T(\alpha);\alpha)) - \log\alpha = \frac{\partial f}{\partial \alpha}(w_T(\alpha);\alpha) - \log\alpha = \log(\mathrm{e}^{w_T(\alpha)} - 1) - \log\alpha \tag{18}$$

and it follows that

$$\mathrm{e}^{-g_T'(\alpha)} = \frac{\alpha}{\mathrm{e}^{w_T} - 1} = \frac{\alpha}{\mathrm{e}^{w_T}(1 - \mathrm{e}^{-w_T})} = \frac{1}{w_T \mathrm{e}^{w_T}} = \frac{1}{W_0^{\leftarrow}(w_T(\alpha))}.$$

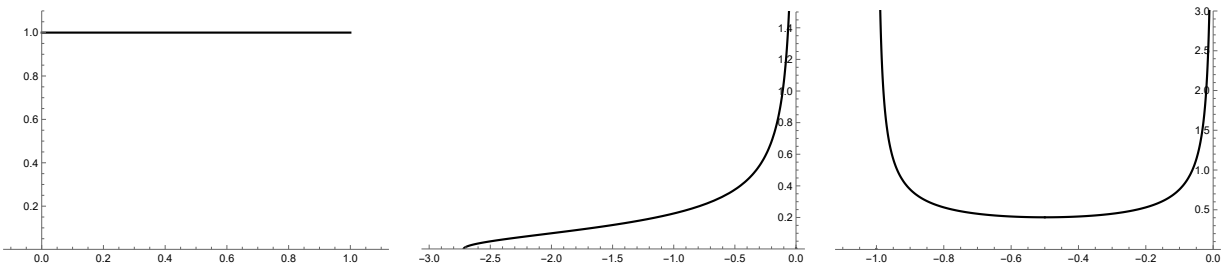

FIGURE 1. The limiting probability densities of zeros for generating polynomials of Stirling numbers of the first kind (left), Touchard polynomials (center) and Fubini polynomials (right).

Inverting this function we obtain

$$G_T(t) = t^{-1} w_T^{\leftarrow}(W_0(1/t)) = \frac{1-\mathrm{e}^{-W_0(1/t)}}{tW_0(1/t)} = \frac{\mathrm{e}^{W_0(1/t)}-1}{tW_0(1/t)\mathrm{e}^{W_0(1/t)}} = \mathrm{e}^{W_0(1/t)} - 1. \tag{19}$$

Formula (15) for the density p_T follows by the Stieltjes–Perron inversion (4). To prove formula (16) for the moments of μ_T, we compare the formula

$$G_T(t) = \int_{[-\mathrm{e},0]} \frac{\mu_T(\mathrm{d}z)}{t-z} = \frac{1}{t}\int_{[-\mathrm{e},0]} \sum_{k=0}^{\infty} \frac{z^k \mu_T(\mathrm{d}z)}{t^k} = \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} \int_{[-\mathrm{e},0]} z^k \mu_T(\mathrm{d}z) \qquad |t| > \mathrm{e},$$

to the following expansion, see Eq. (75) in the Appendix,

$$G_T(t) = \mathrm{e}^{W_0(1/t)} - 1 = \sum_{k=0}^{\infty} \frac{(-k)^k}{(k+1)!} t^{-k-1}, \qquad |t| > \mathrm{e}.$$

In order to verify formula (17), we differentiate and compare it to $2\frac{d}{dt}U_T = G_T$. Indeed, by (73) we obtain

$$2\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\left(\frac{1}{W_0(1/t)} + W_0(1/t) - t + \log t + C\right) = \frac{1}{tW_0(1/t)}$$

and $C = -1$ follows from $\lim_{t\to\infty}(U_T(t) - \log t) = 0$. Alternatively, the antiderivative $\int G_T(t)\mathrm{d}t$ can be calculated from (10) applied with $f(t) = G_T^{\leftarrow}(t) = ((t+1)\log(1+t))^{-1}$. Here the second equality is a consequence of (19). □

Remark 4.3. Edrei [19] also studied zeros of Touchard polynomials (or, more precisely, of successive derivatives of $\mathrm{e}^{-\mathrm{e}^z}$), but in his setting there is no rescaling of the variable by n.

4.3. **Fubini polynomials.** The Fubini numbers $k!\left\{{n \atop k}\right\}$ count partitions of n distinct elements into k nonempty blocks, where the order of blocks is taken into account. Their basic properties are discussed in [47, Chapter 6]. Fubini polynomials are defined by

$$F_n(x) = \sum_{k=0}^{n} \left\{{n \atop k}\right\} k! x^k.$$

It is known [47, p. 147] that these polynomials are real-rooted. This fact can be deduced from Rolle's theorem knowing [47, p. 160] the formula

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\tanh x = 2^n(\tanh x - 1)F_n\left(\frac{\tanh x - 1}{2}\right).$$

Theorem 4.4. *The sequence of probability measures $[\![F_n(x)]\!]_n$ converges weakly to a probability measure μ_F on the interval $(-1,0)$ with the following*

$$\textit{Cauchy transform:} \qquad G_F(t) = \frac{1}{t(t+1)\log(1+\frac{1}{t})}, \qquad t \in \mathbb{C}\backslash[-1,0], \tag{20}$$

$$\textit{Lebesgue density:} \qquad p_F(x) = \frac{1}{|x|(x+1)\left(\pi^2 + \log^2\left|1+\frac{1}{x}\right|\right)}, \qquad x \in (-1,0), \tag{21}$$

$$\textit{moments:} \qquad \int_{-1}^{0} x^k \mu_{\mathrm{F}}(\mathrm{d}x) = (-1)^k \frac{C_k}{k!}, \qquad k \in \mathbb{N}_0, \tag{22}$$

$$\textit{log-potential:} \qquad U_F(t) = -\log\left|\log\left(1+\frac{1}{t}\right)\right|, \qquad t \in \mathbb{C}\backslash[-1,0]. \tag{23}$$

where $C_k = \int_0^1 x(x+1)\dots(x+k-1)\mathrm{d}x$, $k \in \mathbb{N}_0$, are the Cauchy numbers; see [45], [14, pp. 293–294], [47, Section 5.3] *for their properties.*

Proof. We aim to apply Theorem 2.4 to the polynomials $F_n(x)/n!$ with the coefficients $\frac{k!}{n!}\left\{{n \atop k}\right\}$. Using the notation already introduced in the context of the Touchard polynomials, the profile is given by

$$g_F(\alpha) = \lim_{n\to\infty} \frac{1}{n}\log\left(\left\{{n \atop \lfloor \alpha n \rfloor}\right\}\frac{\lfloor \alpha n \rfloor!}{n!}\right) = -f(w_T;\alpha), \quad \alpha \in (0,1).$$

Using (18) we conclude that

$$\mathrm{e}^{-g_F'(\alpha)} = \frac{1}{\mathrm{e}^{w_T} - 1} = L_S(w_T(\alpha)),$$

where we recall the notation $L_S^{\leftarrow}(t) = \log(1+1/t)$ and $w_T^{\leftarrow}(x) = (1-\mathrm{e}^{-x})/x$. Thus,

$$G_F(t) = t^{-1} w_T^{\leftarrow}(L_S^{\leftarrow}(t)) = \frac{1-\mathrm{e}^{-L_S^{\leftarrow}(t)}}{tL_S^{\leftarrow}(t)} = \frac{1}{t(t+1)\log(1+\frac{1}{t})}, \quad t > 0.$$

The formula (21) for the density of μ_F follows from the Stieltjes–Perron inversion formula (4). For $x \in (-1,0)$,

$$p_F(x) = -\frac{1}{\pi}\operatorname{Im} G_{\mathrm{F}}(x+\mathrm{i}0) = -\frac{1}{\pi}\operatorname{Im}\frac{1}{x(x+1)\left(\log\left|1+\frac{1}{x}\right| - \mathrm{i}\pi\right)} = -\frac{1}{x(x+1)\left(\pi^2 + \log^2|1+\frac{1}{x}|\right)}$$

since

$$\log\left(1+\frac{1}{x+\mathrm{i}0}\right) = \log\left(1+\frac{1}{x}-\mathrm{i}0\right) = \log\left|1+\frac{1}{x}\right| - \mathrm{i}\pi.$$

More precisely, part (vii) of [51, Proposition 2.1.2] yields the explicit formula for the density on $(-\infty,0)$ (and the existence of this density), whereas part (v) implies the absence of the atom at 0 taking into account that $G_{\mathrm{F}}(x) = o(1/x)$ as $x \uparrow 0$.

To prove formula (22) for the moments of μ_F, we compare the formula

$$G_F(t) = \int_{[-1,0]} \frac{\mu_F(\mathrm{d}z)}{t-z} = \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} \int_{[-1,0]} z^k \mu_F(\mathrm{d}z), \qquad |t| > 1,$$

with the expansion

$$G_F(t) = \frac{1}{t^2\left(1+\frac{1}{t}\right)\log(1+\frac{1}{t})} = \sum_{k=0}^{\infty} \frac{(-1)^k C_k}{k!}\frac{1}{t^{k+1}}, \qquad |t|>1,$$

where we used the formula [47, p. 135]:

$$\frac{1}{(1+y)\log(1+y)} = \sum_{k=0}^{\infty} \frac{(-1)^k C_k}{k!} y^{k-1}, \qquad |y|<1.$$

As before, the logarithmic potential in formula (23) follows upon noticing that the Wirtinger derivative $\frac{d}{dt}$ of the right-hand side of (23) coincides with $G_F/2$. The additive constant is recovered from the asymptotic relation $U_F(t) = \log t + o(1)$, as $t \to +\infty$. □

4.4. **Eulerian polynomials.** The Eulerian polynomials $(E_n)_{n\in\mathbb{N}}$, are defined by means of the formula

$$\sum_{k=1}^{\infty} k^n x^k = (xD)^n \frac{1}{1-x} = \frac{xE_n(x)}{(1-x)^{n+1}}, \quad n\in\mathbb{N}.$$

Alternatively, if $\left\langle {n \atop j} \right\rangle$ denotes an Eulerian number defined as the number of permutations of $1,\dots,n$ with n ascents, then

$$E_n(x) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle x^j, \quad n\in\mathbb{N}.$$

For the basic properties of Eulerian numbers and polynomials we refer to [37] and [47, § 6.3–6.7]. Polynomials $xE_n(x)$ are also known in the literature as Jonquière polynomials, see [25] and also [27, 28]. Theorem 4.5 below is known and can be found in many papers including those of Sobolev and Sirazhdinov [57, 58, 59] (see Theorem 2 in [57]), Gawronski and Stadtmüller [27, Theorem 1], Gawronski [25, Theorem 5]; see also [22, 28] for further results and generalizations. It has recently been rediscovered by Melotti [44].

Theorem 4.5. *The sequence of probability measures $[\![E_n(x)]\!]_n$ converges weakly to a probability measure μ_E on the interval $(-\infty,0)$ with the following*

Cauchy transform:
$$G_E(t) = \frac{1}{t-1} - \frac{1}{t\log t}, \qquad t\in\mathbb{C}\backslash(-\infty,0], \tag{24}$$

Lebesgue density:
$$p_E(x) = \frac{1}{|x|(\pi^2+\log^2|x|)}, \qquad x\in(-\infty,0), \tag{25}$$

Normalized log-potential:
$$\Psi_E(t) = \log\left|\frac{t-1}{\log t}\right|, \qquad t\in\mathbb{C}\backslash(-\infty,0]. \tag{26}$$

Remark 4.6. Formula (25) for p_E implies that μ_E is invariant under the map $x\mapsto 1/x$. If X is a random variable with a scaled Cauchy distribution having density $\mathbb{R}\ni x\mapsto(\pi^2+x^2)^{-1}$, then μ_E is the distribution of $-\mathrm{e}^X$, that is, up to a sign change, is a log-Cauchy distribution. All power moments of μ_E are infinite. In fact, even its logarithmic potential $U_E(t) := \int_{-\infty}^0 \log|t-x| p_E(x)\mathrm{d}x$ equals $+\infty$. Finally, the invariance of μ_E under $x\mapsto 1/x$ implies the invariance of μ_F, the weak limit of $[\![F_n(x)]\!]_n$ in Theorem 4.4, under the map $(-1,0)\ni x\mapsto -1-x\in(-1,0)$. This is also easily seen from (21). Although not directly related to Theorem 4.5, the log-Cauchy distribution $\mu_E(-\cdot)$ arises

as the limit of free multiplicative self-convolutions of $\boxtimes$-infinitely divisible probability measures. For further details, we refer the reader to Theorem 4.1 in [4].

Proof. As in the previous examples we note that

$$g_E(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log\left(\left\langle \begin{matrix} n \\ \lfloor \alpha n \rfloor \end{matrix} \right\rangle \frac{1}{n!} \right) = -\alpha w_E - \log w_E + \log(\mathrm{e}^{w_E} - 1), \quad \alpha \in (0,1), \tag{27}$$

where $w_E = w_E(\alpha)$ is the unique solution to

$$\frac{\mathrm{e}^{w_E}}{\mathrm{e}^{w_E} - 1} - \frac{1}{w_E} = \alpha, \tag{28}$$

see [7, Eq. (5.7)]. A straightforward differentiation of (27) leads to

$$g_E'(\alpha) = -w_E - \alpha w_E' - w_E' \left(\frac{1}{w_E} - \frac{\mathrm{e}^{w_E}}{\mathrm{e}^{w_E} - 1} \right) = -w_E(\alpha).$$

Now, Theorem 2.4 implies

$$G_E(t) = t^{-1} w_E^{\leftarrow}(\log t) = t^{-1} \left(\frac{t}{t-1} - \frac{1}{\log t} \right) = \frac{1}{t-1} - \frac{1}{t \log t}, \quad t > 0.$$

The normalized log-potential Ψ_E is recovered by taking the antiderivative of G_E and using that $\Psi_E(1) = 0$.

Here is another argument which is based on the fact that the Eulerian polynomials can be expressed through the Fubini polynomials as follows:

$$E_n(x) = (x-1)^n F_n\left(\frac{1}{x-1} \right),$$

see [47, Eq. (6.22) on p. 156]. If we denote the zeros of F_n by $0, x_1, \ldots, x_{n-1}$, then the zeros of E_n are $y_i := 1 + (1/x_i)$, $i = 1, \ldots, n-1$. Note that $x_i \in (-1, 0)$ implies that $y_i \in (-\infty, 0)$. Therefore, if ξ_F is a random variable whose distribution is the weak limit of $[\![F_n(x)]\!]_n$ appearing in Theorem 4.4, then $[\![E_n(x)]\!]_n$ converges weakly to the distribution of the random variable $\xi_E := 1 + (1/\xi_F)$. Its Cauchy transform is

$$G_E(t) := \mathbb{E} \frac{1}{t - \frac{1}{\xi_F} - 1} = \frac{1}{t-1} - \frac{1}{(t-1)^2} \mathbb{E} \frac{1}{\frac{1}{t-1} - \xi_F} = \frac{1}{t-1} - \frac{1}{t-1}^2 G_F\left(\frac{1}{t-1} \right) = \frac{1}{t-1} - \frac{1}{t \log t},$$

where we used the formula for G_F from Theorem 4.4. This formula is true if $1/(t-1) \notin (-1, 0)$, which is equivalent to $t \notin (-\infty, 0)$.

The formula for the density of ξ_{E} follows from the Perron inversion formula: For $x < 0$,

$$p_E(x) = -\frac{1}{\pi} \operatorname{Im} G_{\mathrm{E}}(x + \mathrm{i}0) = \frac{1}{\pi} \operatorname{Im} \frac{1}{x \log(x + \mathrm{i}0)} = \frac{1}{\pi} \operatorname{Im} \frac{1}{x(\log|x| + \pi\mathrm{i})} = -\frac{1}{x(\pi^2 + \log^2|x|)},$$

which completes the alternative proof. □

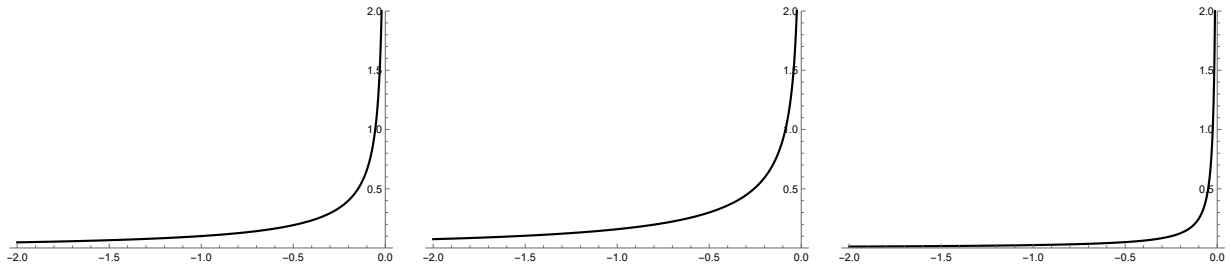

FIGURE 2. The limiting probability densities of zeros of Eulerian polynomials (left), Narayana polynomials for $\gamma = 2$ (center) and $\gamma = 10$ (right).

4.5. **Narayana polynomials.** The Narayana numbers and the corresponding polynomials are defined by

$$N_{n,k} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}, \qquad k \in \{0,\dots,n-1\}, \qquad N_n(x) = \sum_{k=1}^{n} N_{n,k-1}x^k,$$

see [37, Chapter 2] for their definition and properties. It is known [37, p. 90, Exercise 4.7] that all roots of N_n are real (and hence nonpositive).

For any integer $\gamma \geq 2$, define also the polynomials

$$B_{n,\gamma}(x) = \sum_{k=0}^{n} \binom{n}{k}^{\gamma} x^k,$$

which do have only real (nonpositive) roots, see Problem 155 in Part V of Volume II [52] or p. 366 in [36] (for the case $\gamma = 2$, the general case follows by induction from Theorem 6.4.1 in [36]). It is easy to see that the exponential profiles of the Narayana polynomials and polynomials $B_{n,2}$ coincide.

Theorem 4.7. *For integer $\gamma \geq 2$, the sequence of probability measures $[\![B_{n,\gamma}(x)]\!]_n$ converges weakly to a probability measure $\mu_{B,\gamma}$ on the interval $(-\infty, 0)$ with the following*

$$\textit{Cauchy transform:} \qquad G_{B,\gamma}(t) = \frac{1}{t(t^{-1/\gamma}+1)}, \qquad t \in \mathbb{C}\backslash(-\infty,0], \tag{29}$$

$$\textit{Lebesgue density:} \qquad p_{B,\gamma}(x) = \frac{\sin(\pi/\gamma)}{\pi|x|(|x|^{-1/\gamma} + 2\cos(\pi/\gamma) + |x|^{1/\gamma})}, \qquad x \in (-\infty,0), \tag{30}$$

$$\textit{log-potential:} \qquad U_{B,\gamma}(t) = \gamma \log|1 + t^{1/\gamma}|, \qquad t \in \mathbb{C}\backslash(-\infty,0]. \tag{31}$$

In (29) *and* (31) *we take the branch of a multi-valued function $t \mapsto t^{1/\gamma}$ which attains real values on the positive half-line. The sequence of probability measures $[\![N_n(x)]\!]_n$ converges weakly to $\mu_{B,2}$.*

Proof. The profile is given by

$$g_{B,\gamma}(\alpha) = \lim_{n\to\infty} \frac{\gamma}{n} \log\left(\binom{n}{\lfloor n\alpha \rfloor}\right) = -\gamma(\alpha\log\alpha + (1-\alpha)\log(1-\alpha)), \quad \alpha \in (0,1).$$

It follows that $g'_{B,\gamma}(\alpha) = \gamma\log(1-\alpha) - \gamma\log\alpha$ and hence $\mathrm{e}^{-g'_{B,\gamma}(\alpha)} = (\alpha/(1-\alpha))^\gamma$. Inverting this function we obtain (29). To prove (30), we apply Perron's inversion formula. For $x < 0$,

$$p_{B,\gamma}(x) = -\frac{1}{\pi}\operatorname{Im} G_{B,\gamma}(x+\mathrm{i}0) = -\frac{1}{\pi x}\operatorname{Im}\frac{1}{1+|x|^{-1/\gamma}\mathrm{e}^{-\mathrm{i}\pi/\gamma}} = \frac{|x|^{1/\gamma-1}\sin(\pi/\gamma)}{\pi(1+2\cos(\pi/\gamma)|x|^{1/\gamma}+|x|^{2/\gamma})}.$$

Formula (31) for the logarithmic potential follows immediately from the antiderivative of $G_{B,\gamma}(t)$. □

Remark 4.8. In fact, formulas (29)-(31) define a probability measure for all *real* values $\gamma > 1$. This can be seen by integrating $p_{B,\gamma}$ via the change of variables $y = |x|^{1/\gamma}$, $x < 0$. Also, numerical simulations suggest the following conjecture: The polynomials $B_{n,\gamma}$ are real-rooted for every $n \in \mathbb{N}$ and every real $\gamma \geq 2$. On the other hand, simulations show that, for $\gamma \in (1,2)$, the polynomial $B_{n,\gamma}$ is not real-rooted if n is sufficiently large.

4.6. **Hypergeometric polynomials.** Let us consider a fairly general parametrized class of polynomials that covers many classical orthogonal polynomials. The results on their limiting root distribution follow those of [42, §5], see also [43, Theorem 3.9] and [3, §10.3].

For $i, j, n \in \mathbb{N}$, $\mathbf{a} = (a_s)_{s\leq i} \in \mathbb{R}^i$, $\mathbf{b} = (b_s)_{s\leq j} \in \mathbb{R}^j$, we define the rising and falling factorials

$$\mathbf{a}^{\overline{n}} := \prod_{s=1}^{i}\frac{\Gamma(a_s+n)}{\Gamma(a_s)}, \quad \mathbf{b}^{\underline{n}} := \prod_{s=1}^{j}\frac{\Gamma(b_s+1)}{\Gamma(b_s-n+1)}$$

and study the (rescaled) hypergeometric polynomial

$$\begin{aligned}\mathcal{H}_n^{\mathbf{a},\mathbf{b}}(x) &= n^{(i-j)n}\frac{(\mathbf{b}n)^{\underline{n}}}{(\mathbf{a}n)^{\underline{n}}}\,{}_{i+1}F_j\left(\begin{matrix}-n, & \mathbf{a}n-n+1\\ & \mathbf{b}n-n+1\end{matrix}\,;-n^{j-i}x\right)\\ &= \sum_{k=0}^{n}\binom{n}{k}n^{(i-j)(n-k)}\frac{(\mathbf{b}n)^{\underline{n-k}}}{(\mathbf{a}n)^{\underline{n-k}}}x^k.\end{aligned}$$

Here we used the notation

$${}_{i+1}F_j\left(\begin{matrix}-n, & \mathbf{a}\\ & \mathbf{b}\end{matrix}\,;x\right) := \sum_{k=0}^{n}\frac{(-n)^{\overline{k}}(\mathbf{a})^{\overline{k}}}{(\mathbf{b})^{\overline{k}}}\frac{x^k}{k!} = \sum_{k=0}^{n}\binom{n}{k}\frac{(a_1)^{\overline{k}}\cdots(a_i)^{\overline{k}}}{(b_1)^{\overline{k}}\cdots(b_j)^{\overline{k}}}(-x)^k, \tag{32}$$

for a (generalized) hypergeometric polynomial. If for some choice of parameter $\mathbf{a}$ and $\mathbf{b}$, polynomials $\mathcal{H}_n^{\mathbf{a},\mathbf{b}}$ have positive coefficients and nonpositive roots, at least for all large enough n, then we describe their limiting root distribution as follows.

Theorem 4.9. *Let $\mathbf{a} \in \mathbb{R}^i, \mathbf{b} \in \mathbb{R}^j$ be such that:*

(i) $a_s, b_s \notin [0,1)$;

(ii) the number of negative reals among $\{a_1,\dots,a_i,b_1,\dots,b_j\}$ is even;

(iii) $\mathcal{H}_n^{\mathbf{a},\mathbf{b}}$ has only real nonpositive roots for all n large enough.

Then, $\alpha \mapsto \frac{\alpha}{1-\alpha}\frac{\prod_{s=1}^{j}(b_s-1+\alpha)}{\prod_{s=1}^{i}(a_s-1+\alpha)}$ is positive and invertible on $(0,1)$ with inverse $\Phi^{\mathbf{a},\mathbf{b}}$ and $[\![\mathcal{H}_n^{\mathbf{a},\mathbf{b}}]\!]_n$ converges weakly to a distribution with the Cauchy transform $G^{\mathbf{a},\mathbf{b}}(t) := \Phi^{\mathbf{a},\mathbf{b}}(t)/t$.

For an overview on choices of the parameters $\mathbf{a}, \mathbf{b}$ which lead to real, nonpositive roots of hypergeometric polynomials, we refer to [42, §4].

Proof. Another application of Stirling's formula shows that the profile is given by

$$\begin{aligned} g_H(\alpha) &= \lim_{n\to\infty} \frac{1}{n} \log\left(\binom{n}{\lfloor \alpha n \rfloor} \frac{n^{-j(n-\lfloor \alpha n\rfloor)} (\mathbf{b}n)^{\underline{n-\lfloor \alpha n\rfloor}}}{n^{-i(n-\lfloor \alpha n\rfloor)} (\mathbf{a}n)^{\underline{n-\lfloor \alpha n\rfloor}}} \right) \\ &= -\alpha\log\alpha - (1-\alpha)\log(1-\alpha) - \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-\lfloor \alpha n\rfloor -1} \log\left(\frac{\prod_{s=1}^{i}(a_s - \frac{m}{n})}{\prod_{s=1}^{j}(b_s - \frac{m}{n})} \right) \\ &= -\alpha\log\alpha - (1-\alpha)\log(1-\alpha) - \int_0^{1-\alpha} \log\left(\frac{\prod_{s=1}^{i}(a_s - x)}{\prod_{s=1}^{j}(b_s - x)} \right) \mathrm{d}x, \quad \alpha \in (0,1), \end{aligned}$$

by a Riemann summation in the last step. Note that the ratio under the last logarithm ought to be positive and bounded away from zero for all $x \in [0, 1-\alpha]$ by the assumptions (i) and (ii). The derivative of g_H is given by

$$g_H'(\alpha) = \log\left(\frac{1-\alpha}{\alpha} \frac{\prod_{s=1}^{i}(a_s - 1 + \alpha)}{\prod_{s=1}^{j}(b_s - 1 + \alpha)} \right),$$

hence the inverse of $\mathrm{e}^{-g_H'(\alpha)} = \frac{\alpha}{1-\alpha} \frac{\prod_{s=1}^{j}(b_s - 1 + \alpha)}{\prod_{s=1}^{i}(a_s - 1 + \alpha)}$ is equal to $tG^{\mathbf{a},\mathbf{b}}(t)$ by Theorem 2.4. □

Remark 4.10. The positivity and invertibility of the rational function $\alpha \mapsto \frac{\alpha}{1-\alpha} \frac{\prod_{s=1}^{j}(b_s - 1 + \alpha)}{\prod_{s=1}^{i}(a_s - 1 + \alpha)}$ on $(0,1)$ is a necessary condition for $\mathcal{H}_n^{\mathbf{a},\mathbf{b}}$ to have nonpositive roots for all large enough $n \in \mathbb{N}$.

Using Theorems 4.6 and 4.7 in [42] we obtain the following corollary.

Corollary 4.11. *The polynomials $\mathcal{H}_n^{\mathbf{a},\mathbf{b}}$ have only nonpositive roots for all $n \geq 1$ and the sequence of probability measures $[\![\mathcal{H}_n^{\mathbf{a},\mathbf{b}}]\!]_n$ converges weakly to a probability measure whose Cauchy transform is as described in Theorem 4.9 in the following scenarios:*

(a) $i, j \geq 0$, $a_1, \ldots, a_i < 0$, $b_1, \ldots, b_j \geq 1$ and i is even;
(b) $j \geq i$, $b_1, \ldots, b_j \geq 1$ and $a_s \geq b_s + 1$, $s \in \{1, 2, \ldots, i\}$;

Remark 4.12. Observe that, if $a_s, b_s \notin [0,1)$ are replaced by sequences $a_s^{(n)}, b_s^{(n)} \notin [0,1)$ converging to a_s, b_s, respectively, then the proof of Theorem 4.9 remains the same and the resulting weak convergence continues to hold for $[\![\mathcal{H}_n^{\mathbf{a}^{(n)},\mathbf{b}^{(n)}}]\!]_n$. Indeed, the Riemann approximation in the proof of Theorem 4.9 persists, since assumption (i) implies uniform (Lipschitz) equicontinuity of the integrand. In this case, also the statement of Corollary 4.11 continues to hold if $b_s \geq 1$ is replaced by $b_s^{(n)} > 1 - 1/n$. Actually, we shall see below that this condition can further be weakened to $b_s^{(n)} \geq 1 - 1/n$, by setting $n^{\underline{n}} = 0$ or formally allowing for $\Gamma(0) = \infty$. We are going to use these facts in the special cases to follow.

Corollary 4.13. *In the setting of Corollary 4.11(a) with $i = 0$, the following holds true. Assume that $b_s^{(n)} \geq 1 - 1/n$, $s = 1, \ldots, j$ and $\lim_{n\to\infty} b_s^{(n)} = b_s \geq 1$, $s = 1, \ldots, j$. Then the polynomials*

$\mathcal{H}_n^{\varnothing,\mathbf{b}^{(n)}}$ *still have nonpositive roots and* $[\![\mathcal{H}_n^{\varnothing,\mathbf{b}^{(n)}}]\!]_n$ *converges weakly to a probability measure with the Cauchy transform* $G^{\varnothing,\mathbf{b}}$.

Proof. Let $\mathbf{b}^{(n)} \in [1-1/n,\infty)^j$ and set $b_0^{(n)} = 1$. Rewrite the hypergeometric polynomial

$$\mathcal{H}_n^{\varnothing,\mathbf{b}^{(n)}}(x) = \sum_{k=0}^{n} \binom{n}{k} n^{-j(n-k)} (\mathbf{b}^{(n)} n)^{\underline{n-k}} x^k = \sum_{k=0}^{n} \frac{n^k}{k!} \prod_{s=0}^{j} \frac{\Gamma(b_s^{(n)} n + 1)}{n^k \Gamma(b_s^{(n)} n - k + 1)} x^{n-k}.$$

Assume now that $b_{s_0}^{(n)} = 1 - 1/n$ for some s_0, then

$$\begin{aligned}\mathcal{H}_n^{\varnothing,\mathbf{b}^{(n)}}(x) &= x \left(\frac{n-1}{n}\right)^{(n-1)j} \sum_{k=0}^{n-1} \frac{(n-1)^k}{k!} \prod_{s=0}^{j} \frac{\Gamma(b_s^{(n)} n + 1)}{(n-1)^k \Gamma(b_s^{(n)} n - k + 1)} \left(\left(\frac{n}{n-1}\right)^j x\right)^{n-1-k} \\ &= x \left(\frac{n-1}{n}\right)^{(n-1)j} \mathcal{H}_{n-1}^{\varnothing,\tilde{\mathbf{b}}^{(n)}} \left(\left(\frac{n}{n-1}\right)^j x\right)\end{aligned}$$

where $\tilde{b}_s^{(n)} = b_s^{(n)}(1 + \frac{1}{n-1})$ and the new vector $\tilde{\mathbf{b}}^{(n)}$ consists of $j-1$ components $\tilde{b}_s^{(n)}$ for $s = 1,\dots,j$, $s \neq s_0$. One may easily check that $\tilde{b}_s^{(n)} > 1 - \frac{1}{n-1}$ and $\tilde{b}_{s_0}^{(n)} = 1$, hence $\mathcal{H}_{n-1}^{\varnothing,\tilde{\mathbf{b}}^{(n)}}$ has nonpositive roots with limiting distribution as given in Corollary 4.11. Thus, the roots of $\mathcal{H}_n^{\varnothing,\mathbf{b}^{(n)}}$ are nonpositive as well and have the same limit distribution as that of $\mathcal{H}_{n-1}^{\varnothing,\tilde{\mathbf{b}}^{(n)}}$. □

4.7. **Laguerre polynomials.** Consider the associated Laguerre polynomials with parameter $\gamma \in \mathbb{R}$:

$$L_n^{(\gamma)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\gamma}{n-k} \frac{x^k}{k!}.$$

The Laguerre polynomial $L_n^{(n\gamma_*)}(-nx) = \frac{n^n}{n!} \mathcal{H}_n^{\varnothing,1+\gamma_*}(x)$ is a rescaled hypergeometric function with $i = 0$, $j = 1$ and $\mathbf{b} = 1 + \gamma_*$. Recall that the Marchenko–Pastur law with parameters $\lambda > 0$ and $\sigma^2 > 0$ is given by

$$\mathsf{MP}_{\sigma^2,\lambda}(\mathrm{d}x) = \lambda^* \delta_0(\mathrm{d}x) + \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{\lambda x} \mathbb{1}_{(\lambda_-,\lambda_+)}(x)\mathrm{d}x, \tag{33}$$

where $\lambda_\pm := \sigma^2(1 \pm \sqrt{\lambda})^2$ and the mass at 0 is $\lambda^* := \max(1 - \lambda^{-1}, 0)$. It is known that the empirical root distribution of rescaled Laguerre polynomials converge to a Marchenko–Pastur law $\mathsf{MP}_{\sigma^2,\lambda}$ with appropriately chosen parameters σ^2 and λ, see [18, Theorem 3.1] or [26, Theorem 5]. According to [5, Lemma 3.11] the Cauchy transform is equal to

$$G_{\mathsf{MP}_{\sigma^2,\lambda}}(t) = \frac{t - \sigma^2(1-\lambda) - \sqrt{(\lambda_+ - t)(\lambda_- - t)}}{2\sigma^2 \lambda t}, \quad t \in \mathbb{C}\backslash[\lambda_-,\lambda_+], \tag{34}$$

where the branch is chosen such that $t \mapsto \sqrt{(\lambda_+ - t)(\lambda_- - t)} \sim t$, as $|t| \to +\infty$.

Corollary 4.14. *Let* $(\gamma_n)_{n\in\mathbb{N}}$ *with* $\gamma_n \in [0,\infty)$ *or* $\gamma_n \in \{-1,\dots,-n+1\}$ *be a sequence such that* $\gamma_* := \lim_{n\to\infty} \frac{1}{n}\gamma_n > -1$. *Then* $[\![L_n^{(\gamma_n)}(nx)]\!]_n$ *converges weakly to a Marchenko–Pastur law with* $\sigma^2 = \gamma_* + 1$, $\lambda = \frac{1}{\gamma_*+1}$ *and* $\lambda_\pm = \gamma_* + 2 \pm 2\sqrt{\gamma_* + 1}$.

Proof. Let us first consider the case $\gamma_n \geq 0$. By Theorem 4.9, the exponential profile g satisfies $\mathrm{e}^{-g'(\alpha)} = \frac{\alpha(\alpha+\gamma_*)}{1-\alpha}$ and $[\![L_n^{(\gamma_n)}(-nx)]\!]_n$ converges weakly to a distribution with the Cauchy transform

$$G(t) := G^{\varnothing,1+\gamma^*}(t) = \frac{-\gamma_* - t + \sqrt{\gamma_*^2 + 2(\gamma_*+2)t + t^2}}{2t},$$

where the branch of $t \mapsto \sqrt{\gamma_*^2 + 2(\gamma_*+2)t + t^2}$ is chosen such that it is asymptotically equivalent to t, as $|t| \to +\infty$. This choice ensures that $G(t) \sim 1/t$, as $|t| \to +\infty$. Comparing the above formula to (34) reveals that G is the Stieltjes transform of $\mathsf{MP}_{1+\gamma_*,(1+\gamma_*)^{-1}}(-\cdot)$.

Assume now that $-\gamma_n \in \{1, 2, \ldots, n-1\}$ and $\gamma_n/n \to \gamma^* \in (-1, 0]$. Using the relation

$$L_n^{(\gamma_n)}(x) = (-x)^{-\gamma_n} \frac{(n+\gamma)!}{n!} L_{n+\gamma_n}^{(-\gamma_n)}(x)$$

we conclude that the roots of $L_n^{(\gamma_n)}(x)$ are nonnegative and

$$[\![L_n^{(\gamma_n)}(nx)]\!]_n = [\![x^{-\gamma_n} L_{n+\gamma_n}^{(-\gamma_n)}(nx)]\!]_n = \frac{-\gamma_n}{n}\delta_0 + \frac{n+\gamma_n}{n}\left[\!\!\left[L_{n+\gamma_n}^{(-\gamma_n)}\left(\frac{n}{n+\gamma_n}(n+\gamma_n)x\right)\right]\!\!\right]_{n+\gamma_n}.$$

According to the case $\gamma_n \geq 0$, we have

$$[\![L_n^{(\gamma_n)}(nx)]\!]_n \xrightarrow[n\to\infty]{w} -\gamma^*\delta_0 + (1+\gamma^*)\mathsf{MP}_{(1+\gamma^*)^{-1},1+\gamma^*}\left((1+\gamma^*)^{-1}(\cdot)\right) = \mathsf{MP}_{1+\gamma_*,(1+\gamma_*)^{-1}}(\cdot),$$

where the last step follows from a straightforward calculation of the rescaled density (33). □

For completeness, let us derive the logarithmic potential of $\mathsf{MP}_{\sigma^2,\lambda}$. It is easy to see, that $G_{\mathsf{MP}_{\sigma^2,\lambda}}$ is a solution to the quadratic equation

$$(\sigma^2\lambda t)x^2 - (t - \sigma^2(1-\lambda))x + 1 = 0$$

and thereupon is an inverse of the function

$$x \mapsto -(1+\sigma^2(1-\lambda)x)/(x(\sigma^2\lambda x - 1)) = \frac{1}{x} - \frac{\sigma^2}{\sigma^2\lambda x - 1}. \tag{35}$$

on an appropriate interval of monotonicity (whose exact form is of no importance for our purposes). Applying (10) with f given by (35) we conclude that

$$\int G_{\mathsf{MP}_{\sigma^2,\lambda}}(t)\mathrm{d}t = tG_{\mathsf{MP}_{\sigma^2,\lambda}}(t) - \log|G_{\mathsf{MP}_{\sigma^2,\lambda}}(t)| + \lambda^{-1}\log|1 - \sigma^2\lambda G_{\mathsf{MP}_{\sigma^2,\lambda}}(t)| + C_{\lambda,\sigma^2}.$$

Thus, for large enough positive real t,

$$\int_{\mathbb{R}} \log(t-x)\mathsf{MP}_{\sigma^2,\lambda}(\mathrm{d}x) = tG_{\mathsf{MP}_{\sigma^2,\lambda}}(t) - \log|G_{\mathsf{MP}_{\sigma^2,\lambda}}(t)| + \lambda^{-1}\log|1 - \sigma^2\lambda G_{\mathsf{MP}_{\sigma^2,\lambda}}(t)| - 1, \tag{36}$$

where the constant $C_{\lambda,\sigma^2} = -1$ was determined using $tG_{\mathsf{MP}_{\sigma^2,\lambda}}(t) \to 1$ and the fact that the right-hand side of the last centered formula must satisfy $\log t + o(1)$, as $t \to +\infty$.

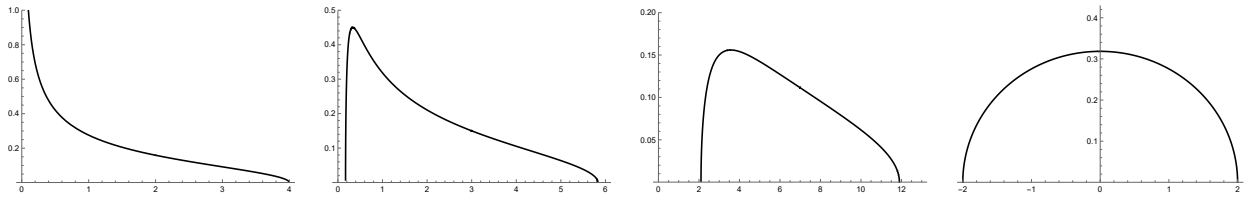

FIGURE 3. The limiting probability densities of zeros of Laguerre polynomials for $\gamma^* \in \{0, 1, 5\}$ (from left to right) and Hermite polynomials (right).

4.8. **Hermite polynomials.** The classical (probabilist) Hermite polynomials are given by

$$\mathrm{He}_n(x) = \mathrm{e}^{-\frac{1}{2}D^2} x^n = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m! 2^m} \cdot \frac{x^{n-2m}}{(n-2m)!}, \tag{37}$$

where we used the notation $D = \frac{\mathrm{d}}{\mathrm{d}x}$. It is known that Hermite polynomials have only real roots and these roots are symmetric around the origin. In particular, they are not all nonpositive and Theorem 2.4 is not directly applicable. Nevertheless, using the following relation between Hermite and Laguerre polynomials

$$\mathrm{He}_n(x) = (-2)^{\lfloor \frac{n}{2} \rfloor} \lfloor \tfrac{n}{2} \rfloor! x^{\epsilon} L_{\lfloor \frac{n}{2} \rfloor}^{(\epsilon - 1/2)}(x^2/2), \tag{38}$$

where $\epsilon = 1$ if n is odd and $\epsilon = 0$ if n is even, we can recover the well known semicircle distribution as the limiting root distribution of Hermite polynomials. Indeed, formula (38) and Corollary 4.14 applied with $\gamma_n = \epsilon - 1/2$ and $\gamma^* = 0$, imply

$$\begin{aligned} [\![\mathrm{He}_n(\sqrt{n}x)]\!]_n &= [\![L_{\lfloor \frac{n}{2} \rfloor}^{(\epsilon - 1/2)}(nx^2/2)]\!]_n + \epsilon \tfrac{1}{n} \delta_0 \\ &= n^{-1} \lfloor \tfrac{n}{2} \rfloor [\![L_{\lfloor \frac{n}{2} \rfloor}^{(\epsilon - 1/2)}(nx^2/2)]\!]_{\lfloor \frac{n}{2} \rfloor} + \epsilon \tfrac{1}{n} \delta_0 \xrightarrow[n \to \infty]{w} \frac{1}{2}\big(\mathsf{MP}_{1,1}(-(\cdot)^2) + \mathsf{MP}_{1,1}((\cdot)^2) \big) = \mathsf{sc}_1. \end{aligned}$$

Here, sc_1 is the standard semi-circle distribution with the density (40) and the last line follows from a direct calculation using that $\mathsf{MP}_{1,1}((\cdot)^2)$ is the quarter-circular law. Thus, we obtain the following

Corollary 4.15. *The weak limit of* $[\![\mathrm{He}_n(\sqrt{n}x)]\!]_n$ *is the standard semicircle distribution* sc_1 *with*

$$\textit{Cauchy transform:} \quad G_{\mathsf{sc}_1}(t) = \frac{1}{2}\big(t - \sqrt{t^2 - 4}\big), \qquad t \in \mathbb{C} \backslash [-2, 2], \tag{39}$$

$$\textit{Lebesgue density:} \quad p_{\mathsf{sc}_1}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}, \qquad x \in [-2, 2], \tag{40}$$

$$\textit{log-potential:} \quad U_{\mathsf{sc}_1}(t) = \frac{1}{4} \operatorname{Re}\left(t^2 - t\sqrt{t^2 - 4} \right) + \log \left| \frac{t + \sqrt{t^2 - 4}}{2} \right| - \frac{1}{2}, \qquad t \in \mathbb{C}. \tag{41}$$

Proof. The convergence to sc_1 has already been proved. Formula (41) follows from (10) by noting that G_{sc_1} is a solution to the equation $x^2 - tx + 1 = 0$, hence is an inverse of $(-\infty, -1) \ni x \mapsto x + x^{-1} \in$

$(-\infty,-2)$. Thus,

$$\int \log(t-x)\mathsf{sc}_1(\mathrm{d}x) = \int G_{\mathsf{sc}_1}(t)\mathrm{d}t \overset{(10)}{=} tG_{\mathsf{sc}_1}(t) - \frac{1}{2}G^2_{\mathsf{sc}_1}(t) - \log|G_{\mathsf{sc}_1}(t)| + C$$
$$= \frac{1}{2}tG_{\mathsf{sc}_1}(t) + \frac{1}{2} - \log|G_{\mathsf{sc}_1}(t)| + C = \frac{1}{4}(t^2 - t\sqrt{t^2-4}) + \frac{1}{2} + \log\left|\frac{t+\sqrt{t^2-4}}{2}\right| + C.$$

The constant $C=-1$ is determined from $U_{\mathsf{sc}_1}(t) = \log t + o(1)$, as $t\to+\infty$. □

Remark 4.16. As we have already mentioned, Theorem 2.4 is not directly applicable because He_n have also positive roots. However, the semicircle limiting distribution for He_n could have been derived directly (without a resort to Laguerre polynomials) by using a trick discussed in Remark 2.6 in conjunction with the classic upper bound for zeros of He_n, see Theorem 6.32 in [60].

4.9. **Jacobi polynomials.** For $u,v>-1$ the Jacobi polynomials $J_n^{(u,v)}$ are defined by

$$J_n^{(u,v)}(x) = \frac{(u+1)^{\overline{n}}}{n!}\,{}_2F_1\left(\begin{matrix}-n, & 1+u+v+n\\ & u+1\end{matrix}; \frac{1-x}{2}\right), \quad n\in\mathbb{N}.$$

In terms of the hypergeometric polynomials this implies (with $i=j=1$)

$$n!\frac{\Gamma((u+v+1)n+1)}{\Gamma((u+v+2)n+1)}J_n^{(un,vn)}(2x+1) = \mathcal{H}_n^{2+u+v,1+u}(x).$$

Thus, part (b) of Corollary 4.11 and the remark after it yield that, for any sequences $u_n, v_n \ge 0$ such that $u_n/n\to u\ge 0, v_n/n\to v\ge 0$,

$$[\![J_n^{(u_n,v_n)}]\!]_n \xrightarrow[n\to\infty]{w} \mu_{J^{(u,v)}},$$

where $\mu_{J^{(u,v)}}$ is the pushforward under $x\mapsto 2x+1$ of the probability measure $\hat{\mu}_{J^{(u,v)}}$ with the Cauchy transform

$$G_{\hat{\mu}_{J^{(u,v)}}}(t) = \frac{-t(u+v)-u+\sqrt{(t(u+v)+u)^2+4(1+u+v)t(1+t)}}{2t(1+t)}.$$

The above formula follows by solving the quadratic equation

$$\mathrm{e}^{-g'(\alpha)} = t \quad\Longleftrightarrow\quad \frac{\alpha(u+\alpha)}{(1-\alpha)(1+u+v+\alpha)} = t \quad\Longleftrightarrow\quad (1+t)\alpha^2 + (t(u+v)+u)\alpha - t(1+u+v) = 0$$

and picking out of two solutions the one satisfying $G_{\hat{\mu}_{J^{(u,v)}}}(t)\sim t^{-1}$ as $t\to+\infty$. Therefore,

$$G_{\mu_{J^{(u,v)}}}(t) = \frac{1}{2}G_{\hat{\mu}_{J^{(u,v)}}}\left(\frac{t-1}{2}\right) = \frac{-t(u+v)-(u-v)+\sqrt{(t(u+v)+(u-v))^2+4(1+u+v)(t^2-1)}}{t^2-1}. \tag{42}$$

From this equation one case deduce the density of $\mu_{J^{(u,v)}}$ by an appeal to Stieltjes-Perron formula (4). We omit these standard calculations and refer to [18] or Section 5.3 in [42].

The special case $u=v=\gamma-1/2$ corresponds to the Gegenbauer polynomials with parameter $\gamma\ge 1/2$, then

$$G_{\mu_{J^{(\gamma-1/2,\gamma-1/2)}}}(t) = \frac{-t(2\gamma-1)+\sqrt{t^2(2\gamma-1)^2+8\gamma(t^2-1)}}{t^2-1}. \tag{43}$$

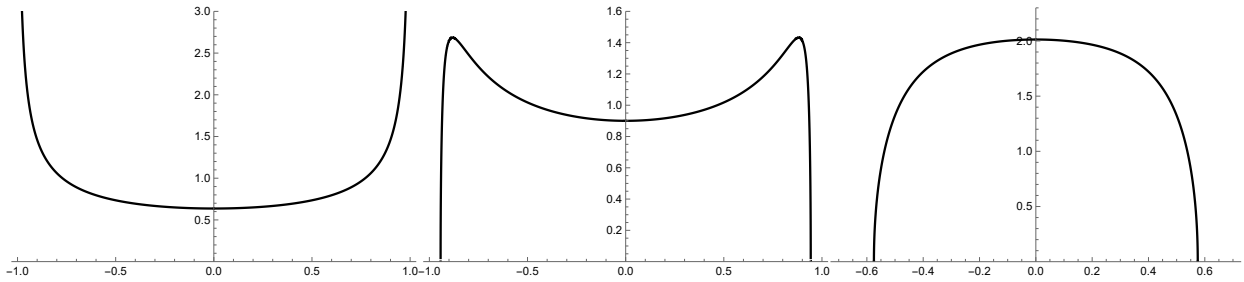

FIGURE 4. The limiting distribution of zeros of Legendre polynomials (left, $\gamma = 1/2$), Gegenbauer polynomials for $\gamma = 1$ (center) and for $\gamma = 5$ (right).

The case $u = v = 0$ corresponds to the Legendre polynomials with

$$G_{\mu_{J^{(0,0)}}}(t) = \frac{2}{\sqrt{t^2-1}}, \quad t \notin [-1,1], \tag{44}$$

which means that $\mu_{J^{(0,0)}}$ is the arcsine distribution with the density $x \mapsto 2\pi^{-1}(1-x^2)^{-1/2}$, $x \in [-1,1]$.

4.10. **Little q-Laguerre polynomials.** The q-analogues of classical orthogonal polynomials can be interpreted as quantized versions of these polynomials, where continuous variables are replaced by discrete ones taking values in a q-geometric lattice $\{q^k\}_{k\geq 0}$, and derivatives become by q-difference operators. We refer to the monograph [35] for an exhaustive overview of all such q-orthogonal polynomials, which are real rooted with roots in the support of the weight function. Their empirical limiting zero distribution can be obtained using exponential profiles, but we restrict ourselves to one particularly illustrative family. One possible q-deformed model of Laguerre polynomials leads to the *little q-Laguerre polynomials* $P_n(x;a|q)$, whose definition involves 'q-factorials' and 'q-hypergeometric series' ${}_2\phi_1$, and is explicitly given by

$$\begin{aligned} P_n(-x;a|q) &= {}_2\,\phi_1\left(\begin{matrix} q^{-n} & 0 \\ aq & \end{matrix}; q, -qx\right) = \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k\,(0;q)_k}{(aq;q)_k\,(q;q)_k}(-qx)^k \\ &= \sum_{k=0}^{\infty}\prod_{j=1}^{k}\frac{q^{j-n}-q}{(1-aq^j)(1-q^j)}x^k = \sum_{k=0}^{n}\prod_{j=1}^{k}\frac{q^{j-n}-q}{(1-aq^j)(1-q^j)}x^k, \quad q\in(0,1), \quad a\in[0,1]. \end{aligned}$$

We chose negative arguments $-x$ so that the coefficients

$$a_{k:n} = \prod_{j=1}^{k}\frac{q^{j-n}-q}{(1-aq^j)(1-q^j)} \geq 0$$

are nonnegative for $a \leq 1 < q^{-1}$. We are interested in a regime when $q \to 1-$. It turns out that a non-trivial profile is obtained if we let $q := \mathrm{e}^{-\lambda/n}$ for some fixed $\lambda > 0$. Then the exponential profile

is, for $\alpha \in (0,1)$,

$$\begin{aligned} g(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log a_{\lfloor \alpha n \rfloor : n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\lfloor \alpha n \rfloor} \log\left(\frac{\mathrm{e}^{-\lambda(j/n-1)} - \mathrm{e}^{-\lambda/n}}{(1-a\mathrm{e}^{-\lambda j/n})(1-\mathrm{e}^{-\lambda j/n})} \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\lfloor \alpha n \rfloor} \log\left(\frac{\mathrm{e}^{-\lambda(j/n-1)} - 1}{(1-a\mathrm{e}^{-\lambda j/n})(1-\mathrm{e}^{-\lambda j/n})} \right) \\ &= \int_0^\alpha \log\left(\frac{\mathrm{e}^{\lambda(1-x)} - 1}{(1-a\mathrm{e}^{-\lambda x})(1-\mathrm{e}^{-\lambda x})} \right) \mathrm{d}x, \end{aligned}$$

by a Riemann summation in the last step. By Theorem 2.4, we have weak convergence of the empirical zero distributions

$$[\![P_n(-x; a|\mathrm{e}^{-\lambda/n})]\!]_n \xrightarrow[n\to\infty]{w} \mu_{a|\lambda}(-\cdot),$$

where $\mu_{a|\lambda}(-\cdot)$ has a Cauchy transform $G^-_{a|\lambda}$ such that $t \mapsto tG^-_{a|\lambda}(t)$ is the inverse of $\alpha \mapsto \mathrm{e}^{-g'(\alpha)}$. Differentiating and exponentiating g yields

$$\mathrm{e}^{-g'(\alpha)} = \frac{(1-a\mathrm{e}^{-\lambda\alpha})(1-\mathrm{e}^{-\lambda\alpha})}{\mathrm{e}^{\lambda}\mathrm{e}^{-\lambda\alpha} - 1} = f(h(\alpha)),$$

where

$$h(\alpha) = \mathrm{e}^{-\lambda\alpha}, \qquad f(x) = \frac{(1-ax)(1-x)}{\mathrm{e}^{\lambda}x - 1}.$$

Observe that $h^{\leftarrow}(x) = -\frac{1}{\lambda}\log x$. Solving a quadratic equation we find the inverse of f as

$$f^{\leftarrow}(t) = \frac{1}{2a}\left(a + 1 + \mathrm{e}^{\lambda}t \pm \sqrt{(a+1+\mathrm{e}^{\lambda}t)^2 - 4a - 4at} \right).$$

Hence, $tG^-_{a|\lambda}(t) = h^{\leftarrow}(f^{\leftarrow}(t))$ implies that the Cauchy transform $G_{a|\lambda}(t) = -G^-_{a|\lambda}(-t)$ of $\mu_{a|\lambda}$ is given by

$$G_{a|\lambda}(t) = -\frac{1}{\lambda t}\log\left[\frac{1}{2a}\left(a + 1 - \mathrm{e}^{\lambda}t + \sqrt{(a+1-\mathrm{e}^{\lambda}t)^2 + 4a(t-1)} \right) \right], \tag{45}$$

where the branch of the square root is chosen such that $G_{a|\lambda}(t) \sim \frac{1}{t}$, as $t \to -\infty$. In order to find the support of $\mu_{a|\lambda}$ observe that the right-hand side of (45) has a branch cut on the positive real line either if

$$(a+1-\mathrm{e}^{\lambda}t)^2 + 4a(t-1) \le 0 \tag{46}$$

or when the argument of the logarithm is negative. By solving the quadratic inequality we find that (46) holds if and only if $t \in [t_-, t_+]$, where

$$t_\pm = \mathrm{e}^{-\lambda}\big(a + 1 - 2a\mathrm{e}^{-\lambda} \pm 2\sqrt{a(1-a\mathrm{e}^{-\lambda})(1-\mathrm{e}^{-\lambda})} \big) = \mathrm{e}^{-\lambda}\big(\sqrt{1-a\mathrm{e}^{-\lambda}} \pm \sqrt{a(1-\mathrm{e}^{-\lambda})} \big)^2.$$

Observe that $0 \le t_- \le t_+$ and, by the AM-GM inequality $t_+ \le 4\mathrm{e}^{-\lambda}\sqrt{a(1-\mathrm{e}^{-\lambda})(1-a\mathrm{e}^{-\lambda})}$. The right-hand side, as a function of $a \in [0,1]$, attains the maximum at $a^* = \mathrm{e}^{\lambda}/2$, if $\mathrm{e}^{\lambda} \le 2$ or at $a^* = 1$, otherwise. In both cases, this yields $t_+ \le 1$. Moreover, for $t \in \mathbb{R} \setminus [t_-, t_+]$, the argument

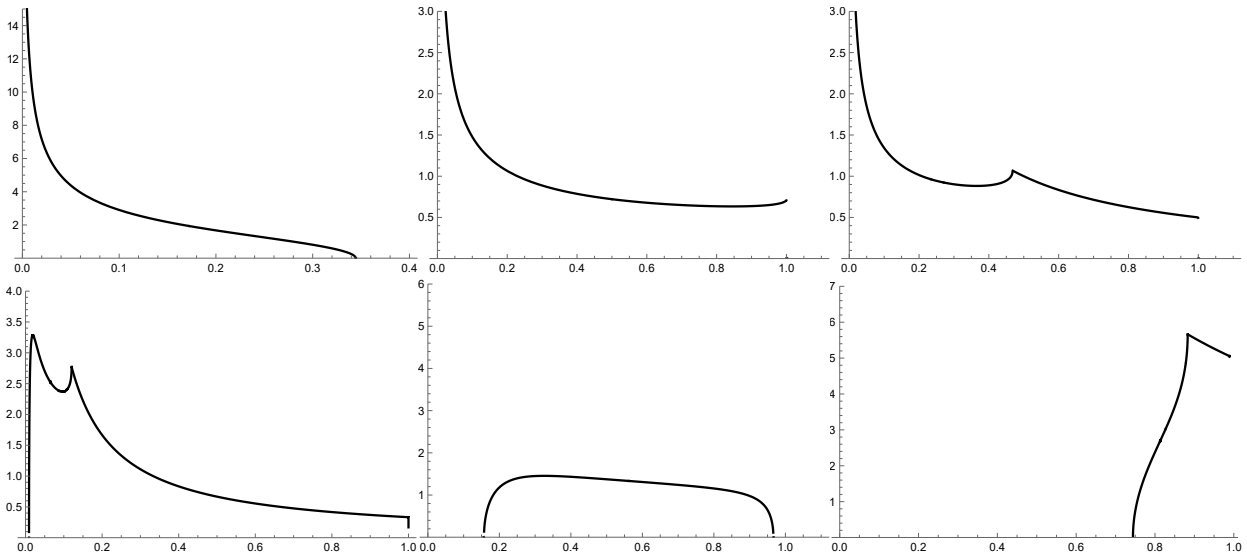

FIGURE 5. The limiting probability densities of zeros of little q-Laguerre polynomials for $a = 1$ in the top row with $\lambda \in \{1/10, \log(2), 2\}$ (from left to right) and in the bottom row for $a = 1/3, \lambda = 3$ (left), $a = 1/2, \lambda = 1/4$ (center) and $a = 1/100, \lambda = 1/5$ (right).

of the logarithm in (45) is nonpositive if and only if $t \geq \mathrm{e}^{-\lambda}(a+1)$ and $t \leq 1$. Using also that $t_- \leq \mathrm{e}^{-\lambda}(a+1)$, we conclude that the natural domain of analyticity of $G_{a|\lambda}$ is

$$\begin{cases} \mathbb{C}\backslash[t_-, t_+], & \text{if } \lambda \leq \log(1+a), \\ \mathbb{C}\backslash[t_-, 1], & \text{if } \lambda > \log(1+a). \end{cases}$$

By the inversion formula (4) we conclude that the density $p_{a|\lambda}(x) = -\frac{1}{\pi}\lim_{y\to 0}\operatorname{Im} G_{a|\lambda}(x+\mathrm{i}y)$ of μ on $[0,+\infty)$ is

$$p_{a|\lambda}(x) = \begin{cases} \frac{1}{\lambda x}, & \text{if } x \in [t_+, 1], \quad \lambda > \log(1+a), \\ \frac{1}{\lambda \pi x} \arg\left(a+1-\mathrm{e}^{\lambda}x + \mathrm{e}^{\lambda}\mathrm{i}\sqrt{(x-t_-)(t_+-x)}\right), & \text{if } x \in [t_-, t_+], \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

Here an additional term appearing when $\lambda > \log(1+a)$ is due to the logarithm crossing the branch cut. In this case $\log(w) = \log|w| + i(\arg(w) \pm \pi\epsilon)$ and $\epsilon \in \{0,1\}$ depends on the approaching direction.

Let us now discuss some illustrative simplifications. Set $a = 1$, then $t_- = 0$ and consider the symmetrized distribution that is obtained under the push-forward of $x \mapsto \pm\sqrt{x}$ (similar to the link (38)). The corresponding density is

$$\tilde{p}_\lambda(x) = \frac{1}{\lambda|x|}\Big(\frac{1}{\pi}\arg\left(2-\mathrm{e}^{\lambda}x^2 + \mathrm{e}^{\lambda}\mathrm{i}x\sqrt{t_+ - x^2}\right)\mathbb{1}_{[0,t_+]}(x^2) + \mathbb{1}_{[t_+,1]}(x^2)\Big)$$

where $t_+ = 4\mathrm{e}^{-\lambda}(1-\mathrm{e}^{-\lambda})$. Interestingly, this is precisely the density of q-deformed GUE matrices, that has been uncovered in [12]. Therein, discrete q-Hermite I polynomials have been investigated,

which share the same exponential profile as $P_n(-x^2;1|q)$ and, hence, the same limiting empirical zero distribution.

Moreover, we can recover the continuous analogue in the $\lambda \to 0$ limit. Indeed, zooming into $\mu_{1|\lambda}$ as $\hat{\mu}_{1|\lambda} = \mu_{1|\lambda}(\lambda\cdot)$, the corresponding Cauchy transform satisfies $\hat{G}_{1|\lambda}(t) = \lambda G_{1|\lambda}(\lambda t)$. Using standard approximations like $\mathrm{e}^{\lambda} = 1 + \lambda + o(\lambda)$, $\log(1+w) = w + o(w)$, we obtain from (45)

$$\lim_{\lambda\to 0+} \hat{G}_{1|\lambda}(t) = -\lim_{\lambda\to 0} \frac{1}{\lambda t} \log\Big[\frac{1}{2}\Big(2 - \mathrm{e}^{\lambda}\lambda t + \sqrt{(2-\mathrm{e}^{\lambda}\lambda t)^2 + 4(\lambda t - 1)}\Big)\Big] = \frac{1}{2t}\Big(t - \sqrt{t(t-4)}\Big).$$

The right-hand side is the Cauchy transform of the standard Marchenko–Pastur law $\mathsf{MP}_{1,1}$, see (34).

5. Polynomials related to free probability

5.1. **Polynomials related to ⊠-infinitely divisible distributions.** For any two probability measures ν_1 and ν_2 on $[0,\infty)$, their multiplicative free convolution $\nu_1 \boxtimes \nu_2$ is another probability measure on $[0,\infty)$ defined by Voiculescu in [61, 62]. The class of probability measures that are infinitely divisible with respect to $\boxtimes$ has been described by Bercovici and Voiculescu [8, 9] in terms of a Lévy–Khintchine-type representation. The basic building blocks of these ⊠-divisible distributions are the free multiplicative analogues of the normal and the Poisson distributions. In this section, we introduce polynomials which may be considered as finite free probability [41, 40] analogues of these basic ⊠-divisible distributions in the same way as Hermite (respectively, Laguerre) polynomials are finite free analogues of the free normal (semicircle) and free Poisson (Marchenko–Pastur) distributions. In particular, the empirical zero distributions of the polynomials we introduce converge to ⊠-divisible probability measures.

Let us first briefly recall some notions from free probability; see, for example, [63, § 3.6], [6, 9]. Let $\mathcal{M}_{[0,\infty)}$ be the set of probability measures on $[0,\infty)$. The *ψ-transform* of a probability measure $\nu \in \mathcal{M}_{[0,\infty)}$ is the function $\psi_\nu(z)$ defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by

$$\psi_\nu(z) = \int_{[0,\infty)} \frac{uz}{1-uz}\nu(\mathrm{d}u), \qquad z \in \mathbb{D}. \tag{48}$$

The *S-transform* and the *Σ-transform* of $\nu \neq \delta_0$ are defined by

$$S_\nu(z) = \frac{1+z}{z}\psi_\nu^{-1}(z), \qquad \Sigma_\nu(z) = S_\nu\Big(\frac{z}{1-z}\Big) = \frac{1}{z}\psi_\nu^{-1}\Big(\frac{z}{1-z}\Big), \tag{49}$$

respectively, if $|z|$ is sufficiently small. The free multiplicative convolution of $\nu_1, \nu_2 \in \mathcal{M}_{[0,\infty)}\backslash\{\delta_0\}$ can be characterized by

$$S_{\nu_1\boxtimes\nu_2}(z) = S_{\nu_1}(z)S_{\nu_2}(z), \qquad \Sigma_{\nu_1\boxtimes\nu_2}(z) = \Sigma_{\nu_1}(z)\Sigma_{\nu_2}(z). \tag{50}$$

The class of ⊠-infinitely divisible distributions has been described in [8, 9] as follows; see also [4, Theorem 2.3]. A probability measure $\nu \neq \delta_0$ on $[0,\infty)$ is ⊠-infinitely divisible if and only if its Σ-transform admits a Lévy–Khintchine-type representation

$$\Sigma_\nu(z) = \exp(v_\nu(z)), \quad v_\nu(z) = c_0 + \int_{[0,+\infty]} \frac{1+tz}{z-t}\rho(\mathrm{d}t), \quad z \in (-\infty, 0), \tag{51}$$

for some finite measure ρ on $[0,+\infty]$ and some $c_0 \in \mathbb{R}$. If ρ has an atom at $+\infty$, we interpret $\frac{1+tz}{z-t}$ at $t = +\infty$ as $-z$.

In finite free probability [40, 41], the analogue of $\boxtimes$ is an operation $\boxtimes_n$ defined on polynomials of degree at most n as follows:

$$\sum_{k=0}^{n} a'_k x^k \boxtimes_n \sum_{k=0}^{n} a''_k x^k = \sum_{k=0}^{n} (-1)^{n-k} \frac{a'_k a''_k}{\binom{n}{k}} x^k.$$

Next we state a general result allowing to construct sequences of polynomials whose empirical zero distributions converge.

Proposition 5.1. *For every $n \in \mathbb{N}$ let $Q_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_{k:n} x^k$ be a polynomial with real nonnegative roots and $b_{0:n} \geq 0, \ldots, b_{n:n} \geq 0$. Suppose that for every $\alpha \in (0,1)$, we have*

$$\lim_{n\to\infty} b_{\lfloor \alpha n \rfloor : n} = \mathrm{e}^{\tilde{g}(\alpha)}$$

with some function $\tilde{g} : (0,1) \to \mathbb{R}$. Then, the polynomials

$$P_n(x) := \underbrace{Q_n(x) \boxtimes_n \cdots \boxtimes_n Q_n(x)}_{n \text{ times}} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_{k:n}^n x^k$$

also have real nonnegative roots and $[\![P_n]\!]_n$ converges weakly on $[0,\infty)$ to a probability measure ν with

$$S_\nu(t) = \mathrm{e}^{\tilde{g}'(1+t)}, \quad t \in (-1,0) \qquad \Sigma_\nu(z) = \mathrm{e}^{\tilde{g}'\left(\frac{1}{1-z}\right)}, \quad z \in (-\infty, 0).$$

Proof. The fact that all roots of P_n are nonnegative follows from a theorem of Szegő, see [41, Theorem 1.6]. The sequence $((-1)^n P_n(-x))_{n\in\mathbb{N}}$ has exponential profile

$$g(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log \left(\binom{n}{\lfloor \alpha n \rfloor} b_{\lfloor \alpha n \rfloor : n}^n \right) = \tilde{g}(\alpha) - \alpha \log \alpha - (1-\alpha) \log(1-\alpha), \qquad \alpha \in (0,1).$$

Recall that the profile is automatically infinitely differentiable. Its derivative is $g'(\alpha) = \log(\frac{1-\alpha}{\alpha}) + \tilde{g}'(\alpha)$. By Theorem 2.4, $[\![P_n(-x)]\!]_n$ converges weakly to some probability measure μ on $[-\infty, 0]$ such that $\alpha \mapsto \mathrm{e}^{-g'(\alpha)}$ is the inverse of $t \mapsto tG_\mu(t)$, where G_μ is the Cauchy transform of μ. Moreover, we have $\mu(\{0\}) = \mu(\{-\infty\}) = 0$ since the profile is defined on the whole of $(0,1)$. Let μ_+ be a reflection of μ with respect to 0 given by $\mu_+(A) := \mu(-A)$, for Borel sets $A \subseteq \mathbb{R}$. Now, Lemma 6.1 in [31] states that the S-transform S_{μ_+} of μ_+ is related to the profile g by

$$S_{\mu_+}(t) = -\frac{1+t}{t} \mathrm{e}^{g'(1+t)}, \qquad t \in (-1,0). \tag{52}$$

Thus, $[\![P_n]\!]_n$ converges weakly on $[0,\infty)$ to the probability measure $\nu := \mu_+$ with the S- and Σ-transforms

$$S_\nu(t) = -\frac{1+t}{t} \mathrm{e}^{g'(1+t)} = \mathrm{e}^{\tilde{g}'(1+t)}, \quad \Sigma_\nu(z) = S_\nu\left(\frac{z}{1-z}\right) = \mathrm{e}^{\tilde{g}'\left(\frac{1}{1-z}\right)},$$

for all $t \in (-1,0)$ and $z \in (-\infty, 0)$. Note that ν is determined uniquely by its S- and Σ-transforms on these intervals since these transforms are analytic inside some disk centered at 0. □

Remark 5.2. We are mostly interested in the case when ν is a $\boxtimes$-infinitely divisible distribution on $[0,\infty)$ with representation (51). Then, $\tilde{g}'(\alpha) = v_\nu(1-1/\alpha)$.

In the following, we shall construct explicit polynomials related to the free multiplicative normal and Poisson distributions. These distributions are $\boxtimes$-infinitely divisible with $\rho = a\delta_1$, $a > 0$ (in the normal case) and $\rho = a\delta_t$, $t \in [0,+\infty]\backslash\{1\}$, $a > 0$ (in the Poisson case).

5.1.1. *Free multiplicative Hermite polynomials.* For $\sigma > 0$ consider the polynomials

$$G_n(x;\sigma^2) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathrm{e}^{\frac{\sigma^2 k(n-k)}{2}} x^k, \qquad n \in \mathbb{N}.$$

Theorem 5.3. *For every $\sigma > 0$ and $n \in \mathbb{N}$, all zeros of $G_n(x;\sigma^2)$ are real and positive. The empirical distribution of zeros of $G_n(x;\sigma^2/n)$ converges, as $n \to \infty$, to a probability measure ν_{σ^2} on $[0,\infty)$ with*

$$S_{\nu_{\sigma^2}}(t) = \mathrm{e}^{-\sigma^2(t+\frac{1}{2})}, \quad t \in (-1,0), \qquad \Sigma_{\nu_{\sigma^2}}(z) = \mathrm{e}^{\frac{\sigma^2}{2}\frac{z+1}{z-1}}, \quad z \in (-\infty,0).$$

Remark 5.4. The distribution ν_{σ^2} is called the free multiplicative normal distribution with parameter σ^2. It appeared in [8, Lemma 7.1] and is $\boxtimes$-infinitely divisible with $\rho = \frac{1}{2}\sigma^2\delta_1$ and $c_0 = 0$ in (51). The moments and the density of ν_{σ^2} have been determined by Biane [10, 11]. For more information, see [66], [65, Sections 4,5], [2, 17, 16].

Remark 5.5. For a fixed $n \in \mathbb{N}$, the polynomials $G_n(x;\sigma^2)$, $\sigma^2 > 0$, form a semigroup with respect to $\boxtimes_n$ in the sense that $G_n(x;\sigma_1^2) \boxtimes_n G_n(x;\sigma_2^2) = G_n(x;\sigma_1^2+\sigma_2^2)$ for all $\sigma_1^2, \sigma_2^2 > 0$.

We need some preparations before we can start with the proof of Theorem 5.3. An entire function $f(z)$ belongs to the Laguerre–Pólya class if it possesses a representation as a Hadamard product

$$f(z) = Cz^m \mathrm{e}^{cz-\frac{\sigma^2}{2}z^2} \prod_{j=1}^{N} \Big(1-\frac{z}{x_j}\Big)\mathrm{e}^{\frac{z}{x_j}} \tag{53}$$

for some $C, c \in \mathbb{R}$, $\sigma^2 \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $N \in \mathbb{N}\cup\{\infty\}$ and $x_j \in \mathbb{R}\backslash\{0\}$ satisfying $\sum_{j=1}^{N} x_j^{-2} < \infty$. The following theorem of Laguerre can be found in [53, Section 5.7 and Theorem 5.6.12] or [64, 25:Theorem].

Theorem 5.6 (Laguerre)**.** *If $f(z)$ is a function from the Laguerre–Pólya class with only negative zeros and if all zeros of the polynomial $p(x) = \sum_{k=0}^{n} c_k x^k \in \mathbb{R}[x]$ are real, then all zeros of the polynomial $\sum_{k=0}^{n} f(k) c_k x^k$ are also real.*

Remark 5.7. Since all zeros of the polynomial $p(x) = (x-1)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k$ are real, Laguerre's theorem implies that $\sum_{k=0}^{n}(-1)^{n-k} f(k)\binom{n}{k}x^k$ is real-rooted.

Proof of Theorem 5.3. The function $f(z) = \mathrm{e}^{\frac{1}{2}\sigma^2 z(n-z)}$ belongs to the Laguerre–Pólya class by (53). Remark 5.7 implies that all zeros of $G_n(x;\sigma^2) = \sum_{k=0}^{n}(-1)^{n-k} f(k)\binom{n}{k}x^k$ are real. Since the coefficients of $G_n(x;\sigma^2)$ alternate signs and $G_n(0;\sigma^2) \neq 0$, all roots are in fact positive.

To identify the weak limit of $[\![G_n(x;\sigma^2/n)]\!]_n$, we write $G_n(x;\sigma^2/n) = \sum_{k=0}^n (-1)^{n-k}\binom{n}{k} b_{k:n}^n x^k$ with $b_{k:n} = \mathrm{e}^{\frac{\sigma^2}{2}\frac{k}{n}(1-\frac{k}{n})}$ and observe that $b_{\lfloor \alpha n\rfloor:n} \to \mathrm{e}^{\frac{\sigma^2}{2}\alpha(1-\alpha)}$ for every $\alpha \in (0,1)$. We can therefore apply Proposition 5.1 with $\tilde g(\alpha) = \frac{\sigma^2}{2}\alpha(1-\alpha)$ and $\tilde g'(\alpha) = \sigma^2(\frac12 - \alpha)$. It follows that $[\![G_n(x;\sigma^2/n)]\!]_n$ converges weakly to a probability measure $\nu = \nu_{\sigma^2}$ with

$$S_\nu(t) = \mathrm{e}^{\tilde g'(1+t)} = \mathrm{e}^{-\sigma^2(t+\frac12)}, \qquad \Sigma_\nu(z) = \mathrm{e}^{\tilde g'(\frac{1}{1-z})} = \mathrm{e}^{\sigma^2(\frac12 - \frac{1}{1-z})} = \mathrm{e}^{\frac{\sigma^2}{2}\frac{z+1}{z-1}},$$

for all $t \in (-1,0)$ and $z \in (-\infty, 0)$. □

Remark 5.8. The polynomials $G_n(x;\sigma^2)$ are similar to unitary Hermite polynomials [33, 48] defined by

$$H_n(x;\sigma^2) := \sum_{k=0}^n (-1)^{n-k}\binom{n}{k}\mathrm{e}^{-\frac{\sigma^2 k(n-k)}{2}} x^k,$$

with the only difference being the sign in the exponent. It is known that all zeros of $H_n(x;\sigma^2)$ lie on the unit circle, see Lemma 2.1 in [33]. The empirical distribution of zeros of $H_n(x;\sigma^2/n)$ converges, as $n\to\infty$, to the free *unitary* normal distribution on the unit circle with parameter σ^2, see Theorem 2.2 in [33].

5.1.2. *Polynomials related to free multiplicative Poisson distributions.* For $n \in \mathbb{N}$ and parameters $b \in \mathbb{R}\backslash(-n,0)$, $c \ge 0$ we consider the polynomial

$$P_n(x;b,c) = \sum_{k=0}^n (-1)^{n-k}\binom{n}{k}|k+b|^c x^k.$$

Lemma 5.9. *Let $c \in \mathbb{N}_0$ and either $b \ge 0$ or $b \le -n$. Then, all roots of $P_n(x;b,c)$ are real.*

Proof. We apply Theorem 5.6 and Remark 5.7. Let first $b \ge 0$. Then, for all $k \in \{0,\dots,n\}$ we have $|k+b|^c = (k+b)^c$. Since $f(z) := (z+b)^c$ is a polynomial with nonpositive roots (in particular, $f(z)$ belongs to the Laguerre–Pólya class), Laguerre's theorem applies and proves that $P_n(x;b,c)$ is real-rooted. Let now $b \le -n$. Then, for all $k \in \{0,\dots,n\}$ we have $|k+b|^c = (-b-k)^c$. Since $f(z) := (-b-z)^c$ is a polynomial with nonpositive roots, Laguerre's theorem implies that $P_n(x;b,c)$ is real-rooted. A different proof of the lemma will be given in Remark 5.13. □

Remark 5.10. The assumption that c is integer is essential. For example, the polynomial $P_3(x;0,0.5)$ is not real-rooted. On the other hand, numerical simulations support the following conjecture. The polynomial $P_n(x;b,c)$ is real-rooted for every $n \in \mathbb{N}$, $b \in \mathbb{R}\backslash(-n,0)$ and every real $c \ge n$.

Remark 5.11. The polynomials $P_n(x;b,c)$, $c \ge 0$, form a semigroup with respect to $\boxtimes_n$ in the sense that $P_n(x;b,c') \boxtimes_n P_n(x;b,c'') = P_n(x;b,c'+c'')$ for all $c', c'' \ge 0$.

Theorem 5.12. *Fix $\beta \in \mathbb{R}\backslash(-1,0)$ and $\gamma > 0$. Let $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}_0$ be sequences such that $b_n/n \to \beta$ and $c_n/n \to \gamma$ as $n\to\infty$. Let also $b_n \in \mathbb{R}\backslash(-n,0)$ for every $n \in \mathbb{N}$. Then, $[\![P_n(x;b_n,c_n)]\!]_n$ converges weakly to a probability measure $\nu = \nu_{\beta,\gamma}$ on $[0,\infty)$ with*

$$S_\nu(t) = \mathrm{e}^{\frac{\gamma}{t+\beta+1}}, \quad t \in (-1,0), \qquad \Sigma_\nu(z) = \mathrm{e}^{\frac{\gamma(1-z)}{1+\beta(1-z)}}, \quad z \in (-\infty, 0).$$

Proof. Write $n^{-c_n}P_n(x;b_n,c_n) = \sum_{k=0}^n (-1)^{n-k}\binom{n}{k}b_{k:n}^n x^k$ with $b_{k:n} = |(k+b_n)/n|^{c_n/n}$. Note that $b_{\lfloor \alpha n\rfloor:n} \to |\alpha+\beta|^\gamma$ as $n\to\infty$, for all $\alpha\in(0,1)$. We can apply Proposition 5.1 with $\tilde{g}(\alpha) = \gamma\log|\alpha+\beta|$. If $\beta \geq 0$, then $\tilde{g}(\alpha) = \gamma\log(\alpha+\beta)$, whereas for $\beta\leq -1$ we have $\tilde{g}(\alpha) = \gamma\log(-\alpha-\beta)$. In both cases, $\tilde{g}'(\alpha) = \gamma/(\alpha+\beta)$. It follows from Proposition 5.1 that $[\![P_n(x;b_n,c_n)]\!]_n$ converges weakly to a probability measure $\nu = \nu_{\beta,\gamma}$ with

$$S_\nu(t) = \mathrm{e}^{\tilde{g}'(1+t)} = \mathrm{e}^{\frac{\gamma}{t+\beta+1}}, \qquad \Sigma_\nu(z) = \mathrm{e}^{\tilde{g}'(\frac{1}{1-z})} = \mathrm{e}^{\frac{\gamma}{\frac{1}{1-z}+\beta}} = \mathrm{e}^{\frac{\gamma(1-z)}{1+\beta(1-z)}},$$

for all $t\in(-1,0)$ and $z\in(-\infty,0)$. □

The distribution $\nu_{\beta,\gamma}$ appeared in [8, Lemma 7.2]. It is the free mutiplicative analogue of the Poisson distribution. If $\beta\neq 0$, then it is $\boxtimes$-infinitely divisible with $\rho = \frac{\gamma}{\beta}\frac{t_0-1}{t_0^2+1}\delta_{t_0}$ and $c_0 = \frac{\gamma}{\beta}\frac{t_0+1}{t_0^2+1}$ in (51), where $t_0 = \beta^{-1}+1$. If $\beta = 0$, it is $\boxtimes$-infinitely divisible as well with $c_0 = \gamma$ and $\rho = \gamma\delta_{+\infty}$.

Remark 5.13. The polynomials $P_n(x;b,c)$ can be considered as a finite free *multiplicative* analogue of the Poisson distribution in the same way as the Laguerre polynomials are the finite free *additive* analogue of the Poisson distribution [40]. Let us provide motivation for this. Let U_n be a random vector with uniform distribution on the unit sphere in $\mathbb{R}^n$. Consider the linear operator $\mathbf{A}_n : \mathbb{R}^n \to \mathbb{R}^n$ given by $\mathbf{A}_n := \mathbf{Id}_n + (\mu-1)U_nU_n^T$, which satisfies $\mathbf{A}_n v = v$ for $v\in U_n^\perp$ and $\mathbf{A}_nU_n = \mu U_n$, where $\mu\geq 0$, $\mu\neq 1$, and $U_n^\perp$ denotes the orthogonal complement of the one-dimensional subspace spanned by U_n. The characteristic polynomial of $\mathbf{A}_n$ is deterministic and given by

$$\begin{aligned}\det(\mathbf{A}_n - \mathbf{Id}_n x) &= (1-x)^{n-1}(\mu-x) = \sum_{\ell\in\mathbb{Z}}(-1)^{\ell+1}\binom{n-1}{\ell}x^\ell(x-\mu)\\ &= \sum_{k\in\mathbb{Z}}(-1)^k x^k\left(\binom{n-1}{k-1}+\mu\binom{n-1}{k}\right) = \sum_{k\in\mathbb{Z}}(-1)^k x^k\binom{n}{k}\left(\mu+(1-\mu)\frac{k}{n}\right)\\ &= \frac{1-\mu}{n}(-1)^n(\operatorname{sgn} b)P_n(x;b,1), \quad b := \frac{\mu n}{1-\mu}\in\mathbb{R}\backslash[-n,0).\end{aligned}$$

It follows that $P_n(x;b,1)$ is real-rooted, which by Remark 5.11 and Szegő's theorem [41, Theorem 1.6] implies that $P_n(x;b,c)$ is real-rooted for every $c\in\mathbb{N}$. This gives another proof of Lemma 5.9 (assuming $b\neq -n$).

Note that the normalized spectral measure of $\mathbf{A}_n$ is $\frac{n-1}{n}\delta_1 + \frac{1}{n}\delta_\mu$. The classical Poisson limit theorem implies that the classical $\lfloor\lambda n\rfloor$-th convolution power of $\frac{n-1}{n}\delta_0+\frac{1}{n}\delta_m$ converges, as $n\to\infty$, to $m\cdot\xi$, where the random variable ξ has Poisson distribution with parameter $\lambda>0$ and $m\neq 0$. For a finite free multiplicative analogue of this result, let $\mathbf{A}_{1;n},\dots,\mathbf{A}_{c;n}$ be c independent realizations of $\mathbf{A}_n$, where $c\in\mathbb{N}$, and note that their distribution is invariant under conjugation with a Haar-unitary matrix. By [41, Theorem 1.5], the expected characteristic polynomial of the product $\mathbf{A}_{1;n}\dots\mathbf{A}_{c;n}$

is

$$\begin{aligned}\mathbb{E}\det(\mathbf{A}_{1;n}\dots\mathbf{A}_{c;n}-\mathbf{Id}_n x) &= (-1)^n\mathbb{E}\det(\mathbf{Id}_n x-\mathbf{A}_{1;n}\dots\mathbf{A}_{c;n})\\ &= (-1)^n\underbrace{(\mathbb{E}\det(\mathbf{Id}_n x-\mathbf{A}_n)\boxtimes_n\cdots\boxtimes_n(\mathbb{E}\det(\mathbf{Id}_n x-\mathbf{A}_n))}_{c\text{ times}}\\ &= (-1)^n\left(\frac{1-\mu}{n}\operatorname{sgn} b\right)^c\underbrace{P_n(x;b,1)\boxtimes_n\cdots\boxtimes_n P_n(x;b,1)}_{c\text{ times}}\\ &= (-1)^n\left(\frac{1-\mu}{n}\operatorname{sgn} b\right)^c P_n(x;b,c),\end{aligned}$$

where the last step follows from the semigroup property stated in Remark 5.11. This explains why it is natural to expect the free multiplicative Poisson distribution to appear as a limit in Theorem 5.12. Another approach to a free multiplicative analogue of the Poisson limit theorem can be found in [3, Section 10.5]. For polynomials that are similar to $P_n(x;b,c)$ but have zeros on the unit circle, see [32]. Sidi polynomials studied in [38, 39, 54, 55, 56] are also closely related to $P_n(x;b,c)$.

5.2. **Special polynomials generated by differential operators of hypergeometric type.** In this and the next subsection, we explore two distinct generalizations of the formula $\mathrm{He}_n(x) = \mathrm{e}^{-\frac{1}{2}D^2}x^n$ each yielding real-rooted polynomials that can be expressed as differential flows of monomials. First, we turn to differential operators of the form e^{-cnH} for some $c>0$ and a word $H=\prod_s B_s$ of operators B_s, each of which either decreases the degree of a polynomial by one or preserves the degree. More precisely, fix a parameter $m\in\mathbb{N}$ describing the degree decrease of H. For $j+1\geq m$, $b_1,\dots,b_j\in[1,+\infty)$, $b_0=1$ and define a collection of operators

$$\begin{aligned}B_s &:= \frac{D}{nm}+\frac{b_s-1}{x}, \quad 0\leq s<m,\\ B_s &:= \frac{xD}{nm}+b_s-1, \quad m\leq s\leq j.\end{aligned}$$

Thus, the operators $B_0=\frac{D}{nm}, B_1,\dots,B_{m-1}$ are degree decreasing and the operators $B_m,\dots,B_j$ are degree preserving. The following formula holds true

$$B_s x^\ell = \left(\frac{\ell}{nm}+b_s-1\right)x^{\ell-1+1_{\{s\geq m\}}}, \quad s=0,\dots,j, \quad \ell\geq 1. \tag{54}$$

A somewhat unusual choice of scaling will become apparent from the result below. As we shall see, the order in which we apply the operators B_s, $s=0,1,\dots,j$, is irrelevant in the limit, provided (for technical reasons) only that the first operator is the differentiation $B_0=D/nm$. Hence, we fix a permutation σ of $\{1,2,\dots,j\}$ and set

$$H = B_{\sigma(j)}\dots B_{\sigma(1)}B_0. \tag{55}$$

Using the notation $t_i := \sum_{k=1}^{i} 1_{\{\sigma(k)\geq m\}}$, $i = 0, \ldots, m$, formula (54) implies

$$Hx^{\ell} = \frac{\ell}{nm} \prod_{s=1}^{j} \left(\frac{\ell - 1 + t_{s-1}}{nm} + b_{\sigma(s)} - 1 \right) x^{\ell - m}, \quad \ell \geq m. \tag{56}$$

We shall show below that under the stated notation and assumptions, for all sufficiently large n, we have

$$\mathrm{e}^{-cnH} x^{nm} = (-c)^n \mathcal{H}_n^{\varnothing, \mathbf{b}^{(\mathbf{n})}}(-x^m/c) \tag{57}$$

for some appropriate $\mathbf{b}^{(n)} \in [1 - 1/n, +\infty)^j$ such that $\mathbf{b}^{(n)} \to (b_1, \ldots, b_j) =: \mathbf{b}$, as $n \to \infty$. By Corollary 4.13 the roots of $\mathcal{H}_n^{\varnothing, \mathbf{b}^{(\mathbf{n})}}$ are nonpositive, hence the zeros of $\mathrm{e}^{-cnH} x^{nm}$ are concentrated on the rays $\{r\mathrm{e}^{2\pi i/m} : r \geq 0\}$ through the mth roots of unity. The function $S(x) = -x^m/c$ maps the union of these rays to the closed negative half-line. Thus, (57) implies

$$[\![\mathrm{e}^{-cnH} x^{nm}]\!]_{nm} \circ S^{-1} = [\![\mathcal{H}_n^{\varnothing, \mathbf{b}^{(\mathbf{n})}}(x)]\!]_n.$$

Corollary 4.13 yields the following result.

Theorem 5.14. *Under the stated notation and assumptions, we have the convergence of the push-forwards measures*

$$[\![\mathrm{e}^{-cnH} x^{nm}]\!]_{nm} \circ S^{-1} \xrightarrow[n\to\infty]{w} \mu, \tag{58}$$

where μ is a probability measure on $(-\infty, 0]$ with the Cauchy transform $G^{\varnothing, \mathbf{b}}$ such that the inverse of $t \mapsto tG^{\varnothing, \mathbf{b}}(t)$ is $\alpha \mapsto \frac{\alpha}{1-\alpha} \prod_{s=1}^{j} (b_s - 1 + \alpha)$.

Proof. We only need to check (57). The hypergeometric polynomial on the right hand side can be rewritten as

$$\begin{aligned}(-c)^n \mathcal{H}_n^{\varnothing, \mathbf{b}^{(n)}}(-x^m/c) &= \sum_{k=0}^{n} \binom{n}{k} n^{-j(n-k)} (\mathbf{b}^{(n)} n)^{\underline{n-k}} (-c)^{n-k} x^{mk} \\ &= \sum_{k=0}^{n} \frac{(-c)^k n^k}{k!} \prod_{s=0}^{j} \frac{\Gamma(b_s^{(n)} n + 1)}{n^k \Gamma(b_s^{(n)} n - k + 1)} x^{m(n-k)}. \end{aligned} \tag{59}$$

On the other hand

$$\mathrm{e}^{-cnH} x^{nm} = \sum_{k=0}^{\infty} \frac{(-c)^k n^k}{k!} H^k x^{nm} = \sum_{k=0}^{n} \frac{(-c)^k n^k}{k!} H^k x^{nm},$$

because each application of H reduces the degree of x^{nm} by m. Hence $H^n x^{nm}$ is a constant, $B_0 H^n x^{nm} = 0$ and, therefore, $H^{n+1} x^{nm} = 0$. Thus, it remains to prove that

$$H^k x^{nm} = \prod_{s=0}^{j} \frac{\Gamma(b_s^{(n)} n + 1)}{n^k \Gamma(b_s^{(n)} n - k + 1)} x^{m(n-k)}, \quad k = 0, \ldots, n, \tag{60}$$

for some choice of $b_s^{(n)}$, $s = 1, \dots, j$ which depends on the permutation σ. Iterating formula (56) we conclude that

$$\begin{aligned} H^k x^{nm} &= \left(\prod_{\ell=0}^{k-1} \frac{m(n-\ell)}{nm}\right) \prod_{s=1}^{j} \prod_{\ell=0}^{k-1} \left(\frac{m(n-\ell)-1+t_{s-1}}{nm} + b_{\sigma(s)} - 1\right) x^{m(n-k)} \\ &= \left(\prod_{\ell=0}^{k-1} \frac{m(n-\ell)}{nm}\right) \prod_{s=1}^{j} \prod_{\ell=0}^{k-1} \left(\frac{m(n-\ell)-1+t_{\sigma^{-1}(s)-1}}{nm} + b_s - 1\right) x^{m(n-k)}. \end{aligned}$$

The first product on the right-hand side is equal to $n^{-k}\Gamma(b_0^{(n)}n+1)/\Gamma(b_0^{(n)}n-k+1)$ if we set $b_0^{(n)} = 1$ and matches the product with $s = 0$ on the right-hand side of (60). Put

$$b_s^{(n)} := b_s + \frac{t_{\sigma^{-1}(s)-1} - 1}{mn}, \quad s = 1, \dots, j, \tag{61}$$

and observe that the corresponding product on the right-hand side of (60) is equal to

$$\frac{\Gamma(b_s^{(n)}n+1)}{n^k\Gamma(b_s^{(n)}n-k+1)} = \prod_{\ell=0}^{k-1} \frac{b_s^{(n)}n-\ell}{n} = \prod_{\ell=0}^{k-1} \left(\frac{m(n-\ell)-1+t_{\sigma^{-1}(s)-1}}{nm} + b_s - 1\right).$$

Thus, (60) holds true under the choice in (61). It remains to note that

$$b_s^{(n)} := b_s + \frac{t_{\sigma^{-1}(s)-1} - 1}{mn} \geq b_s - \frac{1}{n} \geq 1 - \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} b_s^{(n)} = b, \quad s = 1, \dots, j.$$

□

Remark 5.15. Note that in the case $m = 1, 2$, it follows that $\mathrm{e}^{-cnH}x^{nm}$ is real-rooted. One may extend the model to factors $(B_s)_{s=0,1,\dots,2j+1-m} = \left(\frac{D}{nm}, \frac{D}{nm} + \frac{b_1-1}{x}, \dots, \frac{D}{nm} + \frac{b_j-1}{x}, x, \dots, x\right)$, to reorder separated multiplication and degree decreasing operators, or even including integration operators, and the parameters b_s could also mildly depend on n. However, in these extensions it is crucial to restrict the permutation σ and hence we skip this extension for not overcomplicating the setting.

We shall now discuss some particular cases of Theorem 5.14.

Case I in which $m = j = 1$, $b_0 = 1$ and $b_1 = b_1^{(n)} = 1 + (\gamma_n + 1)/n \geq 1$ for a sequence of real numbers (γ_n) such that $\lim_{n\to\infty} \frac{1}{n}\gamma_n = \gamma_* \geq 0$. Then,

$$B_0 = \frac{D}{n}, \quad B_1 = \frac{xD}{n} + \frac{\gamma_n + 1}{n}$$

and we obtain a differential flow that maps monomials to suitably rescaled Laguerre polynomials

$$\mathrm{e}^{-cn(x\frac{D^2}{n^2} + (\gamma_n+1)\frac{D}{n^2})}x^n = (-c)^n \mathcal{H}_n^{\varnothing, 1+\gamma_n/n}(-x/c) = \tfrac{(-c)^n n!}{n^n} L_n^{(\gamma_n)}(nx/c),$$

where we have used (57) and (61). The weak convergence of the empirical zero distribution to a Marchenko–Pastur distribution has already been discussed in Corollary 4.14.

Case II in which $m = 2$, $j = 1$, $b_0 = b_1 = 1$ and $c = 1$. Thus,

$$B_0 = \frac{D}{2n}, \quad B_1 = \frac{D}{2n}$$

and the resulting operator e^{-cH} is the heat flow. Using (57) and (61) we conclude that the even degree Hermite polynomials are

$$e^{-nH}x^{2n} = e^{-\frac{1}{4n}D^2}x^{2n} = (-1)^n\mathcal{H}_n^{\varnothing,1-\frac{1}{2n}}(-x^2) = \tfrac{n!}{(-n)^n}L_n^{(-\frac{1}{2})}(nx^2) = \tfrac{1}{(2n)^n}\mathrm{He}_{2n}(\sqrt{2n}x),$$

where the last equality follows from (38). Note that in this case the rays through the roots of unity is just the real line, where the semicircle distribution is supported, see (40).

Case III in which $m = j+1 \in \mathbb{N}$ and $c = b_1 = \ldots = b_j = 1$. Then $H = (\frac{D}{nm})^m$. Then, $[\![e^{-\frac{1}{n^{m-1}m^m}D^m}x^{nm}]\!]_{nm}$ and its weak limit are supported on $\{re^{2\pi i/m} : r \geq 0\}$. After pushing forward to the negative half-line, the limit distribution μ is the same as in the following case.

Case IV in which $m = 1$, $j \geq 1$ and $c = b_1 = \ldots = b_j = 1$. The operator H is $(xD)^jD/n^{j+1}$. By Theorem 5.14, we have

$$[\![e^{-n^{-j+1}(xD)^jD}x^n]\!]_n \xrightarrow[n\to\infty]{w} \mu_+,$$

where $\mu_+(\cdot) := \mu(-(\cdot))$ and the measure μ on $(-\infty, 0]$ is determined by the profile $e^{-g'(\alpha)} = \frac{\alpha^{j+1}}{1-\alpha}$, see Corollary 4.13. By [Part I, Lemma 6.1], the S-transform S_{μ_+} of the limit μ_+ is

$$S_{\mu_+}(z) = -\frac{1+z}{z}e^{g'(1+z)} = (1+z)^{-j}. \tag{62}$$

According to Theorem 2.2 in [4] this is an S-transform of a $\boxtimes$-infinitely divisible distribution.

5.3. **Appell polynomials and differential operators of Laguerre–Pólya class.** The differential operators of hypergeometric type are not the only ones which map monomials to polynomials with roots located on rays, such as the nonpositive real line, another class are those of Laguerre–Pólya type. Let us revisit a recent work by Andrew Campbell [13], where the limiting distribution of rescaled Appell polynomials have been identified to be freely infinitely divisible distributions. We won't provide a full introduction into the tools from free probability, and instead refer to [31], and [9, 13, 50, 63].

Let f be an entire function of the Laguerre–Pólya class, see (53), with $C = 1, m = 0$. Appell polynomials A_n are real rooted, satisfy $A_n' = nA_{n-1}$ and can be represented as

$$A_n(x) = f(D)x^n, \tag{63}$$

where $D = \frac{d}{dx}$ is the differential operator. Recall that the R-transform of a distribution μ is defined by $R_\mu(t) = tG_\mu^{\leftarrow}(t) - 1$, where $G_\mu^{\leftarrow}$ is the inverse of the Cauchy transform of μ. Note that this definition differs from that of [13] by a factor of t.

Theorem 5.16 ([13, Theorem 2.1])**.** *Define $\hat{A}_n(x) = \left(f(\frac{D}{n})\right)^n x^n$. Suppose that $N < \infty$, then the sequence $[\![\hat{A}_n]\!]$ converges weakly to an $\boxplus$-infinitely divisible distribution μ_A with the R-transform*

$$R_A(t) = \gamma t + \sigma^2 t^2 + \int \frac{t(t+x)}{1-tx}\frac{x^2}{x^2+1}d\nu(x),$$

where $\nu = \sum_{j=1}^N \delta_{1/x_j}$ and $\gamma = -c - \sum_{j=1}^N \frac{1}{x_j^3+x_j}$.

Proof. Let us write $f = f_1 f_2 f_3$, where $f_1(z) = \exp(z(c + \sum_j \frac{1}{x_j}))$, $f_2(z) = \exp(-\frac{\sigma^2}{2} z^2)$ and $f_3(z) = \prod_j \left(1 - \frac{z}{x_j}\right)$, such that $\hat{A}_n(x) = \left(f_3(\frac{D}{n})\right)^n \left(f_2(\frac{D}{n})\right)^n \left(f_1(\frac{D}{n})\right)^n x^n$. First note that

$$[\![\left(f_2(\tfrac{D}{n})\right)^n x^n]\!]_n = [\![\left(x + c + \sum_j \tfrac{1}{x_j}\right)^n]\!]_n = \delta_{-c-\sum_j \frac{1}{x_j}}.$$

By Corollary 4.15, we have

$$[\![\left(f_1(\tfrac{D}{n})\right)^n x^n]\!]_n = [\![\mathrm{He}_n(\sqrt{\tfrac{n}{\sigma^2}} x)]\!]_n \xrightarrow[n\to\infty]{w} \mathsf{sc}_{\sigma^2},$$

where $\mathsf{sc}_{\sigma^2} = \mathsf{sc}_1(\frac{\cdot}{\sigma})$ is the semicircle distribution of variance σ^2. Furthermore, by Corollary 4.14 and using the Rodrigues formula for Laguerre polynomials, we conclude that

$$[\![\left(1 - \tfrac{D}{n x_j}\right)^n x^n]\!]_n = [\![L_n^{(0)}(n x_j x)]\!]_n \xrightarrow[n\to\infty]{w} \mathsf{MP}_{x_j^{-1},1}, \quad j = 1, \dots, N.$$

Combining these, leads to the finite free convolution, see for instance Eq. (2) in [41], that is,

$$\hat{A}_n(x) = \left(x + c + \sum_j \tfrac{1}{x_j}\right)^n \boxplus_n \mathrm{He}_n(\sqrt{\tfrac{n}{\sigma^2}} x) \boxplus_n L_n^{(0)}(n x_1 x) \boxplus_n \cdots \boxplus_n L_n^{(0)}(n x_N x).$$

Observe that by Theorem 1.3 in [41] in combination with Theorems 6.32 and 6.31.3 in [60], the roots of $\hat{A}_n(x)$ are upper bounded by an absolute constant. Thus, Theorem 3.5 in [31], yields that the distribution $[\![\hat{A}_n]\!]$ converges weakly to the free additive convolution $\delta_{-c-\sum_j \frac{1}{x_j}} \boxplus \mathsf{sc}_{\sigma^2} \boxplus \mathsf{MP}_{x_1^{-1},1} \boxplus \cdots \boxplus \mathsf{MP}_{x_N^{-1},1}$. Its R transform is a sum of the individual R-transforms $R_{\delta_x}(t) = xt$, $R_{\mathsf{sc}_{\sigma^2}}(t) = \sigma^2 t^2$, and $R_{\mathsf{MP}_{x_j^{-1},1}}(t) = \frac{t}{x_j - t}$. Summing up and rearranging yields the claim. □

6. Characteristic polynomial of the sample covariance matrix

The goal of this section is to demonstrate how the converse Theorem 2.5 can be used to derive the asymptotic behavior of the coefficients. We focus on random polynomials arising from a well-known ensemble of random matrices. Fortunately, the empirical zero distributions of these polynomials converge almost surely, which allows us to apply Theorem 2.5 pathwise.

Let $(X_{i,j})_{i,j\geq 1}$ be a double array of independent identically distributed real random variables satisfying

$$\mathbb{E}[X_{i,j}] = 0, \quad \mathrm{Var}[X_{i,j}] =: \sigma^2 \in (0, +\infty).$$

For $m, n \in \mathbb{N}$, put $\mathbf{X} := [X_{i,j}]_{1\leq i\leq n, 1\leq j\leq m}$. The symmetric random matrix

$$\mathbf{M}(m,n) := m^{-1} \mathbf{X}\mathbf{X}^T \tag{64}$$

is called the sample covariance matrix. Let $(\xi_{j:n})_{1\leq j\leq n} \subset [0, \infty)$ be the set of its eigenvalues and $D_{m,n}(x) := \det(\mathbf{M}(m,n) - x\mathbf{Id}_n) = \prod_{j=1}^n (\xi_{j:n} - x)$ be its characteristic polynomial. Assume that $m = m(n)$ is such that

$$\lim_{n\to\infty} \frac{n}{m(n)} = \lambda \in (0, \infty). \tag{65}$$

Put $\lambda^* := \max(1-\lambda^{-1}, 0)$. According to [5, Theorem 3.6] and assuming (65), with probability one

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{-\xi_{j:n}} = [\![D_{m,n}(-x)]\!]_n \xrightarrow[n\to\infty]{w} \mathsf{MP}_{\sigma^2,\lambda}(-(\cdot)), \tag{66}$$

where $\mathsf{MP}_{\sigma^2,\lambda}$ is the Marchenko–Pastur distribution with the density (33) if $\lambda \le 1$, and is a sum of an atom $\lambda^*\delta_0$ at zero and the distribution with the same density (33) of the total mass λ^{-1} if $\lambda > 1$.

We shall now apply Theorem 2.5 to deduce a strong law of large numbers for the coefficients $a_{k:n}$ of the normalized polynomial $D_{m,n}(-x)$. Observe that the limiting measure μ satisfies $\underline{m} = \mu(\{0\}) = \max(1-\lambda^{-1}, 0)$ and $\overline{m} = 1 - \mu(\{-\infty\}) = 1$.

Theorem 6.1. *Recall* $\lim_{n\to\infty} \frac{n}{m(n)} = \lambda \in (0,\infty)$ *and* $\lambda^* = \max(1-\lambda^{-1}, 0)$. *Define the coefficients of the normalized characteristic polynomial of covariance matrices as*

$$a_{k:n} := [x^k]\frac{\det(\mathbf{M}(m,n) + x\mathbf{Id}_n)}{\det(\mathbf{M}(m,n) + \mathbf{Id}_n)}, \quad k \in \{0,1,\dots,n\}.$$

Then, for all sufficiently small $\varepsilon > 0$, *with probability one,*

$$\sup_{(\lambda^*+\varepsilon)n \le k \le (1-\varepsilon)n} \left| \frac{1}{n}\log a_{k:n} - g_{\mathbf{M}}\left(\frac{k}{n}\right)\right| \xrightarrow[n\to\infty]{} 0, \tag{67}$$

as well as

$$\sup_{0 \le k \le (\lambda^*-\varepsilon)n} \frac{1}{n}\log a_{k:n} \xrightarrow[n\to\infty]{} -\infty. \tag{68}$$

Here $g_{\mathbf{M}} : (\lambda_*, 1) \to (-\infty, 0]$ *is a strictly concave infinitely differentiable function, defined by formulas* (70)-(72) *below and satisfying* $\mathrm{e}^{-g'_{\mathbf{M}}(\alpha)} = \frac{\sigma^2\alpha(\lambda\alpha+1-\lambda)}{1-\alpha}$, $\lambda^* < \alpha < 1$.

Naturally, for $\sigma^2 = \lambda^{-1}$, this coincides with the exponential profile of Laguerre polynomials of Corollary 4.14 corresponding to $\gamma^* = \lambda^{-1} - 1$.

Proof. We only need to compute $g_{\mathbf{M}}$. According to [5, Lemma 3.11], see also Eq. (34), the Cauchy transform of the right-hand side of (66) is equal to[3]

$$G(t) = -\frac{t + \sigma^2(1-\lambda) + \sqrt{(\lambda_+ + t)(\lambda_- + t)}}{2\sigma^2\lambda t}, \quad t \in \mathbb{C}\backslash[-\lambda_+, -\lambda_-], \tag{69}$$

where the branch is chosen such that $t \mapsto \sqrt{(\lambda_+ + t)(\lambda_- + t)} \sim -t$, as $|t| \to +\infty$. One can check that $t \mapsto tG(t)$ is a solution to the quadratic equation

$$\sigma^2\lambda\alpha^2 + (t + \sigma^2(1-\lambda))\alpha - t = 0.$$

Solving this equation for t we find that the inverse of $t \mapsto tG(t)$ is

$$\mathrm{e}^{-g'_{\mathbf{M}}(\alpha)} = \frac{\sigma^2\lambda\alpha^2 + \sigma^2(1-\lambda)\alpha}{1-\alpha} = -\sigma^2\lambda\alpha + \frac{\sigma^2\alpha}{1-\alpha} = \frac{\sigma^2\alpha(\lambda\alpha + 1 - \lambda)}{1-\alpha}.$$

[3]Observe that (34) is stated for $\lambda \in (0,1]$ only. If $\lambda > 1$, then the right-hand side of (34) is the Cauchy transform of $\max(1-\lambda^{-1}, 0)\delta_0 + \mathsf{MP}_{\sigma^2,\lambda}$, see [5, Lemma 3.11] and their definition of the Marchenko–Pastur law in Eq. (3.1.1).

Observe that this function if positive and nondecreasing for $\alpha \in (\lambda^*, 1)$. Taking the logarithms and integrating we obtain,

$$g_{\mathbf{M}}(\alpha) = -2\alpha \log \sigma + \left(-\alpha + 1 - \frac{1}{\lambda}\right) \log(\lambda\alpha + 1 - \lambda) + \alpha - \alpha \log \alpha - (1-\alpha)\log(1-\alpha) + C. \tag{70}$$

The additive constant C can be determined from the relation

$$\sup_{\alpha \in (\lambda^*, 1)} g_{\mathbf{M}}(\alpha) = 0,$$

see formula (8). After elementary but lengthy calculations this yields

$$C = \frac{1}{\lambda}\log(1-\alpha^*) - \frac{1}{\lambda}\log\alpha^* + \log\alpha^* - \alpha^* - 2\left(\frac{1}{\lambda} - 1\right)\log\sigma, \tag{71}$$

where α^* is the unique solution on $(\lambda^*, 1)$ to the quadratic equation

$$\frac{1-\alpha^*}{\alpha^*(\lambda\alpha^* + 1 - \lambda)} = \sigma^2. \tag{72}$$

The proof is complete. □

Appendix A. Lambert function

Here we collect some relevant properties of the Lambert function. We refer to [15] and [46] for more details. The Lambert function is a multivalued analytic function defined by the implicit equation

$$W(z)\mathrm{e}^{W(z)} = z, \qquad z \in \mathbb{C}.$$

It has a branch point at $z = -1/\mathrm{e}$ and infinitely many branches. We need only the principal branch W_0 which is defined on the whole complex plane with a branch cut at $(-\infty, -1/\mathrm{e}]$. On the real line, the function $w \mapsto w\mathrm{e}^w = z$ has a unique minimum $-1/\mathrm{e}$ attained at $w = -1$. Thus, there is a well-defined inverse function $W_0 : (-1/\mathrm{e}, \infty) \to (-1, \infty)$, called the principal branch, which is monotone increasing, and satisfies

$$W_0(z)\mathrm{e}^{W_0(z)} = z, \qquad \lim_{x \downarrow -1/\mathrm{e}} W_0(x) = -1, \qquad W_0(0) = 0, \qquad \lim_{x \uparrow +\infty} W_0(x) = +\infty.$$

The principal branch W_0 can be analytically continued to the slitted complex plane $\mathbb{C}\backslash(-\infty, -1/\mathrm{e}]$. In Theorem 4.1, we need the limit values of $W_0(z)$ when the complex variable z approaches the branch cut $(-\infty, -1/\mathrm{e}]$. Such limit value depends on whether z stays in the upper half-plane or in the lower half-plane. We can define

$$W_0(x + \mathrm{i}0) := \lim_{\varepsilon \downarrow 0} W_0(x + \mathrm{i}\varepsilon), \qquad W_0(x - \mathrm{i}0) := \lim_{\varepsilon \downarrow 0} W_0(x - \mathrm{i}\varepsilon) = \overline{W_0(x + \mathrm{i}0)}, \qquad x \in (-\infty, -1/\mathrm{e}).$$

It is also known that $\operatorname{Im} W_0(x + \mathrm{i}0)$ is a decreasing function of $x \in (-\infty, -1/\mathrm{e})$ and that

$$\lim_{x \downarrow -\infty} \operatorname{Im} W_0(x + \mathrm{i}0) = \pi, \qquad \lim_{x \uparrow -1/\mathrm{e}} \operatorname{Im} W_0(x + \mathrm{i}0) = 0.$$

We need the following formula for the derivative of the Lambert function:

$$W_0'(z) = \frac{W_0(z)}{z(1 + W_0(z))}, \qquad z \in \mathbb{C}\backslash(-\infty, -1/\mathrm{e}], \quad z \neq 0. \tag{73}$$

It is well known that for every $v \in \mathbb{C}$ and for all complex $|z| < 1/\mathrm{e}$ the following expansion holds:

$$\left(\frac{W_0(z)}{z}\right)^v = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{v(n+v)^{n-1}}{n!} z^n \tag{74}$$

This identity easily follows from the Lagrange inversion formula in Bürmann's form [24, pp. 732–733]. This formula states that if the formal power series $y(z)$ is defined as the solution to $y/\phi(y) = z$, where $\phi(u) = \sum_{k=0}^{\infty} \phi_k u^k$ is a formal power series with $\phi_0 \neq 0$, then

$$[z^n]\left(\frac{y(z)}{z}\right)^v = \frac{v}{n+v}[u^n]\phi(u)^{n+v}.$$

Taking $\phi(u) = \mathrm{e}^{-u}$, one easily arrives at (74).

Taking $v = -1$ in (74), and multiplying by z, we obtain the formula

$$\mathrm{e}^{W_0(z)} = \frac{z}{W_0(z)} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)^{n-1}}{n!} z^n = 1 + \sum_{k=0}^{\infty} \frac{(-k)^k}{(k+1)!} z^{k+1}. \tag{75}$$

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References

[1] N. I. Akhiezer. *The classical moment problem and some related questions in analysis*. Hafner Publishing Co., New York, 1965.

[2] M. Anshelevich, J.-C. Wang, and P. Zhong. Local limit theorems for multiplicative free convolutions. *J. Funct. Anal.*, 267(9):3469–3499, 2014.

[3] O. Arizmendi, K. Fujie, D. Perales, and Y. Ueda. S-transform in finite free probability. *arXiv preprint arXiv:2408.09337*, 2024.

[4] O. Arizmendi and T. Hasebe. Limit theorems for free Lévy processes. *Electronic Journal of Probability*, 23:1 – 36, 2018.

[5] Zhidong Bai and Jack W Silverstein. *Spectral analysis of large dimensional random matrices*, volume 20. Springer, 2010.

[6] S. T. Belinschi and H. Bercovici. Partially defined semigroups relative to multiplicative free convolution. *International Mathematics Research Notices*, 2005(2):65–101, 2005.

[7] E. A. Bender. Central and local limit theorems applied to asymptotic enumeration. *J. Combinatorial Theory Ser. A*, 15:91–111, 1973.

[8] H. Bercovici and D. V. Voiculescu. Lévy-Hinčin type theorems for multiplicative and additive free convolution. *Pacific J. Math.*, 153(2), 1992.

[9] H. Bercovici and D. V. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, 42(3):733–773, 1993.

[10] P. Biane. Free Brownian motion, free stochastic calculus and random matrices. In *Free probability theory (Waterloo, ON)*, volume 12 of *Fields Inst. Commun.*, pages 1–19. Amer. Math. Soc., Providence, RI, 1995.

[11] P. Biane. Segal–Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems. *J. Funct. Anal.*, 144(1):232–286, 1997.

[12] S.-S. Byun, P. J. Forrester, and J. Oh. q-deformed gaussian unitary ensemble: spectral moments and genus-type expansions. *arXiv preprint arXiv:2404.03400*, 2024.

[13] A. Campbell. Free infinite divisibility, fractional convolution powers, and Appell polynomials. *arXiv preprint arXiv:2412.20488*, 2024.

[14] L. Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[15] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.*, 5(4):329–359, 1996.

[16] N. Demni and T. Hamdi. Spectral distribution of large-size Brownian motions on the complex linear group. *arXiv preprint arXiv:1307.4027*, 2016.

[17] N. Demni and T. Hmidi. Spectral distribution of the free unitary Brownian motion: another approach. *arXiv preprint arXiv:1103.4693*, 2011.

[18] H. Dette and W. J. Studden. Some new asymptotic properties for the zeros of Jacobi, Laguerre, and Hermite polynomials. *Constr. Approx.*, 11(2):227–238, 1995.

[19] A. Edrei. Zeros of successive derivatives of entire functions of the form $h(z)\exp(-e^z)$. *Trans. Amer. Math. Soc.*, 259(1):207–226, 1980.

[20] C. Elbert. Strong asymptotics of the generating polynomials of the Stirling numbers of the second kind. *J. Approx. Theory*, 109(2):198–217, 2001.

[21] C. Elbert. Weak asymptotics for the generating polynomials of the Stirling numbers of the second kind. *J. Approx. Theory*, 109(2):218–228, 2001.

[22] J. Faldey and W. Gawronski. On the limit distributions of the zeros of Jonquière polynomials and generalized classical orthogonal polynomials. *J. Approx. Theory*, 81(2):231–249, 1995.

[23] G. Fano, F. Ortolani, and W. Van Assche. Asymptotic behaviour of the coefficients of orthogonal polynomials. In *Orthogonal polynomials and their applications (Erice, 1990)*, volume 9 of *IMACS Ann. Comput. Appl. Math.*, pages 57–59. Baltzer, Basel, 1991.

[24] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.

[25] W. Gawronski. On the asymptotic distribution of the zeros of Hermite, Laguerre, and Jonquière polynomials. *J. Approx. Theory*, 50(3):214–231, 1987.

[26] W. Gawronski. Strong asymptotics and the asymptotic zero distributions of Laguerre polynomials $L_n^{(an+\alpha)}$ and Hermite polynomials $H_n^{(an+\alpha)}$. *Analysis*, 13(1-2):29–67, 1993.

[27] W. Gawronski and U. Stadtmüller. On the zeros of Jonquière's function with a large complex parameter. *Michigan Math. J.*, 31(3):275–293, 1984.

[28] W. Gawronski and U. Stadtmüller. On the zeros of Lerch's transcendental function with real parameters. *J. Approx. Theory*, 53(3):354–364, 1988.

[29] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, second edition, 1994.

[30] L. H. Harper. Stirling behavior is asymptotically normal. *Ann. Math. Statist.*, 38:410–414, 1967.

[31] J. Jalowy, Z. Kabluchko, and A. Marynych. Zeros and exponential profiles of polynomials I: Limit distributions, finite free convolutions and repeated differentiation. *arXiv preprint arXiv:2504.11593*, 2025.

[32] Z. Kabluchko. Repeated differentiation and free unitary Poisson process. *Trans. Amer. Math. Soc, to appear. arXiv preprint arXiv:2112.14729*, 2024.

[33] Z. Kabluchko. Lee – Yang zeroes of the Curie – Weiss ferromagnet, unitary Hermite polynomials, and the backward heat flow. *Annales Henri Lebesgue*, 8:1–34, 2025.

[34] Z. Kabluchko, A. Marynych, and H. Pitters. Mod-φ convergence of stirling distributions and limit theorems for zeros of their generating functions. *Journal of Mathematical Analysis and Applications*, 529(1):127571, 2024.

[35] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. Hypergeometric orthogonal polynomials. In *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, pages 183–253. Springer, 2010.

[36] J. P. S. Kung, G.-C. Rota, and C. H. Yan. *Combinatorics: the Rota way*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2009.

[37] T. Kyle Petersen. *Eulerian numbers*. Birkhäuser/Springer, New York, 2015.

[38] D. S. Lubinsky and A. Sidi. Strong asymptotics for polynomials biorthogonal to powers of $\log x$. *Analysis*, 14(4):341–379, 1994.

[39] D. S. Lubinsky and A. Sidi. Zero distribution of composite polynomials and polynomials biorthogonal to exponentials. *Constr. Approx.*, 28(3):343–371, 2008.

[40] A. W. Marcus. Polynomial convolutions and (finite) free probability. *arXiv preprint arXiv:2108.07054*, 2021.

[41] A. W. Marcus, D. A. Spielman, and N. Srivastava. Finite free convolutions of polynomials. *Probab. Theory Relat. Fields*, 182(3-4):807–848, 2022.

[42] A. Martínez-Finkelshtein, R. Morales, and D. Perales. Real roots of hypergeometric polynomials via finite free convolution. *International Mathematics Research Notices*, 2024(16):11642–11687, 06 2024.

[43] A. Martínez-Finkelshtein, R. Morales, and D. Perales. Zeros of generalized hypergeometric polynomials via finite free convolution: Applications to multiple orthogonality. *Constr. Approx.*, pages 1–70, 2025.

[44] P. Melotti. Distribution of roots of Eulerian polynomials. *arXiv preprint arXiv:2507.15908*, 2025.

[45] D. Merlini, R. Sprugnoli, and M. C. Verri. The Cauchy numbers. *Discrete Math.*, 306(16):1906–1920, 2006.

[46] I. Mező. *The Lambert W function—its generalizations and applications*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2022.

[47] I. Mező. *Combinatorics and number theory of counting sequences*. CRC Press, 2020.

[48] B. B. P. Mirabelli. *Hermitian, non-hermitian and multivariate finite free probability*. PhD thesis, Princeton University, 2021.

[49] L. Moser and M. Wyman. Asymptotic development of the Stirling numbers of the first kind. *J. London Math. Soc.*, 33:133–146, 1958.

[50] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[51] L. Pastur and M. Shcherbina. *Eigenvalue distribution of large random matrices*, volume 171 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[52] G. Pólya and G. Szegő. *Problems and Theorems in Analysis, Vols. I and II*. Springer-Verlag, Berlin, Heidelberg, 1976.

[53] Q. I. Rahman and G. Schmeisser. *Analytic theory of polynomials*, volume 26 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, Oxford, 2002.

[54] A. Sidi. Numerical quadrature and nonlinear sequence transformations; unified rules for efficient computation of integrals with algebraic and logarithmic endpoint singularities. *Math. Comp.*, 35(151):851–874, 1980.

[55] A. Sidi. Numerical quadrature rules for some infinite range integrals. *Math. Comp.*, 38(157):127–142, 1982.

[56] A. Sidi and D. S. Lubinsky. On the zeros of some polynomials that arise in numerical quadrature and convergence acceleration. *SIAM J. Numer. Anal.*, 20(2):400–405, 1983.

[57] S. H. Siraždinov. The limit distribution of the roots of Euler polynomials. *Dokl. Akad. Nauk SSSR*, 239(1):56–59, 1978.

[58] S. L. Sobolev. Roots of Euler polynomials. *Dokl. Akad. Nauk SSSR*, 235(2):277–280, 1977.

[59] S. L. Sobolev. More on roots of Euler polynomials. *Dokl. Akad. Nauk SSSR*, 245(4):801–804, 1979.

[60] G. Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, Rhode Island, 1939.

[61] D. V. Voiculescu. Symmetries of some reduced free product C^*-algebras. Operator algebras and their connections with topology and ergodic theory, Proc. Conf., Buşteni/Rom. 1983, Lect. Notes Math. 1132, 556-588 (1985)., 1985.

[62] D. V. Voiculescu. Multiplication of certain noncommuting random variables. *J. Operator Theory*, 18(2):223–235, 1987.

[63] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992.

[64] G. Warner. ZEROS, 2024. arXiv: 2408.00961.

[65] P. Zhong. Free Brownian motion and free convolution semigroups: multiplicative case. *Pacific J. Math.*, 269(1):219–256, 2014.

[66] P. Zhong. On the free convolution with a free multiplicative analogue of the normal distribution. *J. Theoret. Probab.*, 28(4):1354–1379, 2015.

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