CONVOLUTION POWERS OF UNBOUNDED MEASURES ON THE POSITIVE HALF-LINE

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ABSTRACT. For a right-continuous nondecreasing and unbounded function V of at most exponential growth, which vanishes on the negative half-line, we investigate the asymptotic behavior of the Lebesgue-Stieltjes convolution powers $V^{*(j)}(t)$ as both j and t tend to infinity. We obtain a comprehensive asymptotic formula for $V^{*(j)}(t)$, which is valid across different regimes of simultaneous growth of j and t. Our main technical tool is an exponential change of measure, which is a standard technique in the large deviations theory. Various applications of our result are given.

1. INTRODUCTION

Let $V:\mathbb{R}\to[0,\infty)$ be a right-continuous and nondecreasing function vanishing on $(-\infty,0)$. The function V can be thought of as the distribution function of a measure μ on the nonnegative half-line, that is,

$$\mu([0,t])=V(t),\quad t\geq 0.$$

Let $\mu^{*(j)}$ (respectively, $V^{*(j)}$) denote the jth convolution power in the Lebesgue–Stieltjes sense of measure μ (respectively, of function V), so that

$$\mu^{*(j)}([0,t])=V^{*(j)}(t)=\int_{[0,t]}V^{*(j-1)}(t-y)\mathrm{d}V(y),\quad t\geq 0,\quad j\geq 2.$$

Let $(t_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers which diverges to $+\infty$ as $j\to\infty$. In this note we are interested in the asymptotic behavior of $V^{*(j)}(t_j)$ as $j\to\infty$. If

$$\lim_{x\to\infty}V(x)\in(0,\infty),\tag{1.1}$$

then, by simple rescaling, we may assume that the limit is equal to 1 and μ is a probability measure. A basic result of probability theory, called the weak law of large numbers, then tells us that, for every $\varepsilon>0$,

$$\lim_{j\to\infty}V^{*(j)}((\mathtt{m}-\varepsilon)j)=0,\quad \lim_{j\to\infty}V^{*(j)}((\mathtt{m}+\varepsilon)j)=1,\tag{1.2}$$

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where $\mathtt{m} := \int_{[0,\infty)} x\mathrm{d}V(x)$ is assumed finite. Under the additional assumption $\mathtt{s}^2 := \int_{[0,\infty)} x^2\mathrm{d}V(x) - \mathtt{m}^2 \in (0,\infty)$, the celebrated central limit theorem provides a refinement of (1.2):

$$\lim_{j\to\infty} V^{*(j)}(\mathtt{m}j + x\mathtt{s}\sqrt{j}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y, \quad x \in \mathbb{R}. \tag{1.3}$$

If $\mathtt{s}^2 = \infty$ or even $\mathtt{m} = \infty$, so-called 'stable' versions of (1.3) are still available under conditions of regular variation. This involves replacing $t_j = t_j(x) = \mathtt{m}j + x\mathtt{s}\sqrt{j}$ on the left-hand side with another appropriate sequence defined by the rate at which $V(t)$ tends to 1. Further results on the asymptotic behavior of $V^{*(j)}(t_j)$, for other classes of sequences $(t_j)_{j\in\mathbb{N}}$, follow from the classic large- and moderate-deviations theory. With these at hand, the picture of asymptotic regimes of $V^{*(j)}(t_j)$ under assumption (1.1) becomes more or less complete. For this circle of questions we refer to the book [15].

A related problem has been addressed in the articles [7, 11, 17, 19]. These are concerned with the asymptotic behavior of $W^{*(j)}(t_j)$ as $j \to \infty$. Here, $W : \mathbb{Z} \to \mathbb{C}$ is a function of a finite support, and $W^{*(k)}$ is the k-fold Lebesgue convolution of W with itself defined by $W^{*(1)} := W$ and $W^{*(k)}(t) := \sum_{s\in\mathbb{Z}} W^{*(k-1)}(t-s)W(s)$ for $k \geq 2$ and $t \in \mathbb{C}$. In this setting the connections with probability theory vanish. One particularly interesting aspect of these works is the appearance in the asymptotic behavior of $W^{*(j)}(t_j)$ of the functions H_m with $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}zx} H_m(x)\mathrm{d}x = \exp(-z^m)$ for $z \in \mathbb{R}$ and even $m \geq 2$ and $= \exp(\mathrm{i}z^m)$ for $z \in \mathbb{R}$ and odd $m \geq 3$. If the function W is nonnegative, as in our setting, then the asymptotic shape is driven by the Gaussian density H_2, similarly to (1.3), whereas if the function W is complex-valued or real-valued and takes values of both signs, then H_m may arise with any $m \geq 2$. In [5, 16, 18] the asymptotic behavior of $W^{*(j)}(t_j)$ as $j \to \infty$ was investigated for some particular subclasses of functions $W : \mathbb{Z}^d \to \mathbb{C}$ satisfying $\sum_{x\in\mathbb{Z}^d} |W(x)| < \infty$. The arguments of all the papers mentioned in this paragraph are heavily based on Fourier analysis.

Returning to our setting of nonnegative nondecreasing functions V, assume now that the limit in (1.1) is equal to $+\infty$, meaning that the measure μ is infinite. As far as we know, under this assumption, the problem of our interest has not been investigated so far, and the present paper aims at (partly) closing this gap. Although our results differ significantly from those obtained in the aforementioned articles, they share some similarities with the classic theory of large deviations for probability measures, see, for instance, Theorem 1 in [1] or Theorem 1 in [14]. A more detailed comparison with the existing theory, in particular with results in the aforementioned references, will be given in Section 6.

2. The setting

Throughout the paper we assume that

$$\lim_{x\to\infty} V(x) = +\infty, \tag{2.1}$$

and that the sequence $(t_j)_{j\in\mathbb{N}}$ grows at least linearly. Also we shall normally write t for t_j. The standing assumption in what follows is

$$\hat{V}(s) := \int_{[0,\infty)} \mathrm{e}^{-sx}\mathrm{d}V(x) < \infty \quad \text{for some } s > 0, \tag{2.2}$$

which means that $\hat{V}(s')$ is also finite for all $s' \geq s$. Depending on V the function $\hat{V}$ may be finite for all $s > 0$ or $s \geq s_1$ or $s > s_1$ for some $s_1 > 0$. Denote by $\mathscr{D}$ the domain of $\hat{V}$, that is,

$$\mathscr{D} := \{s > 0 : \hat{V}(s) < \infty\}.$$

It is a standard fact that if $\hat{V}(s)$ is defined for $s > s_0$, then it possesses an analytic continuation to the domain $\{s \in \mathbb{C} : \mathrm{Re}\, s > s_0\}$, see [22]. Thus, the function

$$s \mapsto \lambda(s) := \log \hat{V}(s) = \log \int_{[0,\infty)} \mathrm{e}^{-sx}\mathrm{d}V(x)$$

is well-defined in a sufficiently small complex domain containing $\mathrm{Int}\,\mathscr{D}$, the interior of $\mathscr{D}$. Here log is the principal branch of the complex logarithm.

The function λ is infinitely differentiable on $\mathrm{Int}\,\mathscr{D}$. In particular, $-\lambda'$ is a continuous function on $\mathrm{Int}\,\mathscr{D}$. Assumption (2.1) excludes the situation in which $V(x) = a_1 \mathbb{1}_{[a_2,\infty)}(x)$ for $x \geq 0$ and some positive a_1 and a_2. The function λ is then strictly convex on $\mathrm{Int}\,\mathscr{D}$ which entails that the function $-\lambda'$ is strictly decreasing on $\mathrm{Int}\,\mathscr{D}$. Thus, for every $s \in (s_-, s_+)$, where

$$s_- := \inf_{s\in\mathrm{Int}\,\mathscr{D}} (-\lambda'(s)) \quad \text{and} \quad s_+ := \sup_{s\in\mathrm{Int}\,\mathscr{D}} (-\lambda'(s)),$$

the equation $-\lambda'(x) = s$ has a unique solution. According to Lemma 7.1 in the Appendix

$$s_- = \inf\{x > 0 : V(x) > 0\}.$$

In most applications $s_+ = +\infty$, and the equation $-\lambda'(x) = s$ then has a unique solution for all $s > s_-$. Nevertheless, the situation where $s_+ < \infty$ is also possible, see Example 2.7 below.

Throughout the paper we assume that[1]

$$t/j \in (s_-, s_+) \quad \text{for all large enough } j \in \mathbb{N}, \tag{2.3}$$

and let $\kappa(j) = \kappa(j,t)$ be uniquely defined by the equation

$$-\lambda'(\kappa(j)) = t/j. \tag{2.4}$$

Here are some examples of a function V and the corresponding quantities λ and κ. Later on some of these will be explored further.

Example 2.1. Let $V(x) = bx^\alpha$ for $x \geq 0$ and some positive b and α. Then $\lambda(s) = \log(b\Gamma(\alpha+1)) - \alpha \log s$, $-\lambda'(s) = \alpha/s$ for $s > 0$, $s_- = 0$ and $s_+ = \infty$. Hence, $\kappa(j) = \alpha j/t$ for $j \in \mathbb{N}$.

Example 2.2. Let $V(x) = ax + b$ for $x \geq 0$ and some positive a and b. Then $\lambda(s) = \log(b + a/s)$ and $-\lambda'(s) = a/(bs^2 + as)$ for $s > 0$. Hence, $\kappa(j,t) = ((a^2 + 4abj/t)^{1/2} - a)/(2b)$ for $j \in \mathbb{N}$.

[1] Note that $V^{*(j)}(t) = 0$ for $t < s_- j$.

Example 2.3. Let $V(x) = (\log(x+1))^\alpha$ for $x \geq 0$ and some $\alpha > 0$. Then $\mathscr{D} = \mathrm{Int}\mathscr{D} = (0,\infty)$. By the Karamata Tauberian theorem, see Theorem 1.7.1 in [2], $\hat{V}(s) \sim \log^\alpha(1/s)$ and $\lambda(s) \sim \alpha \log\log(1/s)$, as $s \to 0+$. Moreover, by the monotone density theorem (a version of Theorem 1.7.2 in [2]),

$$-\lambda'(s) \sim \frac{\alpha}{s\log(1/s)}, \quad s \to 0+.$$

Hence, $\kappa(j) \sim \alpha/((t/j)\log(t/j))$ as $j \to \infty$ and $t/j \to \infty$.

Example 2.4. Let $\mathrm{d}V(x) = 2^{-1}x^{-1/2}\exp(x^{1/2})\mathbb{1}_{(0,\infty)}(x)\mathrm{d}x$. Then, as $s \to 0+$, $\lambda(s) \sim 1/(4s)$ and $-\lambda'(s) \sim 1/(4s^2)$. Hence, $\kappa(j,t) \sim (j/t)^{1/2}/2$ as $j \to \infty$ and $t/j \to \infty$.

Example 2.5. Let $V(x) = \mathrm{e}^{ax} - 1$ for $x \geq 0$ and some $a > 0$. Then $\lambda(s) = \log a - \log(s-a)$ and $-\lambda'(s) = 1/(s-a)$ for $s > a$. Hence, $\kappa(j,t) = a + j/t$ for $j \in \mathbb{N}$. Interestingly, there is an explicit formula for the convolutions of V

$$V^{*(j)}(x) = \frac{a^j}{(j-1)!}\int_0^x y^{j-1}\mathrm{e}^{ay}\mathrm{d}y = \frac{1}{(j-1)!}\int_0^{ax} y^{j-1}\mathrm{e}^{y}\mathrm{d}y, \quad t \geq 0, \quad j \in \mathbb{N} \tag{2.5}$$

which follows by induction. Here is an even simpler formula

$$(V_0 * V^{*(j)})(x) = \frac{(ax)^j\mathrm{e}^{ax}}{j!}, \quad x \geq 0, \quad j \in \mathbb{N}_0 := \mathbb{N}\cup\{0\},$$

where $V_0(x) = \mathrm{e}^{ax}$ for $x \geq 0$.

Example 2.6. Let $V(x) = \mathrm{e}^{ax}$ for $x \geq 0$ and some $a > 0$. Then $\lambda(s) = \log s - \log(s-a)$ and $-\lambda'(s) = a/(s(s-a))$ for $s > a$. Hence, $\kappa(j,t) = (a + (a^2 + 4aj/t)^{1/2})/2$ for $j \in \mathbb{N}$.

Example 2.7. Let $\mathrm{d}V(x) = (x+1)^{-2-\alpha}\mathrm{e}^x\mathbb{1}_{(0,\infty)}(x)\mathrm{d}x$ for some $\alpha > 0$. Then $\mathscr{D} = [1,\infty)$, $\hat{V}(1) = 1/(1+\alpha)$ and $-\hat{V}'(1) = \int_0^\infty x(x+1)^{-2-\alpha}\mathrm{d}x = 1/(\alpha(1+\alpha))$. Thus, $-\lambda'(1) = 1/\alpha$ meaning that $s_+ = 1/\alpha$ and $\kappa(j)$ is undefined for $\alpha t > j$.

3. Main results

We call V arithmetic, if it is piecewise-constant with jumps at points belonging to the lattice $(\omega_1 + \omega_2 n)_{n\in\mathbb{N}}$ for some $\omega_1 \geq 0$ and $\omega_2 > 0$, and nonarithmetic, otherwise. Put $a(j) := (j\lambda''(\kappa(j)))^{1/2}$ for $j \in \mathbb{N}$.

Theorem 3.1. *Suppose that $j \to \infty$ and $t = t_j \to \infty$ in such a way that* (2.3) *holds true.*
(a) Assume that $\lim_{j\to\infty}\kappa(j)a(j) = \infty$ and $\lim_{j\to\infty}\mathrm{e}^{-\gamma j}\kappa(j)a(j) = 0$ for all $\gamma > 0$. Put

$$T_j := \frac{|\lambda'(\kappa(j))|^3}{\lambda''(\kappa(j))} + \frac{|\hat{V}'''(\kappa(j))|}{\lambda''(\kappa(j))\hat{V}(\kappa(j))}.$$

Suppose further that

$$T_j = o(a(j)), \quad j \to \infty \tag{3.1}$$

and, for every fixed $\gamma > 0$,

$$\limsup_{j\to\infty} \sup_{|z|\geq\gamma} \left| \frac{\hat{V}(\kappa(j) - \mathrm{i}z/T_j)}{\hat{V}(\kappa(j))} \right| < 1. \tag{3.2}$$

Then

$$V^{*(j)}(t) \;\sim\; \frac{1}{(2\pi)^{1/2}} \frac{(\hat{V}(\kappa(j)))^j \mathrm{e}^{t\kappa(j)}}{\kappa(j)a(j)}, \quad j \to \infty. \tag{3.3}$$

(b) Assume that $\lim_{j\to\infty} \kappa(j) = \theta \in (0,\infty)$, $\lim_{j\to\infty} \lambda''(\kappa(j)) = \sigma^2 \in (0,\infty)$ *and that* V *is nonarithmetic. Then*

$$V^{*(j)}(t) \;\sim\; \frac{1}{(2\pi\sigma^2)^{1/2}\theta} \frac{(\hat{V}(\kappa(j)))^j \mathrm{e}^{t\kappa(j)}}{j^{1/2}}, \quad j \to \infty. \tag{3.4}$$

Remark 3.2. Note the under the assumptions of part (b) condition (3.1) holds automatically.

Remark 3.3. The function $s \mapsto -\lambda(s)$ is strictly concave on (s_0,∞) and therefore possesses the Legendre transform $u \mapsto \lambda^*(u) := \inf_{s>s_0}(us + \lambda(s))$, which is finite at least for $u \in (s_-, s_+)$. The definition of $\kappa(j)$ in (2.4) entails

$$(\hat{V}(\kappa(j)))^j \mathrm{e}^{t\kappa(j)} = \exp\{j((t/j)\kappa(j) + \lambda(\kappa(j)))\} = \mathrm{e}^{j\lambda^*(t/j)},$$

thereby leading to an alternative form of (3.3)

$$V^{*(j)}(t) \;\sim\; \frac{1}{(2\pi)^{1/2}} \frac{\exp(j\lambda^*(t/j))}{\kappa(j)a(j)}, \quad j \to \infty.$$

Taking into account that $\lim_{j\to\infty} \mathrm{e}^{-\gamma j}\kappa(j)a(j) = 0$, for all $\gamma > 0$, this yields a neat formula

$$\log V^{*(j)}(t) \;\sim\; j\lambda^*(t/j), \quad j \to \infty, \tag{3.5}$$

that holds true in both parts (a) and (b) of Theorem 3.1.

Denote by $p(\mathtt{m}, \mathtt{s}^2)$ the density of a normal random variable with mean $\mathtt{m} \in \mathbb{R}$ and variance $\mathtt{s}^2 > 0$ given by

$$p(\mathtt{m}, \mathtt{s}^2; y) = \frac{1}{(2\pi\mathtt{s}^2)^{1/2}} \mathrm{e}^{-\frac{(y-\mathtt{m})^2}{2\mathtt{s}^2}}, \quad y \in \mathbb{R}.$$

As usual, for $k \in \mathbb{N}$, $\lambda^{(k)}$ denotes the kth derivative of λ.

Corollary 3.4. *Assume that* V *is nonarithmetic and that* $t = \alpha j + y$ *for some fixed* $\alpha > s_-$ *and some* $y = y(j)$. *Let* $r \in \mathbb{N}$ *be fixed and* θ *a unique solution to* $-\lambda'(\theta) = \alpha$ *and* $\sigma^2 := \lambda''(\theta)$. *If* $y(j) = o(j^{(r+1)/(r+2)})$ *as* $j \to \infty$, *then*

$$V^{*(j)}(t) \;\sim\; \frac{1}{\theta} \mathrm{e}^{\theta t} (\hat{V}(\theta))^j \frac{1}{(2\pi\sigma^2 j)^{1/2}} \mathrm{e}^{\Sigma_{\ell=1}^r b_\ell \frac{y^{\ell+1}}{j^\ell}}, \quad j \to \infty, \tag{3.6}$$

where $b_1 = -(2\sigma^2)^{-1}$, $b_2 = -\lambda^{(3)}(\theta)/(6\sigma^2)$ *and* $b_3, b_4, \ldots$ *are some*[2] *real constants.*

[2]The constant b_k is a rational function of $\lambda^{(2)}(\theta), \ldots, \lambda^{(k+1)}(\theta)$. Details can be found in the proof of Corollary 3.4.

If $y(j) \sim c j^{(r+1)/(r+2)}$ as $j \to \infty$ for some $c > 0$, then

$$V^{*(j)}(t) \;\sim\; \theta^{-1} \mathrm{e}^{b_{r+1} c^{r+2}} \frac{1}{(2\pi\sigma^2 j)^{1/2}} \mathrm{e}^{\theta t} (\hat{V}(\theta))^j \mathrm{e}^{\sum_{\ell=1}^{r} b_\ell \frac{y^{\ell+1}}{j^\ell}}, \quad j \to \infty. \tag{3.7}$$

In particular, if $y(j) = o(j^{2/3})$ as $j \to \infty$, then

$$V^{*(j)}(t) \;\sim\; \frac{1}{\theta} e^{\theta t} \hat{V}^j(\theta) p(0, \sigma^2 j; y), \quad j \to \infty;$$

if $y(j) \sim c j^{2/3}$ as $j \to \infty$ for some $c > 0$, then

$$V^{*(j)}(t) \;\sim\; \theta^{-1} \mathrm{e}^{-c^3 \lambda^{(3)}(\theta)/(6\sigma^6)} \mathrm{e}^{\theta t} (\hat{V}(\theta))^j p(0, \sigma^2 j; y), \quad j \to \infty.$$

Thus, in the setting of Corollary 3.4 the asymptotic behavior of $V^{*(j)}(t)$ exhibits infinitely many phase transitions, which are indexed by $r \in \mathbb{N}$ and depend on whether $y(j) = o(j^{(r+1)/(r+2)})$ or $y(j) \sim \mathrm{const} \cdot j^{(r+1)/(r+2)}$ as $j \to \infty$.

An interesting problem is to find a sequence $(t_j)_{j\in\mathbb{N}}$ such that $V^{*(j)}(t_j)$ converges to a finite positive constant. This is accomplished in the following corollary which can also be regarded as a version of (1.3).

Corollary 3.5. *Assume that there exists $\theta^* \in \mathscr{D}$ such that $\lambda(\theta^*) = \theta^* \lambda'(\theta^*)$ and V is nonarithmetic. Fix $y \in \mathbb{R}$ and let $t = t(j,y)$ be a function satisfying*

$$-\lambda' \left(\theta^* - \frac{\log j - y + \varepsilon(j)}{2 j \theta^* \lambda''(\theta^*)} \right) = \frac{t}{j}, \quad j \to \infty \tag{3.8}$$

for some $\varepsilon(j) = o(1)$, $j \to \infty$. Then, with $\sigma^2 := \lambda''(\theta^*)$,

$$\lim_{j\to\infty} V^{*(j)}(t) \;=\; \frac{1}{(2\pi\sigma^2)^{1/2} \theta^*} \mathrm{e}^{-y/2}. \tag{3.9}$$

4. Applications

Recall our standing assumption that $V(x) = 0$ for $x < 0$.

Example 2.1 (continuation). Recall that in this example $V(x) = b x^\alpha$ for $x \geq 0$ and some $b, \alpha > 0$, $\hat{V}(s) = b\Gamma(\alpha+1) s^{-\alpha}$, $-\lambda'(s) = \frac{\alpha}{s}$ and $\kappa(j) = \alpha j/t$. Further, $\lambda''(s) = \frac{\alpha}{s^2}$, which implies that

$$\lambda''(\kappa(j)) = \frac{t^2}{\alpha j^2}, \quad a(j) = \frac{t}{\sqrt{\alpha j}}, \quad \kappa(j) a(j) = \sqrt{\alpha j}.$$

Condition (3.1) holds in view of

$$T_j \;=\; c_1 \cdot t/j$$

for a constant $c_1 > 0$. Condition (3.2) reads

$$\sup_{|z|>\gamma} |1 - \mathrm{i}z|^{-\alpha} = (1+\gamma^2)^{-\alpha/2} < 1 \quad \text{for every fixed } \gamma > 0$$

and also holds true. Note that no assumptions have been imposed on $t = t_j$. This ensures that (3.3) holds true for any sequence $(t_j)_{j\in\mathbb{N}}$.

If $t_j \sim c \cdot j$ for some $c > 0$, the same asymptotic also follows from part (b) of Theorem 3.1. After simple manipulations we obtain

$$V^{*(j)}(t) \;\sim\; \frac{1}{(2\pi\alpha j)^{1/2}} \left(\frac{b\Gamma(\alpha+1)\mathrm{e}^{\alpha} t^{\alpha}}{\alpha^{\alpha} j^{\alpha}} \right)^{j}, \quad j \to \infty.$$

Example 2.2 (continuation with $a = b = 1$, Laguerre polynomials). Recall that in this case $V(t) = t+1$ for $t \geq 0$. Put $f(t) = t$ and $g(t) = 1$ for $t \geq 0$ and note that $f^{*(j)}(t) = t^j/(j!)$ and $g^{*(j)}(t) = 1$ for all $j \in \mathbb{N}$ and $t \geq 0$. This entails

$$V^{*(j)}(t) = (f+g)^{*(j)}(t) = \sum_{i=0}^{j} \binom{j}{i} (f^{*(i)} * g^{*(j-i)})(t) = \sum_{i=0}^{j} \binom{j}{i} \frac{t^i}{i!} = L_j(-t) \quad t \geq 0,$$

where L_j is the standard Laguerre polynomial. Clearly,

$$\hat{V}(s) = s^{-1} + 1, \quad \lambda(s) = \log(s+1) - \log(s), \quad \lambda'(s) = \frac{1}{s+1} - \frac{1}{s}, \quad s_- = 0, \quad s_+ = +\infty,$$
$$\lambda''(s) = \frac{1}{s^2} - \frac{1}{(1+s)^2}, \quad \lambda'''(s) = \frac{2}{(1+s)^3} - \frac{2}{s^3}, \quad \kappa(j) = \frac{1}{2}\left(\left(1 + \frac{4j}{t}\right)^{1/2} - 1 \right).$$

We now distinguish three cases according to the limit behavior of the ratio j/t.

CASE I: $j = o(t)$ and $j \to +\infty$. We are in the setting of Theorem 3.1(a), since in this case

$$\kappa(j) = \frac{j}{t} - \frac{j^2}{t^2} + O\left(\frac{j^3}{t^3}\right), \quad j \to \infty,$$

and, moreover,

$$a(j) \;\sim\; t j^{-1/2}, \quad \kappa(j)a(j) \;\sim\; \sqrt{j}, \quad T_j \;\sim\; 7t/j, \quad j \to \infty. \tag{4.1}$$

The only condition which remains to be checked is (3.2). Upon squaring it takes the form

$$\limsup_{j\to\infty} \sup_{|z|\geq\gamma} \left(\frac{1 + \frac{z^2}{(\kappa(j)+1)^2 T_j^2}}{1 + \frac{z^2}{\kappa^2(j) T_j^2}} \right) < 1 \quad \text{for every } \gamma > 0.$$

For $0 < b < a$, the function $z \mapsto (1+z^2/a)/(1+z^2/b)$ is decreasing on $(0,\infty)$. Thus,

$$\limsup_{j\to\infty} \sup_{|z|\geq\gamma} \left(\frac{1 + \frac{z^2}{(\kappa(j)+1)^2 T_j^2}}{1 + \frac{z^2}{\kappa^2(j) T_j^2}} \right) = \limsup_{j\to\infty} \left(\frac{1 + \frac{\gamma^2}{(\kappa(j)+1)^2 T_j^2}}{1 + \frac{\gamma^2}{\kappa^2(j) T_j^2}} \right) = \frac{1}{1+\gamma^2/49} < 1.$$

In view of $t\kappa(j) = j/(1+\kappa(j))$, formula (3.3) reads

$$L_j(-t) = V^{*(j)}(t) \;\sim\; \frac{1}{(2\pi j)^{1/2}} (G(\kappa(j)))^j, \quad j \to \infty,$$

where

$$G(z) := \exp\left\{ \frac{1}{1+z} + \log\left(1 + \frac{1}{z}\right) \right\} = \left(1 + \frac{1}{z}\right) \exp\left(\frac{1}{1+z}\right), \quad z > 0.$$

CASE II: $t \sim c \cdot j$ for some $c > 0$. In this case $\lim_{j\to\infty} \kappa(j) = \phi(c) := 2^{-1}((1+4c^{-1})^{1/2} - 1) > 0$. Since also

$$\lambda''(\kappa(j)) \ \to \ \lambda''(\phi(c)) = \frac{1}{(\phi(c))^2} - \frac{1}{(1+\phi(c))^2} = \frac{1+2\phi(c)}{(\phi(c))^2(1+\phi(c))^2}$$
$$= \frac{1+2\phi(c)}{c^{-2}} = c^2(1+4c^{-1})^{1/2}, \quad j \to \infty$$

and V is nonarithmetic, part (b) of Theorem 3.1 yields

$$L_j(-t) = V^{*(j)}(t) \ \sim \ \frac{(c^2+4c)^{-1/4}}{\phi(c)} \frac{1}{\sqrt{2\pi t}} (G(\kappa(j)))^j, \quad j \to \infty.$$

CASE III: $t = o(j)$. In this case $\kappa(j) \sim (j/t)^{1/2}$ as $j \to \infty$. In particular, $\kappa(j)$ diverges to infinity. Also,

$$\lambda''(\kappa(j)) \ \sim \ 2(t/j)^{3/2}, \quad a(j) \ \sim \ (2t^{3/2} j^{-1/2})^{1/2},$$
$$\kappa(j)a(j) \ \sim \ 2^{1/2}(tj)^{1/4}, \quad T_j \ \sim 3(t/j)^{1/2}, \quad j \to \infty,$$

and (3.1) holds true. Unfortunately, (3.2) does not hold in this regime which means that Theorem 3.1 is not applicable. Nevertheless, a formal substitution of the above ingredients into (3.3) yields

$$L_j(-t) = V^{*(j)}(t) \ \sim \ \frac{1}{2(\pi^2 tj)^{1/4}} (G(\kappa(j))^j, \quad j \to \infty, \tag{4.2}$$

which under the additional assumption $t = o(j^{1/3})$ as $j \to \infty$ simplifies to

$$L_j(-t) = V^{*(j)}(t) \ \sim \ \frac{1}{2(\pi^2 tj)^{1/4}} \exp\big(-t/2 + 2(tj)^{1/2}\big), \quad t \to \infty. \tag{4.3}$$

Formula (4.3) is indeed correct. Actually, it is a particular case of Perron's formula for the Laguerre polynomials, see formula (8.22.3) on p. 199 in [21].

Example 2.2 (general functions of linear growth). Consider a more general situation $V(x) = a \cdot x + \varepsilon(x)$, where $a > 0$ and $x \mapsto \varepsilon(x)$ is a function of finite total variation on $[0,\infty)$. Suppose that[3], for some $p \in \{3,4,5,\ldots\}$, $\int_{[0,\infty)} y^p |\mathrm{d}\varepsilon(y)| < \infty$ and put

$$\beta_k := \int_{[0,\infty)} y^k \mathrm{d}\varepsilon(y), \quad k = 0,1,\ldots,p-1.$$

[3]By definition, $\int_{[0,\infty)} y^p |\mathrm{d}\varepsilon(y)| = \int_{[0,\infty)} y^p \mathrm{d}\Xi_{[0,y]}(\varepsilon)$, where $\Xi_{[0,y]}(\varepsilon)$ is the total variation of ε on $[0,y]$.

Therefore, as $s \to 0+$, by Taylor's expansion,

$$\hat{V}(s) = a/s + \sum_{k=0}^{p-1} \frac{\beta_k (-s)^k}{k!} + O(s^p), \quad -\hat{V}'(s) = a/s^2 + \sum_{k=0}^{p-2} \frac{\beta_{k+1}(-s)^k}{k!} + O(s^{p-1}),$$
$$\hat{V}''(s) = 2a/s^3 + \sum_{k=0}^{p-3} \frac{\beta_{k+2}(-s)^k}{k!} + O(s^{p-2}), \quad -\hat{V}'''(s) = 6a/s^4 + \sum_{k=0}^{p-4} \frac{\beta_{k+3}(-s)^k}{k!} + O(s^{p-3}) \tag{4.4}$$

and

$$\lambda(s) = -\log s + \log a + \sum_{k=1}^{p} \delta_k s^k + O(s^{p+1}), \quad s \to 0+,$$

where δ_k's are the values of certain multivariate polynomials at points β_ℓ/a, $\ell = 0, \dots, p-1$, which can be calculated explicitly. For instance, $\delta_1 = \beta_0/a$, $\delta_2 = -\beta_1/a - \beta_0^2/(2a^2)$, $\delta_3 = \beta_2/(2a) + \beta_0\beta_1/a^2 + \beta_0^3/(3a^3)$. Since the derivatives of λ are the known rational functions of the derivatives of $\hat{V}$, we conclude from (4.4) that

$$\lambda'(s) = -1/s + \sum_{k=1}^{p} k\delta_k s^{k-1} + O(s^p), \quad \lambda''(s) \sim 1/s^2, \quad \lambda'''(s) \sim -2/s^3, \quad s \to 0+.$$

Denote by ℓ a solution to $-\lambda'(\ell(s)) = 1/s$ for $s > 0$. In particular, $\kappa(j) = \ell(j/t)$. By asymptotic inversion, we infer

$$\ell(s) = s + \sum_{k=2}^{p+1} \iota_{k-1} s^k + O(s^{p+2}), \quad s \to 0+ \tag{4.5}$$

for some real coefficients $\iota_1, \dots, \iota_p$. For instance, $\iota_1 = -\delta_1$, $\iota_2 = \delta_1^2 - 2\delta_2$, $\iota_3 = -\delta_1^3 + 6\delta_1\delta_2 - 3\delta_3$. Assume that $j \to +\infty$ and $t/j \to +\infty$. A specialization of (4.5) yields

$$\kappa(j) = j/t + \sum_{k=2}^{p+1} \iota_{k-1} (j/t)^k + O((j/t)^{p+2}). \tag{4.6}$$

By virtue of

$$a(j) \sim t j^{-1/2}, \quad \kappa(j)a(j) \sim j^{1/2}, \quad T_j \sim 7t/j, \quad j \to \infty,$$

relation (3.1) holds true. Next, we check that (3.2) holds without any extra assumptions. Using that $\int_{[0,\infty)} |\mathrm{d}\varepsilon(y)| < \infty$ and writing

$$\hat{V}(\kappa(j) - \mathrm{i}z/T_j) = \frac{a}{\kappa(j) - \mathrm{i}z/T_j} + \int_{[0,\infty)} \mathrm{e}^{-x(\kappa(j) - \mathrm{i}z/T_j)} \mathrm{d}\varepsilon(x),$$

we infer

$$\sup_{|z| \ge \gamma} |\hat{V}(\kappa(j) - \mathrm{i}z/T_j)| \le \sup_{|z| \ge \gamma} \frac{a}{|\kappa(j) - \mathrm{i}z/T_j|} + \int_{[0,\infty)} \mathrm{e}^{-x\kappa(j)} |\mathrm{d}\varepsilon(x)| = \frac{a}{((\kappa(j))^2 + z^2/T_j^2)^{1/2}} + O(1).$$

Thus, (3.2) follows upon noticing that

$$\limsup_{j\to\infty}\sup_{|z|\ge\gamma}\frac{|\hat V(\kappa(j)-\mathrm{i}z/T_j)|}{\hat V(\kappa(j))}=\limsup_{j\to\infty}\frac{j}{at}\sup_{|z|\ge\gamma}|\hat V(\kappa(j)-\mathrm{i}z/T_j)|$$
$$\le\lim_{j\to\infty}\frac{j}{t((\kappa(j))^2+\gamma^2/T_j^2)^{1/2}}=\frac{1}{(1+\gamma^2/49)^{1/2}}<1.$$

Thus, part (a) of Theorem 3.1 is applicable. Using (4.5) in combination with $(\lambda(\ell(s))+\ell(s)/s)'=-\ell(s)/s^2$ we conclude that

$$\lambda(\ell(s))+\ell(s)/s=-\log s+C-\sum_{k=1}^{p}(\iota_k/k)s^k+O(s^{p+1}),\quad s\to 0+,$$

for some $C\in\mathbb{R}$. Using $\lim_{s\to 0+}(\lambda(s)+\log s)=\log a$, we conclude that $C=1+\log a$, whence

$$(\hat V(\kappa(j)))^j\mathrm{e}^{t\kappa(j)}=\mathrm{e}^{j(\lambda(\ell(j/t))+\ell(j/t)/(j/t))}=\frac{t^j}{j^j}a^j\mathrm{e}^j\exp\left(-\sum_{k=1}^{p}(\iota_k/k)j^{k+1}/t^k+O(j^{p+2}/t^{p+1})\right).$$

Using Stirling's approximation we obtain from (3.3)

$$V^{*(j)}(t)\ \sim\ \frac{t^ja^j}{j!}\exp\left(-\sum_{k=1}^{p}(\iota_k/k)j^{k+1}/t^k+O(j^{p+2}/t^{p+1})\right),\quad j\to\infty,\tag{4.7}$$

which is meaningful provided that $j=o(t^{(p+1)/(p+2)})$. A particular case of this formula for $p=1$ was derived by a completely different technique in [3].

Let $\xi_1,\xi_2,\ldots$ be independent copies of an almost surely (a.s.) positive random variable ξ. The function U defined by $U(t):=1+\sum_{n\ge1}\mathbb{P}\{\xi_1+\ldots+\xi_n\le t\}$ for $t\ge0$ is called renewal function. The original motivation behind the present work has been our desire to understand the asymptotic of $U^{*(j)}(t)$ for various asymptotic regimes of divergent j and t. Now we explain that (4.7) does the job under some additional assumptions imposed on the distribution of ξ. Assume that $\mathbb{E}[\xi^{p+2}]<\infty$ for some $p\in\mathbb{N}$ and that some convolution power of the distribution of ξ has an absolutely continuous component. Put $\mathfrak{m}:=\mathbb{E}[\xi]$. By Remark 3.1.7 (ii) in [10], the function $x\mapsto U(x)-x/\mathfrak{m}$ has a finite total variation and, furthermore, $\int_{[0,\infty)}y^p|\mathrm{d}(U(y)-y/\mathfrak{m})|<\infty$. Since

$$\int_{[0,\infty)}\mathrm{e}^{-sy}\mathrm{d}(U(y)-y/\mathfrak{m})=\frac{1}{1-\mathbb{E}[\mathrm{e}^{-s\xi}]}-\frac{1}{\mathfrak{m}s},\quad s>0,$$

an application of Lebesgue's dominated convergence theorem yields

$$\beta_k=\int_{[0,\infty)}y^k\mathrm{d}(U(y)-y/\mathfrak{m})=(-1)^k\lim_{s\to0+}\Big(\frac{1}{1-\mathbb{E}[\mathrm{e}^{-s\xi}]}-\frac{1}{\mathfrak{m}s}\Big)^{(k)},\quad k=0,1,\ldots,p-1.$$

In particular,

$$\beta_0=\mathbb{E}[\xi^2]/(2\mathfrak{m}^2),\quad \beta_1=\mathbb{E}[\xi^3]/(6\mathfrak{m}^2)-(\mathbb{E}[\xi^2])^2/(4\mathfrak{m}^3),$$
$$\beta_2=\mathbb{E}[\xi^4]/(12\mathfrak{m}^2)-\mathbb{E}[\xi^3]\mathbb{E}[\xi^2]/(3\mathfrak{m}^3)+(\mathbb{E}[\xi^2])^3/(4\mathfrak{m}^4).$$

Thus, assuming, for instance, that $p = 3$ we obtain from (4.7)

$$U^{*(j)}(t) \sim \frac{t^j}{\mathfrak{m}^j j!} \exp\left(b_1 j^2/t + b_2 j^3/t^2 + b_3 j^4/t^3\right), \quad j \to \infty$$

whenever $j = o(t^{4/5})$. Here, $b_1 = \mathbb{E}[\xi^2]/(2\mathfrak{m})$, $b_2 = -\mathbb{E}[\xi^3]/(6\mathfrak{m})$,

$$b_3 = \frac{1}{24}\Big(\frac{\mathbb{E}[\xi^4]}{\mathfrak{m}} + \frac{2\mathbb{E}[\xi^2]\mathbb{E}[\xi^3]}{\mathfrak{m}^2} - \frac{(\mathbb{E}[\xi^2])^3}{\mathfrak{m}^3}\Big).$$

The expression for b_3 makes it clear that, unlike b_1 and b_2, b_j with $j \geq 3$ is not a function of $\mathbb{E}[\xi^{j+1}]$ alone.

Example 2.5 (continuation). Recall that $V(t) = \mathrm{e}^{at} - 1$ and there is an explicit formula (2.5) for $V^{*(j)}(t)$. Further

$$\lambda''(\kappa(j)) = t^2/j^2, \quad a(j) = t\,j^{-1/2}, \quad \kappa(j)a(j) = at\,j^{-1/2} + j^{1/2}, \quad T_j = 7t/j.$$

Note that $\kappa(j)a(j)$ always diverges as $j \to \infty$ but in order to ensure subexponential divergence we need to assume that

$$t = o(\mathrm{e}^{\gamma j}), \quad j \to \infty \quad \text{for every } \gamma > 0. \tag{4.8}$$

Condition (3.1) holds true whereas condition (3.2) follows from the formula

$$\frac{\hat{V}(\kappa(j) - \mathrm{i}z/T_j)}{\hat{V}(\kappa(j))} = \frac{1}{1 - \mathrm{i}z/7}.$$

Thus, Theorem 3.1(a) yields

$$V^{*(j)}(t) = \frac{1}{(j-1)!}\int_0^{at} y^{j-1}\mathrm{e}^y \mathrm{d}y \sim \frac{1}{\sqrt{2\pi}}\left(\frac{ae t}{j}\right)^j \mathrm{e}^{at}\left(\frac{at}{\sqrt{j}} + \sqrt{j}\right)^{-1}$$
$$\sim \frac{(at)^{j-1}}{(j-1)!}\mathrm{e}^{at}\frac{at}{at+j}, \quad j \to \infty,$$

where the last equivalence is a consequence of Stirling's approximation. This asymptotic formula holds under the sole assumption (4.8).

Our results have a neat interpretation in the theory of branching processes. Let $\mathscr{P}$ be a random point process on $[0,\infty)$, that is, a random element defined on some underlying probability space, which takes values in the space of locally finite point measures on $[0,\infty)$. Now we define a branching random walk generated by $\mathscr{P}$. A population is initiated by one individual, the ancestor located at the origin. The ancestor generates offspring, constituting the first generation, whose birth locations are given by the point process $\mathscr{P}$. The first generation gives rise to the second generation, where the differences in birth locations between individuals and their respective mothers are distributed in accordance with copies of $\mathscr{P}$, which are independent for distinct mothers. The second generation produces the third one, and so on. Each individual in these successive generations operates independently of others.

By definition, the intensity measure of $\mathscr{P}$ is a deterministic measure given by

$$\mu(A) := \mathbb{E}[\mathscr{P}(A)] \quad \text{for Borel } A \subseteq [0,\infty).$$

Thus, $\mu(A)$ is the average number of individuals of the first generation with birth locations belonging to A. Denote by $N_j(t)$ the number of individuals of the generation j with birth locations in $[0,t]$. Using induction in j, it can be checked that $\mathbb{E}[N_j(t)] = \mu^{*(j)}([0,t])$. Our results provide a robust tool for the asymptotic analysis of these averages in generations j, with j and t tending to infinity.

For the time being, put $V(x) = \mu([0,x])$ for $x \geq 0$. Assume that there exists $\theta^* > 0$ such that $\lambda(\theta^*) = \theta^*\lambda'(\theta^*)$. A specialization of Proposition 3.2 in [12] yields for $\theta \in \operatorname{Int}\mathscr{D}$, $\theta < \theta^*$

$$\limsup_{j\to\infty} \sup_{y\in\mathbb{R}} \left| j^{1/2} \mathrm{e}^{-\theta(-\lambda'(\theta)j+y)} (\hat{V}(\theta))^{-j} N_j(-\lambda'(\theta)j+y) - \frac{W(\theta)}{\theta} \frac{1}{(2\pi\lambda''(\theta))^{1/2}} \mathrm{e}^{-\frac{y^2}{2\lambda''(\theta)j}} \right| = 0 \quad \text{a.s.},$$

where $W(\theta)$ is a random variable with $\mathbb{E}[W(\theta)] = 1$ (the a.s. limit of the uniformly integrable Biggins martingale). In particular, if $y = O(j^{1/2})$ we infer

$$N_j(-\lambda'(\theta)j+y) \;\sim\; \frac{W(\theta)}{\theta} \mathrm{e}^{\theta(-\lambda'(\theta)j+y)} (\hat{V}(\theta))^j p(0,\lambda''(\theta)j;y), \quad j\to\infty \quad \text{a.s.}$$

Formula (3.6) demonstrates that this a.s. convergence is accompanied by the convergence in mean. Finally, we note that we are not aware of any results on the a.s. convergence of $N_j(t)$ with diverging $j = o(t)$.

5. Proofs

Denote by $X_{j,t}$ a random variable with the distribution

$$\mathbb{P}\{X_{j,t} \in \mathrm{d}s\} = \frac{\mathrm{e}^{-\kappa(j)s} V^{*(j)}(\mathrm{d}s)}{(\hat{V}(\kappa(j)))^j}, \quad s \geq 0, \quad j \in \mathbb{N} \tag{5.1}$$

and note that

$$\mathbb{E}[\mathrm{e}^{-uX_{j,t}}] = \frac{(\hat{V}(\kappa(j)+u))^j}{(\hat{V}(\kappa(j)))^j} = \mathrm{e}^{j(\lambda(\kappa(j)+u)-\lambda(\kappa(j)))}, \quad u \geq 0, \quad j \in \mathbb{N}. \tag{5.2}$$

The original choice of $\kappa(j)$ was motivated by this change of measure and more importantly by the fact that $\mathbb{E}[X_{j,t}] = t$ which can be readily seen by differentiating (5.2). Also, observe that $\operatorname{Var}[X_{j,t}] = (a(j))^2$. Put $\bar{X}_{j,t} = t - X_{j,t}$ for $j \in \mathbb{N}$.

Integrating over $s \in [0,t]$ in (5.1) we obtain

$$V^{*(j)}(t) = (\hat{V}(\kappa(j)))^j \mathrm{e}^{t\kappa(j)} \mathbb{E}[\mathrm{e}^{-\kappa(j)\bar{X}_{j,t}} \mathbb{1}_{\{\bar{X}_{j,t}\geq 0\}}], \quad j \in \mathbb{N}.$$

Thus, Theorem 3.1 is equivalent to the following proposition.

Proposition 5.1. *(i) Under the assumptions of part (a) of Theorem 3.1*

$$\lim_{j\to\infty} \kappa(j)a(j)\mathbb{E}[\mathrm{e}^{-\kappa(j)\bar{X}_{j,t}}\mathbb{1}_{\{\bar{X}_{j,t}\geq 0\}}] = p(0,1;0) = (2\pi)^{-1/2}.$$

(ii) Under the assumptions of part (b) of Theorem 3.1

$$\lim_{j\to\infty} \theta j^{1/2}\mathbb{E}[\mathrm{e}^{-\kappa(j)\bar{X}_{j,t}}\mathbb{1}_{\{\bar{X}_{j,t}\geq 0\}}] = p(0,\sigma^2;0) = (2\pi\sigma^2)^{-1/2}.$$

To explain the appearance of the normal densities in Proposition 5.1, and as preparation for its proof, we introduce, for each $j \in \mathbb{N}$, a nonnegative random variable $X'_{j,t}$ with the Laplace transform

$$\mathbb{E}[\mathrm{e}^{-uX'_{j,t}}] = \frac{\hat{V}(\kappa(j)+u)}{\hat{V}(\kappa(j))}, \quad u \geq 0. \tag{5.3}$$

Formula (5.2) shows that $\bar{X}_{j,t}$ is the sum of j independent copies of $t/j - X'_{j,t}$ and it is straightforward to verify that $\bar{X}_{j,t}/a(j)$ converges in distribution to a standard normal law, see (5.8) below. However, this central limit theorem alone is insufficient for deriving the asymptotics in Proposition 5.1, and more subtle properties of $\bar{X}_{j,t}$ are required. Our proof of Proposition 5.1 hinges on Proposition 5.2 below, which is essentially a central *local limit theorem* for a specific triangular array of random variables, where the jth row consists of j independent copies of $t/j - X'_{j,t}$.

Proposition 5.2. *(i) Under the assumptions of part (a) of Theorem 3.1,*

$$\lim_{j\to\infty}\sup_{x\in\mathbb{R}} \left| \frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t} \in [xa(j), xa(j)+\delta(j))\} - p(0,1;x)\right| = 0$$

for any sequence $(\delta(j))_{j\in\mathbb{N}}$ of positive numbers satisfying

$$\lim_{j\to\infty} \frac{\delta(j)}{a(j)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} \frac{\mathrm{e}^{\gamma j}\delta(j)}{a(j)} = \infty$$

for every fixed $\gamma > 0$.
(ii) Under the assumptions of part (b) of Theorem 3.1, locally uniformly in $\delta > 0$,

$$\lim_{j\to\infty}\sup_{x\in\mathbb{R}} \left|\mathbb{P}\{\bar{X}_{j,t} \in [xa(j), (x+\delta)a(j))\} - \delta p(0,1;x)\right| = 0. \tag{5.4}$$

Our proof of Proposition 5.2 is a non-trivial adaptation of the standard proofs of local limit theorems for sums of independent random variables with a fixed distribution, see [14, 20], to the triangular array setting. Surprisingly, not much seems to be known about local limit theorems in this setting. The only general result we are aware of is [13], which, however, does not serve our purposes.

5.1. **Proof of Proposition 5.2.** If the assumptions of part (i) prevail, we argue along the lines of the proof of a local limit theorem for standard random walks (Corollary 1 in [20]).

For $j \in \mathbb{N}$ and $\delta > 0$, put

$$v_{j,\delta} := \delta^{-1}\mathbb{P}\{\bar{X}_{j,t} \in [x, x+\delta)\}, \quad x \in \mathbb{R}.$$

The so defined function is a version of the density of a random variable $\bar{X}_{j,t} - U_\delta$, where the variables $\bar{X}_{j,t}$ and U_δ are independent, and U_δ has a uniform distribution on $(0,\delta)$. In particular,

$$\psi_\delta(z) := \mathbb{E}[\mathrm{e}^{-\mathrm{i}zU_\delta}] = \frac{1-\mathrm{e}^{-\mathrm{i}\delta z}}{\mathrm{i}\delta z}, \quad z \in \mathbb{R}.$$

Thus,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}zx} v_{j,\delta}(xa(j))\mathrm{d}x = \frac{1}{a(j)} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}zx/a(j)} v_{j,\delta}(x)\mathrm{d}x = \frac{1}{a(j)} \mathbb{E}\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)} \psi_\delta(z/a(j)). \tag{5.5}$$

The function φ defined by $\varphi(z) := (1-|z|)\mathbb{1}_{(-1,1)}(z)$ for $z \in \mathbb{R}$ is the characteristic function of an absolutely continuous probability distribution with density g given by

$$g(x) := \frac{1-\cos x}{\pi x^2}, \quad x \in \mathbb{R},$$

see Chapter XV.2 in [9]. For each $b > 0$, put $g_b(x) := bg(bx)$, $x \in \mathbb{R}$ and $\varphi_b(z) := \varphi(z/b)$, $z \in \mathbb{R}$. Then φ_b is the characteristic function of a distribution with density g_b. Using this in combination with (5.5) we infer, for $j \in \mathbb{N}$ and $b > 0$,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}zx} \int_{\mathbb{R}} v_{j,\delta}((x-y)a(j))g_b(y)\mathrm{d}y\mathrm{d}x = \frac{1}{a(j)} \mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}] \psi_\delta(z/a(j))\varphi_b(z), \quad z \in \mathbb{R}.$$

By the Fourier inversion,

$$\int_{\mathbb{R}} v_{j,\delta}((x-y)a(j))g_b(y)\mathrm{d}y = \frac{1}{2\pi a(j)} \int_{-b}^{b} \mathrm{e}^{-\mathrm{i}zx} \mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}] \psi_\delta(z/a(j))\varphi_b(z)\mathrm{d}z, \quad x \in \mathbb{R}.$$

Equivalently, for $x \in \mathbb{R}$,

$$\frac{a(j)}{\delta} \int_{\mathbb{R}} \mathbb{P}\{\bar{X}_{j,t} \in [(x-y)a(j), (x-y)a(j)+\delta)\} g_b(y)\mathrm{d}y = \frac{1}{2\pi} \int_{-b}^{b} \mathrm{e}^{-\mathrm{i}zx} \mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}] \psi_\delta(z/a(j))\varphi_b(z)\mathrm{d}z. \tag{5.6}$$

We shall prove that, with $b = b(j) = \mathrm{e}^{cj}$ for some $c > 0$ to be specified below and $\delta = \delta(j)$, where $(\delta(j))_{j\in\mathbb{N}}$ is any sequence of positive numbers satisfying $\delta(j) = o(a(j))$ as $j \to \infty$,

$$\lim_{j\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{a(j)}{\delta(j)} \int_{\mathbb{R}} \mathbb{P}\{\bar{X}_{j,t} \in [(x-y)a(j), (x-y)a(j)+\delta(j))\} g_{b(j)}(y)\mathrm{d}y - p(0,1;x) \right| = 0. \tag{5.7}$$

As we have already mentioned, $\bar{X}_{j,t}$ is the sum of j independent copies of the random variable $t/j - X'_{j,t}$, where $X'_{j,t}$ has Laplace transform (5.3). According to Lemma 1 on p. 109 in [15],

$$\left| \mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}] - \mathrm{e}^{-z^2/2} \right| \le 16|z|^3 \mathrm{e}^{-z^2/3} L_j, \quad |z| \le (4L_j)^{-1}, \tag{5.8}$$

where

$$L_j := \frac{1}{j^{1/2}}\mathbb{E}\left[\left|\frac{X'_{j,t}-t/j}{(\lambda''(\kappa(j)))^{1/2}}\right|^3\right].$$

Now we check that under the present assumptions

$$\lim_{j\to\infty} L_j = 0. \tag{5.9}$$

Indeed,

$$\mathbb{E}\left[\left|\frac{X'_{j,t}-t/j}{(\lambda''(\kappa(j)))^{1/2}}\right|^3\right] \le \frac{4((t/j)^3+\mathbb{E}[(X'_{j,t})^3])}{(\lambda''(\kappa(j)))^{3/2}}$$
$$= 4\left(\frac{|\lambda'(\kappa(j))|}{(\lambda''(\kappa(j)))^{1/2}}\right)^3 + \frac{4|\widehat{V}'''(\kappa(j))|}{(\lambda''(\kappa(j)))^{3/2}\widehat{V}(\kappa(j))} = \frac{4T_j}{\sqrt{\lambda''(\kappa(j))}},$$

and the right-hand side is $o(j^{1/2})$ as $j\to+\infty$ in view of (3.1). This proves (5.9). As a consequence of (5.9), $X_{j,t}$ satisfies the central limit theorem

$$\frac{X_{j,t}-t}{a(j)} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \quad j\to\infty,$$

where $\mathrm{N}(0,1)$ is a random variable with the standard normal law.

Using $p(0,1;x) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zx}\mathrm{e}^{-z^2/2}\mathrm{d}z$ for $x\in\mathbb{R}$ we obtain from (5.6)

$$2\pi\sup_{x\in\mathbb{R}}\left|\frac{a(j)}{\delta(j)}\int_{\mathbb{R}}\mathbb{P}\{\bar{X}_{j,t}\in[(x-y)a(j),(x-y)a(j)+\delta(j))\}g_{b(j)}\mathrm{d}y - p(0,1;x)\right|$$
$$= \sup_{x\in\mathbb{R}}\left|\int_{-b(j)}^{b(j)}\mathrm{e}^{-\mathrm{i}zx}\mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}]\psi_{\delta(j)}(z/a(j))\varphi_{b(j)}(z)\mathrm{d}z - \int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}zx}\mathrm{e}^{-z^2/2}\mathrm{d}z\right|$$
$$\le \int_{-(4L_j)^{-1}}^{(4L_j)^{-1}}\left|\mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}]\psi_{\delta(j)}(z/a(j))\varphi_{b(j)}(z) - \mathrm{e}^{-z^2/2}\right|\mathrm{d}z$$
$$+\int_{(4L_j)^{-1}<|z|\le b(j)}|\mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}]|\mathrm{d}z + \int_{|z|>(4L_j)^{-1}}\mathrm{e}^{-z^2/2}\mathrm{d}z$$
$$=: A(j)+B(j)+C(j).$$

To analyze $A(j)$, write, for $|z|\le(4L_j)^{-1}$,

$$\left|\mathbb{E}\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}\psi_{\delta(j)}(z/a(j))\varphi_{b(j)}(z) - \mathrm{e}^{-z^2/2}\right|$$
$$\le \left|\mathbb{E}[\mathrm{e}^{\mathrm{i}z\bar{X}_{j,t}/a(j)}] - \mathrm{e}^{-z^2/2}\right| + |\varphi_{b(j)}(z)-1|\mathrm{e}^{-z^2/2} + \left|\psi_{\delta(j)}(z/a(j))-1\right|\mathrm{e}^{-z^2/2}$$
$$\le 16|z|^3\mathrm{e}^{-z^2/3}L_j + |z|\mathrm{e}^{-z^2/2}/b(j) + |z|\mathrm{e}^{-z^2/2}\delta(j)/(2a(j)).$$

We have used (5.8) to obtain the first term on the right-hand side. The third term is a consequence of the inequality $|1-\mathrm{e}^{-\mathrm{i}x}-\mathrm{i}x|\le x^2/2$ for $x\in\mathbb{R}$, see Lemma 1 on p. 512 in [9]. Thus, recalling

also (5.9),

$$A(j) \le L_j \int_{\mathbb{R}} |z|^3 \mathrm{e}^{-z^2/3} \mathrm{d}z + \big(\mathrm{e}^{-cj} + \delta(j)/(2a(j))\big) \int_{\mathbb{R}} |z| \mathrm{e}^{-z^2/2} \mathrm{d}z \ \to \ 0, \quad j \to \infty.$$

Now we work with $B(j)$. As we have shown, $L_j \le 4T_j/a(j)$. Thus,

$$B(j) = \int_{(4L_j)^{-1}<|u|\le b(j)} \Big| \frac{\hat{V}(\kappa(j) - \mathrm{i}u/a(j))}{\hat{V}(\kappa(j))} \Big|^j \mathrm{d}u \le 2b(j) \sup_{|u| \ge (4L_j a(j))^{-1}} \Big| \frac{\hat{V}(\kappa(j) - \mathrm{i}u)}{\hat{V}(\kappa(j))} \Big|^j$$
$$\le 2b(j) \sup_{|u| \ge (16T_j)^{-1}} \Big| \frac{\hat{V}(\kappa(j) - \mathrm{i}u)}{\hat{V}(\kappa(j))} \Big|^j.$$

By virtue of (3.2), for large $j \in \mathbb{N}$,

$$\sup_{|u| \ge 1/16} \Big| \frac{\hat{V}(\kappa(j) - \mathrm{i}u/T_j)}{\hat{V}(\kappa(j))} \Big| \le \mathrm{e}^{-c_1}$$

for some $c_1 > 0$. Hence, recalling that $b(j) = \mathrm{e}^{cj}$ we obtain $B(j) \le 2\mathrm{e}^{(c-c_1)j} \to 0$ as $j \to \infty$ whenever $c \in (0, c_1)$. Since $\lim_{j\to\infty} L_j = 0$, we immediately conclude that $\lim_{j\to\infty} C(j) = 0$. Combining fragments together we arrive at (5.7).

Let $(\rho(j))_{j\in\mathbb{N}}$ be a sequence of positive numbers tending to zero, to be specified below. Replacing in (5.7) first $\delta(j)$ with $\delta(j)(1 \pm 2\rho(j))$ and then x with $x \mp \rho(j)\delta(j)/a(j)$ yields

$$\limsup_{j\to\infty} \sup_{x\in\mathbb{R}} \Big| \frac{a(j)}{(1 \pm 2\rho(j))\delta(j)} \int_{\mathbb{R}} \mathbb{P}\{\bar{X}_{j,t} \in [(x-y)a(j) \mp \rho(j)\delta(j), (x-y)a(j) + (1 \pm \rho(j))\delta(j))\}$$
$$\times g_{b(j)}(y)\mathrm{d}y - p(0,1; x \mp \rho(j)\delta(j)/a(j)) \Big| = 0.$$

Thus, given $\varepsilon > 0$, for all $x \in \mathbb{R}$ and large enough j,

$$D_j(x) := \frac{a(j)}{\delta(j)} \int_{\mathbb{R}} \mathbb{P}\{\bar{X}_{j,t} \in [(x-y)a(j) + \rho(j)\delta(j), (x-y)a(j) + (1-\rho(j))\delta(j))\} g_{b(j)}(y)\mathrm{d}y$$
$$\ge (1 - 2\rho(j)) p(0,1; x + \rho(j)\delta(j)/a(j)) - \varepsilon.$$

Observe that, for any $x \in \mathbb{R}$,

$$[x - y + \rho(j)\delta(j)/a(j), x - y + (1-\rho(j))\delta(j)/a(j)) \subseteq [x, x + \delta(j)/a(j))$$

whenever $|y| \le \rho(j)\delta(j)/a(j)$. This entails

$$D_j(x) \le \frac{a(j)}{\delta(j)} \mathbb{P}\{\bar{X}_{j,t} \in [xa(j), xa(j) + \delta(j))\} \int_{|y| \le \rho(j)\delta(j)/a(j)} g_{b(j)}(y)\mathrm{d}y$$
$$+ \frac{a(j)}{\delta(j)} \int_{|y| > \rho(j)\delta(j)/a(j)} g_{b(j)}(y)\mathrm{d}y. \quad (5.10)$$

Recall that so far the sequence $(\delta(j))_{j\in\mathbb{N}}$ has been an arbitrary sequence satisfying $\delta(j)=o(a(j))$ as $j\to\infty$. We shall now also exploit the assumption that $\delta(j)/a(j)$ decays subexponentially. Let $(\rho(j))_{j\in\mathbb{N}}$ be defined by

$$\frac{a(j)}{(\rho(j))^2\delta(j)}=b(j)=\mathrm{e}^{cj},\quad j\in\mathbb{N}. \tag{5.11}$$

Such a choice and subexponential decay of $\delta(j)/a(j)$ guarantees $\lim_{j\to\infty}\rho(j)=0$. With this choice of ρ and using that $g_{b(j)}(y)=b(j)g(b(j)y)$ we conclude from (5.10) that

$$D_j(x)\le\frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t}\in[xa(j),xa(j)+\delta(j))\}\int_{|y|\le 1/\rho(j)}g(y)\mathrm{d}y+\frac{a(j)}{\delta(j)}\int_{|y|>1/\rho(j)}g(y)\mathrm{d}y$$

and thereupon

$$\begin{aligned}&\frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t}\in[xa(j),xa(j)+\delta(j))\}\\&\ge\Big((1-2\rho(j))p(0,1;x+\rho(j)\delta(j)/a(j))-\varepsilon-\frac{a(j)}{\delta(j)}\int_{|y|>1/\rho(j)}g(y)\mathrm{d}y\Big)\Big(\int_{|y|\le 1/\rho(j)}g(y)\mathrm{d}y\Big)^{-1}.\end{aligned} \tag{5.12}$$

Uniform continuity of p in combination with $\lim_{j\to\infty}\big(\rho(j)\delta(j)/a(j)\big)=0$ entails

$$\lim_{j\to\infty}\sup_{x\in\mathbb{R}}|p(0,1;x+\rho(j)\delta(j)/a(j))-p(0,1;x)|=0.$$

Since ρ converges to 0 and g is the density of a probability distribution, $\lim_{j\to\infty}\int_{|y|\le 1/\rho(j)}g(y)\mathrm{d}y=1$. In view of $\int_{|y|>1/\rho(j)}g(y)\mathrm{d}y\le 4\pi^{-1}\int_{1/\rho(j)}^{\infty}y^{-2}\mathrm{d}y=4\pi^{-1}\rho(j)$ and the facts that ρ exhibits an exponential decay, whereas $a(j)/\delta(j)$ grows subexponentially, we conclude that

$$\lim_{j\to\infty}\frac{a(j)}{\delta(j)}\int_{|y|>1/\rho(j)}g(y)\mathrm{d}y=0.$$

Letting now in (5.12) $j\to\infty$ and then $\varepsilon\to 0+$ we obtain that, uniformly in $x\in\mathbb{R}$,

$$\liminf_{j\to\infty}\frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t}\in[xa(j),xa(j)+\delta(j))\}\ge p(0,1;x).$$

The converse inequality for the upper limit follows analogously but simpler.

Relation (5.4) is essentially Stone's local limit theorem (Corollary 1 in [20]). In view of this we omit a proof. The only thing that needs to be mentioned is that nonarithmeticity of V is equivalent to the inequality

$$\left|\frac{\hat{V}(\theta-\mathrm{i}u)}{\hat{V}(\theta)}\right|<1,\quad u\in\mathbb{R}\setminus\{0\}.$$

The proof of Proposition 5.2 is complete.

Remark 5.3. Recall that a probability measure with the characteristic function ϕ is called strongly nonlattice, if

$$\sup_{|u|>\gamma}|\phi(u)|<1$$

for every fixed $\gamma > 0$, see, for instance, equation (3) in [20]. Our assumption (3.2) is equivalent to

$$\limsup_{j\to+\infty} \sup_{|u|>\gamma} |\mathbb{E}[\mathrm{e}^{iu(X'_{j,t}-t/j)/T_j}]| < 1 \tag{5.13}$$

for every fixed $\gamma > 0$, where the random variables $X'_{j,t}$ are as defined in (5.3). Thus, (3.2) can be regarded as the assumption that the distributions of the normalized random variables $(X'_{j,t}-t/j)/T_j$ are asymptotically strongly nonlattice as $j \to \infty$.

5.2. **Proof of Proposition 5.1.** Suppose that the assumptions of part (a) of Theorem 3.1 hold. Put $\delta(j) = \theta/\kappa(j)$ for a fixed $\theta > 0$ and all $j \in \mathbb{N}$. The so defined $\delta(j)$ satisfies the assumption of part (i) of Proposition 5.2. We use a decomposition

$$\begin{aligned}\kappa(j)a(j)\mathbb{E}[\mathrm{e}^{-\kappa(j)\bar{X}_{j,t}}\mathbb{1}_{\{\bar{X}_{j,t}\geq 0\}}] &= \kappa(j)a(j)\int_{[0,\,\theta a(j))} \mathrm{e}^{-\kappa(j)y}\mathrm{d}\mathbb{P}\{\bar{X}_{j,t}\leq y\}\\ &\quad + \kappa(j)a(j)\int_{[\theta a(j),\infty)} \mathrm{e}^{-\kappa(j)y}\mathrm{d}\mathbb{P}\{\bar{X}_{j,t}\leq y\} =: K(j)+L(j).\end{aligned}$$

In view of $\lim_{j\to\infty}\kappa(j)a(j) = \infty$,

$$L(j) \leq \kappa(j)a(j)\mathrm{e}^{-\theta\kappa(j)a(j)} \;\to\; 0, \quad j\to\infty.$$

We proceed by writing

$$\begin{aligned}K(j) &\leq \kappa(j)a(j)\sum_{i=0}^{\lfloor\kappa(j)a(j)\rfloor-1}\int_{[i\theta,(i+1)\theta)} \mathrm{e}^{-y}\mathrm{d}\mathbb{P}\{\bar{X}_{j,t}\leq y/\kappa(j)\}\\ &\leq \theta\sum_{i=0}^{\lfloor\kappa(j)a(j)\rfloor-1}\mathrm{e}^{-i\theta}\Big(\frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t}\in[i\delta(j),(i+1)\delta(j))\}-p(0,1;i\delta(j)/a(j))\Big)\\ &\quad+\theta\sum_{i=0}^{\lfloor\kappa(j)a(j)\rfloor-1}\mathrm{e}^{-i\theta}\big(p(0,1;i\delta(j)/a(j))-p(0,1;0)\big)+p(0,1;0)\theta\sum_{i=0}^{\lfloor a(j)\kappa(j)\rfloor-1}\mathrm{e}^{-i\theta}\\ &=: K_1(j,\theta)+K_2(j,\theta)+K_3(j,\theta).\end{aligned}$$

Note that

$$\lim_{j\to\infty}K_3(j,\theta) = p(0,1;0)\frac{\theta}{1-\mathrm{e}^{-\theta}} \to p(0,1;0), \quad \theta\to 0+.$$

Further, given $\varepsilon > 0$ and picking $\theta > 0$ sufficiently small, we obtain, for large $j \in \mathbb{N}$ and $i \in [0, \lfloor\kappa(j)a(j)\rfloor - 1]$,

$$|p(0,1;i\delta(j)/a(j)) - p(0,1;0)| \leq \varepsilon.$$

This yields $\limsup_{\theta\to0+}\limsup_{j\to\infty}|K_2(j,\theta)| \leq \varepsilon$. Therefore, $\limsup_{\theta\to0+}\limsup_{j\to\infty}|K_2(j,\theta)| = 0$, since $K_2(j,\theta)$ does not depend on ε. Finally, by Proposition 5.2, given $\varepsilon > 0$, for large $j \in \mathbb{N}$ and all $i \in [0, \lfloor\kappa(j)a(j)\rfloor - 1]$,

$$\Big|\frac{a(j)}{\delta(j)}\mathbb{P}\{\bar{X}_{j,t}\in[i\delta(j),(i+1)\delta(j))\}-p(0,1;i\delta(j)/a(j))\Big| \leq \varepsilon.$$

As a consequence, $\lim_{j\to\infty}K_1(j,\theta) = 0$.

We have proved that $\limsup_{j\to\infty}(K(j)+L(j)) \le p(0,1;0) = (2\pi)^{-1/2}$. The converse inequality for the lower limit follows analogously. The proof of Proposition 5.1 is complete under the assumptions of part (a) of Theorem 3.1.

Under the assumptions of part (b) of Theorem 3.1, $\lim_{j\to\infty}\kappa(j) = \theta \in (0,\infty)$ and $\kappa(j)a(j) \to \infty$ holds automatically. The proof goes by exactly the same reasoning as in case (a), and we omit further details.

5.3. **Proofs of Corollaries 3.4 and 3.5.**

Proof of Corollary 3.4. Note that $t/j \to \alpha > s_-$ as $j \to \infty$ which entails $\lim_{j\to\infty}\kappa(j) = \theta$. Recall the standard notation $[x^k]f(x)$ for the coefficient in front of x^k in the Taylor expansion of f at zero.

For $\ell \in \mathbb{N}$, put $a_\ell := [t^\ell](-\lambda'(\theta - t)) = (-1)^{\ell+1}\lambda^{(\ell+1)}(\theta)/\ell!$. We intend to define the constants b_ℓ for $\ell \ge 2$, which appear in (3.6) and (3.7). To this end, we first recursively define auxiliary constants c_ℓ as follows. Put $c_1 := 1/a_1$. Fix $k \ge 2$ and assume that the values of $c_1, \dots, c_{k-1}$ have been determined. For $i = 2, \dots, k$, put $r_{i,k} := [x^k](c_1 x + \dots + c_{k-1}x^{k-1})^i$. Then c_k is defined by $a_1 c_k + \sum_{i=2}^k a_i r_{i,k} = 0$. For instance, since $r_{2,2} = c_1^2$, we conclude by solving $a_1 c_2 + a_2 r_{2,2} = 0$ that

$$c_2 = -\frac{a_2 c_1^2}{a_1} = -\frac{a_2}{a_1^3} = \frac{\lambda^{(3)}(\theta)}{2\sigma^6}.$$

Fix $r \in \mathbb{N}$. Our construction of the constants c_ℓ ensures that, for each $k = 1, \dots, r$,

$$[x^k]\left(\sum_{i=1}^r a_i\left(\sum_{\ell=1}^r c_\ell x^\ell\right)^i\right) = \sum_{i=1}^r a_i [x^k]\left(\sum_{\ell=1}^r c_\ell x^\ell\right)^i = \sum_{i=1}^k a_i [x^k]\left(\sum_{\ell=1}^k c_\ell x^\ell\right)^i$$
$$= a_1 c_k + \sum_{i=2}^k a_i [x^k]\left(\sum_{\ell=1}^{k-1} c_\ell x^\ell\right)^i = a_1 c_k + \sum_{i=2}^k a_i r_{i,k} = \begin{cases} 1, & k = 1, \\ 0, & k = 2, \dots, r. \end{cases}$$

Hence, for $r \in \mathbb{N}$, we obtain the relation

$$\sum_{\ell=1}^r a_\ell \left(\sum_{i=1}^r c_i x^i\right)^\ell = x + O(x^{r+1}), \quad x \to 0, \tag{5.14}$$

which could have also served as the definition of the constants c_k. Put

$$b_k := -c_k + \sum_{i=1}^k \frac{a_i}{i+1} r_{i+1,k+1}, \quad k \in \mathbb{N}.$$

Since $r_{k,k} = c_1^k$, we infer $b_1 = -c_1 + (a_1/2)c_1^2 = -c_1/2 = -(2\sigma^2)^{-1}$. Using $r_{2,3} = 2c_1 c_2$, we obtain $b_2 = -c_2 + (a_1/2)2c_1 c_2 + (a_2/3)c_1^3 = (a_2/3)c_1^3 = -\lambda^{(3)}(\theta)/(6\sigma^2)$.

Now we prove (3.6). According to part (b) of Theorem 3.1, it suffices to check that

$$(\hat{V}(\kappa(j)))^j \mathrm{e}^{t\kappa(j)} \sim (\hat{V}(\theta))^j \mathrm{e}^{\theta t} \mathrm{e}^{\sum_{\ell=1}^r b_\ell \frac{y^{\ell+1}}{j^\ell}}, \quad j \to \infty.$$

Put $\widetilde{\kappa}(j) = \theta - \kappa(j)$. Using (2.4) and Taylor's expansion we obtain

$$t/j = \alpha + y/j = -\lambda'(\theta - \widetilde{\kappa}(j)) = -\lambda'(\theta) + \sum_{\ell=1}^{r} a_\ell (\widetilde{\kappa}(j))^\ell + O((\widetilde{\kappa}(j))^{r+1})$$
$$= \alpha + \sum_{\ell=1}^{r} a_\ell (\widetilde{\kappa}(j))^\ell + O((\widetilde{\kappa}(j))^{r+1}).$$

Thus, $\sum_{\ell=1}^{r} a_\ell (\widetilde{\kappa}(j))^\ell + O((\widetilde{\kappa}(j))^{r+1}) = \frac{y}{j}$, and (5.14) implies that

$$\widetilde{\kappa}(j) = \sum_{\ell=1}^{r} c_\ell \Big(\frac{y}{j}\Big)^\ell + O\Big(\Big(\frac{y}{j}\Big)^{r+1}\Big), \quad j \to \infty. \tag{5.15}$$

It remains to show that

$$\lambda(\theta - \widetilde{\kappa}(j))j + (\theta - \widetilde{\kappa}(j))t = \theta t + \lambda(\theta)j + \sum_{\ell=1}^{r} b_\ell \frac{y^{\ell+1}}{j^\ell} + o(1), \quad j \to \infty. \tag{5.16}$$

This follows by another application of Taylor's expansion

$$\begin{aligned}
&\lambda(\theta - \widetilde{\kappa}(j))j + (\theta - \widetilde{\kappa}(j))t \\
&= \left(\lambda(\theta) - \lambda'(\theta)\widetilde{\kappa}(j) + \sum_{\ell=1}^{r} \frac{a_\ell}{\ell+1} (\widetilde{\kappa}(j))^{\ell+1} + O\left((\widetilde{\kappa}(j))^{r+2}\right)\right) j + \theta t - \widetilde{\kappa}(j)t \\
&= \theta t + \lambda(\theta)j - y\widetilde{\kappa}(j) + j \sum_{\ell=1}^{r} \frac{a_\ell}{\ell+1} (\widetilde{\kappa}(j))^{\ell+1} + O\left(\frac{y^{r+2}}{j^{r+1}}\right) \\
&\overset{(5.15)}{=} \theta t + \lambda(\theta)j + \sum_{\ell=1}^{r} b_\ell \frac{y^{\ell+1}}{j^\ell} + o(1).
\end{aligned}$$

Here, $O(y^{(r+2)/(r+1)}) = o(1)$ is secured by the assumption $y(j) = o(j^{(r+1)/(r+2)})$. This finishes the proof of both (5.16) and (3.6).

The proof of (3.7) goes the same path and only requires an extra term in Taylor's expansions. A counterpart of (5.15) is

$$\widetilde{\kappa}(j) = \sum_{\ell=1}^{r+1} c_\ell \Big(\frac{y}{j}\Big)^\ell + o\left(\Big(\frac{y}{j}\Big)^{r+1}\right), \quad j \to \infty.$$

With this at hand,

$$\begin{aligned}
&\lambda(\theta-\widetilde{\kappa}(j))j+(\theta-\widetilde{\kappa}(j))t\\
&\quad= \left(\lambda(\theta)-\lambda'(\theta)\widetilde{\kappa}(j)+\sum_{\ell=1}^{r+1}\frac{a_\ell}{\ell+1}(\widetilde{\kappa}(j))^{\ell+1}+O\left((\widetilde{\kappa}(j))^{r+3}\right)\right)j+\theta t-\widetilde{\kappa}(j)t\\
&\quad= \theta t+\lambda(\theta)j+\sum_{\ell=1}^{r+1}b_\ell\frac{y^{\ell+1}}{j^\ell}+o(1)\\
&\quad= \theta t+\lambda(\theta)j+\sum_{\ell=1}^{r}b_\ell\frac{y^{\ell+1}}{j^\ell}+b_{r+1}c^{r+2}+o(1),\quad j\to\infty.
\end{aligned}$$

The proof of Corollary 3.4 is complete. □

Proof of Corollary 3.5. First of all, note that the function $t(j,y)$ exists (at least for large enough j), for (3.8) implies that, for large enough $t>0$, $t/j\geq-\lambda'(\theta^*)\in(s_-,s_+)$.

Formula (2.4) implies that

$$\kappa(j)=\theta^*-\frac{\log j-y+\varepsilon(j)}{2j\theta^*\lambda''(\theta^*)}=:\theta^*-\widetilde{\kappa}(j).$$

In particular, $\lim_{j\to\infty}\kappa(j)=\theta^*$. According to part (b) of Theorem 3.1, it suffices to check that under the assumption of Corollary 3.5

$$\lim_{j\to\infty}\frac{(\hat{V}(\kappa(j)))^j\mathrm{e}^{t\kappa(j)}}{j^{1/2}}=\lim_{j\to\infty}\frac{\mathrm{e}^{j(\lambda(\kappa(j))-\kappa(j)\lambda'(\kappa(j)))}}{j^{1/2}}=\mathrm{e}^{-y/2}. \tag{5.17}$$

Using Taylor's expansion yields

$$\begin{aligned}
&\lambda(\kappa(j))-\kappa(j)\lambda'(\kappa(j))=\lambda(\theta^*-\widetilde{\kappa}(j))-(\theta^*-\widetilde{\kappa}(j))\lambda'(\theta^*-\widetilde{\kappa}(j))\\
&\quad=\lambda(\theta^*)-\lambda'(\theta^*)\widetilde{\kappa}(j)-(\theta^*-\widetilde{\kappa}(j))(\lambda'(\theta^*)-\widetilde{\kappa}(j)\lambda''(\theta^*))+O\left(\frac{(\log j)^2}{j^2}\right)\\
&\quad=\theta^*\sigma^2\widetilde{\kappa}(j)+O\left(\frac{(\log j)^2}{j^2}\right),\quad j\to\infty.
\end{aligned}$$

This entails

$$\begin{aligned}
&-2^{-1}\log j+j(\lambda(\kappa_j(t))-\kappa_j(t)\lambda'(\kappa_j(t)))\\
&\qquad\qquad=-2^{-1}\log j+\theta^*\sigma^2 j\widetilde{\kappa}(j)+O\left(\frac{(\log j)^2}{j}\right)=-y/2+o(1),\quad j\to\infty
\end{aligned}$$

and thereupon (5.17). □

6. BIBLIOGRAPHIC COMMENTS

In this brief closing section, we compare our results and methods with those existing in the literature. Observe that (3.4) formally coincides with (3.3), whenever $\kappa(j) \to \theta$ and $\lambda''(\kappa(j)) \to \sigma^2$. Therefore, we focus on the latter formula. Relation (3.3) resembles a large deviation estimate for standard random walks due to Petrov [14], see the bottom of p. 248 in [15]. However, there are two significant differences. First, since V satisfies (2.1) we had to pass to probability measures, which was achieved with the help of the standard method of exponential change of measure (formula (5.1)). Second, even after applying the exponential tilting, the resulting 'random walk' $X_{j,t}$ turned out to be non-standard, for its step distribution depends on t, see (5.3). This has caused significant technical challenges in the proofs of Propositions 5.1 and 5.2. Informally, this complexity is reflected in the fact that, in our version of Petrov's formula (p. 248 in [15]), both x and h depend on n (in Petrov's notation).

Another related large deviations result is Theorem 3.3 in [6] with relation (3.19) therein resembling our formula (3.3). However, this result is not applicable (even after an exponential change of measure) in the setting of part (a) of our Theorem 3.1 where t/j (which corresponds to m_n in the notation of [6]) diverges to infinity. Proposition 5.1 which is our key technical result can be regarded as a generalization of Lemma 3.2 in [6].

We conclude the discussion by noting Feller's paper [8] on logarithmic large deviations for rowwise sums in triangular arrays of independent random variables. In principle, the results from that paper could have been applied to derive the logarithmic asymptotics in (3.5) under weaker assumptions. Just in case, we stress that we are primarily interested in the precise rather than logarithmic asymptotics.

7. APPENDIX

The following result is known if V is a distribution function, see, for example, Chapter 9.1.2 in [4]. We have not been able to locate in the literature its version for arbitrary nondecreasing unbounded functions. Therefore, we give a proof.

Lemma 7.1. *Let $V : \mathbb{R} \to [0,\infty)$ be a right-continuous and nondecreasing function vanishing on $(-\infty,0)$. Put $x_0 := \inf\{x > 0 : V(x) > 0\}$ and $\lambda(s) = \log \int_{[0,\infty)} \mathrm{e}^{-sx} \mathrm{d}V(x)$ for $s \in \mathscr{D}$. Then*

$$s_- = \inf_{s\in\mathscr{D}}(-\lambda'(s)) = \lim_{s\to+\infty}(-\lambda'(s)) = x_0.$$

Proof. The second equality follows from the monotonicity of $-\lambda'$. For the proof of the last equality we distinguish two cases: $x_0 = 0$ and $x_0 > 0$.

CASE $x_0 = 0$. Fix $\varepsilon > 0$ and note that $\int_{[0,\varepsilon/2]} \mathrm{d}V(x) =: K(\varepsilon) > 0$. Since

$$\frac{\int_{(\varepsilon,\infty)} \mathrm{e}^{-sx}\mathrm{d}V(x)}{\int_{[0,\infty)} \mathrm{e}^{-sx}\mathrm{d}V(x)} \le \frac{\mathrm{e}^{-s\varepsilon/2}\int_{(0,\infty)} \mathrm{e}^{-sx/2}\mathrm{d}V(x)}{\int_{[0,\varepsilon/2]} \mathrm{e}^{-sx}\mathrm{d}V(x)} \le \frac{\mathrm{e}^{-s\varepsilon/2}\int_{(0,\infty)} \mathrm{e}^{-sx/2}\mathrm{d}V(x)}{\mathrm{e}^{-s\varepsilon/2}K(\varepsilon)} \to 0, \quad s\to\infty$$

by the monotone convergence theorem, we conclude that,

$$\int_{[0,\infty)} \mathrm{e}^{-sx}\mathrm{d}V(x) \;\sim\; \int_{[0,\varepsilon]} \mathrm{e}^{-sx}\mathrm{d}V(x), \quad s\to\infty.$$

Similarly, one can check that

$$\int_{[0,\infty)} x\mathrm{e}^{-sx}\mathrm{d}V(x) \;\sim\; \int_{[0,\varepsilon]} x\mathrm{e}^{-sx}\mathrm{d}V(x), \quad s\to\infty$$

for every fixed $\varepsilon>0$. Thus,

$$\lim_{s\to+\infty}(-\lambda'(s)) = \lim_{s\to+\infty}\frac{\int_{[0,\infty)} x\mathrm{e}^{-sx}\mathrm{d}V(x)}{\int_{[0,\infty)}\mathrm{e}^{-sx}\mathrm{d}V(x)} = \lim_{s\to+\infty}\frac{\int_{[0,\varepsilon]} x\mathrm{e}^{-sx}\mathrm{d}V(x)}{\int_{[0,\varepsilon]}\mathrm{e}^{-sx}\mathrm{d}V(x)} \le \varepsilon.$$

Since $\varepsilon>0$ is arbitrary we conclude that $\lim_{s\to+\infty}(-\lambda'(s))=0=x_0$.

CASE $x_0>0$. This situation can be reduced to the previous one by passing to the function $V_1(x):=V(x+x_0)$. Indeed,

$$\lambda(s)=\log\int_{[0,\infty)}\mathrm{e}^{-sx}\mathrm{d}V(x)=\log\int_{[x_0,\infty)}\mathrm{e}^{-sx}\mathrm{d}V(x)=\log\int_{[0,\infty)}\mathrm{e}^{-s(x+x_0)}\mathrm{d}V_1(x)=:-sx_0+\lambda_1(s)$$

and thereupon

$$\inf_{s\in\mathscr{D}}(-\lambda'(s))=x_0+\inf_{s\in\mathscr{D}}(-\lambda_1'(s))=x_0$$

according to the already considered case $x_0=0$. □

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