A law of the iterated logarithm for random multiplicative functions on integers with a large prime factor

Dariusz Buraczewski* and Alexander Iksanov†

Abstract

Let f be a Rademacher random multiplicative function supported on the squarefree integers. We investigate the almost sure fluctuations of the restricted sum $M^{>}(x) = \sum_{n\le x;\ P(n)>x^{1/2}} f(n)$, where $P(n)$ is the largest prime factor of n. With $Q(x)$ being the conditional variance of $M^{>}(x)$ given the values of f at primes not exceeding $x^{1/2}$, we prove that, almost surely, the upper and lower limits of $M^{>}(x)(2Q(x)\log\log Q(x))^{-1/2}$ as $x\to\infty$ are 1 and -1, respectively. Furthermore, it is shown that the almost sure cluster set of the normalized sum is $[-1,1]$.

Key words: conditional variance; law of the iterated logarithm; probabilistic number theory; Rademacher random multiplicative function; random Euler product.

2020 Mathematics Subject Classification: Primary: 60F15; Secondary: 11K65, 11N37, 60G50, 60G42.

1 Introduction and main results

Let $\mathcal{P}$ be the set of primes. On a probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $(f(p))_{p\in\mathcal{P}}$ be independent random variables with $\mathbb{P}\{f(p)=\pm1\}=1/2$. A positive integer is *squarefree* if no square of a prime divides it. Define $f(1)=1$ and, for $n\ge 2$,

$$f(n)=\begin{cases}\prod_{p|n} f(p), & n \text{ squarefree},\\ 0, & n \text{ not squarefree}.\end{cases}$$

Equivalently, $f(n)=(\mu(n))^2\prod_{p|n} f(p)$, where μ is the Möbius function. Thus, for instance, $f(6)=f(2)f(3)$, $f(12)=0$ and $f(30)=f(2)f(3)f(5)$. The function f so defined is called a *Rademacher random multiplicative function*. Put

$$M(x)=\sum_{1\le n\le x} f(n),\quad x\in\mathbb{R}$$

with the usual convention that the sum over the empty set is equal to 0. Plainly, $M(x)=0$ for $x<1$ and $M(x)=M(\lfloor x\rfloor)$ whenever $x\ge 1$.

Let $P(n)$ be the largest prime factor of an integer $n\ge 2$, and put $P(1)=1$. For $x\ge 0$, define

$$M^{>}(x):=\sum_{n\le x;\ P(n)>x^{1/2}} f(n)\quad\text{and}\quad Q(x):=\sum_{x^{1/2}<p\le x}(M(x/p))^2.$$

*Mathematical Institute, University of Wroclaw, Poland; e-mail address: dariusz.buraczewski@math.uni.wroc.pl

†Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Ukraine; e-mail address: iksan@univ.kiev.ua

All sums with the index p are over primes. The sum $M^{>}$ is called a restricted sum. The restriction retains a positive proportion of the integers up to x, while permitting a particularly useful separation of the prime signs.

1.1 A short survey of relevant works

Wintner introduced this model in [28] as a probabilistic counterpart of the Möbius function. He proved that, for each $\varepsilon > 0$, $M(x) = O(x^{1/2+\varepsilon})$ almost surely (a.s.), whereas $M(x) = O(x^{1/2-\varepsilon})$ fails a.s. The former limit relation is a probabilistic counterpart of the fact that the Riemann hypothesis is equivalent to $\sum_{n\le x} \mu(n) = O(x^{1/2+\varepsilon})$ for each $\varepsilon > 0$. These results determine the power of x, but leave open the size of fluctuations on the scale $x^{1/2}$. The problem of finding a counterpart of the law of the iterated logarithm (LIL) for $M(x)$, properly normalized, goes back to Wintner.

Halász in [11] strengthened the earlier upper bounds by proving that, for some $C > 0$,

$$|M(x)| = O\big(x^{1/2} \exp\{C(\log\log x \log\log\log x)^{1/2}\}\big), \quad x \to \infty \quad \text{a.s.}$$

Significantly later, Basquin in [2] and, independently, Lau, Tenenbaum and Wu in [18] obtained, for each $\varepsilon > 0$,

$$|M(x)| = O\big(x^{1/2}(\log\log x)^{2+\varepsilon}\big) \quad \text{a.s.} \tag{1.1}$$

Caich in [5] reduced the exponent $2+\varepsilon$ to $3/4+\varepsilon$. More recently, Durkan and Pearce-Crump [7] proved that, for each $\varepsilon > 0$,

$$|M(x)| = O\big(x^{1/2}(\log\log x)^{1/4+\varepsilon}\big), \quad x \to \infty \quad \text{a.s.}$$

Verreault [27] further refined the upper bound to $O\big(x^{1/2}(\log\log x)^{1/4}(\log\log\log x)^{1+\varepsilon}\big)$, for each $\varepsilon > 0$.

In the opposite direction, Harper in [16] proved that, for any fixed deterministic function V with $\lim_{x\to\infty} V(x) = \infty$,

$$|M(x)| \ge \frac{x^{1/2}(\log\log x)^{1/4}}{V(x)} \quad \text{a.s.}$$

for arbitrarily large x. Together, these results determine the optimal exponent $1/4$ of $\log\log x$, but do not identify an exact deterministic LIL normalization, nor a limit constant. Taking $\varepsilon < 1/4$ in the upper bound of Durkan and Pearce-Crump gives $\lim_{n\to\infty}(n\log\log n)^{-1/2}M(n) = 0$ a.s., showing that the classical LIL normalization is too large for this model. A more extensive account of these developments is given in the accompanying paper [4] (in preparation).

Limit theorems for the restricted sums have also received considerable attention. Mastrostefano proved in Theorem 1.1 of [20] that, for each $\varepsilon > 0$,

$$|M^{>}(x)| = O\big(x^{1/2}(\log\log x)^{1/4+\varepsilon}\big), \quad x \to \infty \quad \text{a.s.}$$

His result is also valid for Steinhaus random multiplicative functions (completely multiplicative functions whose prime values are independent and uniform on the unit circle). The proof combines conditioning on the small primes with estimates for the resulting random variance. Large prime subsums also enter Harper's lower-bound argument, see Section 3.1 and Proposition 2 in [16], where conditional covariance estimates allow one to find sufficiently many weakly correlated sums. Thus, this restriction isolates a part of the multiplicative sum which already exhibits the fluctuations relevant to the full problem.

For Steinhaus random multiplicative functions, Hardy in [14] proved that the restricted sum, multiplied by $x^{-1/2}(\log\log x)^{1/4}$, converges in distribution to a complex Gaussian scale mixture. In Theorem 2 of the same paper, Hardy also obtained an approximation of the

conditional variance by an Euler product integral for both the Steinhaus and Rademacher models. The corresponding distributional limit for the full Steinhaus sum was subsequently proved by Gorodetsky and Wong in [9]. These results describe convergence in distribution as $x \to \infty$. Our purpose is to determine the a.s. fluctuations of the Rademacher restricted sum as x varies.

1.2 Main results

Denote by $\mathcal{F}_y = \sigma(f(p) : p \le y)$ the σ-algebra generated by the collection $(f(p))_{p\le y}$. We write $\mathbb{E}_y$ and $\mathbb{P}_y$ for conditional expectation and conditional probability given $\mathcal{F}_y$, respectively. If $n \le x$ has a prime factor $p > x^{1/2}$, then $n = pm$ with $m \le x/p < x^{1/2} < p$. In particular, p is unique and $p \nmid m$. Consequently $f(pm) = f(p)f(m)$ and

$$M^{>}(x) = \sum_{x^{1/2}<p\le x} f(p)M(x/p). \tag{1.2}$$

Conditionally on $\mathcal{F}_{x^{1/2}}$, the coefficients $M(x/p)$ in this representation are fixed and the signs $(f(p))_{p>x^{1/2}}$ remain independent. Hence

$$\mathbb{E}_{x^{1/2}}[M^{>}(x)] = 0 \quad \text{and} \quad \mathbb{E}_{x^{1/2}}\big[(M^{>}(x))^2\big] = Q(x) \quad \text{a.s.}$$

The restriction $p > x^{1/2}$ ensures that the coefficients $M(x/p)$ depend only on the signs $f(q)$ with $q \le x^{1/2}$. Thus, conditionally on $\mathcal{F}_{x^{1/2}}$, the representation (1.2) is a sum of independent Rademacher variables with fixed coefficients and variance $Q(x)$. This gives exponential tail estimates at each fixed x. To prove an LIL, we must also control the sum uniformly in x and show that large excursions occur infinitely often. For the lower bound of lim sup, we compare sums at many different points. These sums share prime signs, and conditional covariance estimates allow us to control their dependence.

The random variance retains the contribution of the small-prime signs to the size of the sum. This is consistent with the role of random variances in the distributional results just discussed and with Harper's low-moment estimates [15], which show that the second moment alone does not describe the typical size of a multiplicative sum. In the present article we use $Q(x)$ itself in the normalization and obtain the classical signed LIL constants.

Here is our main result.

Theorem 1.1. *We have*

$$\limsup_{x\to\infty} \frac{M^{>}(x)}{(2Q(x)\log\log Q(x))^{1/2}} = 1 \quad \textit{almost surely,}$$

with the corresponding lower limit being equal to -1.

The normalization is well defined for all sufficiently large x. Lemma 2.3 entails $\log\log Q(x) \sim \log\log x$ as $x \to \infty$. Thus, $\log\log Q(x)$ in Theorem 1.1 can be replaced by $\log\log x$.

For a random function A, denote by $\mathrm{Clust}_{x\to\infty}A(x)$ its cluster set as $x \to \infty$, understood realization-wise. Write $\mathcal{C}$ for the cluster set of the normalized function in Theorem 1.1. The theorem gives $\sup\mathcal{C} = 1$ and $\inf\mathcal{C} = -1$ a.s. Our next result identifies all its points.

Corollary 1.2. *We have*

$$\mathrm{Clust}_{x\to\infty} \frac{M^{>}(x)}{(2Q(x)\log\log Q(x))^{1/2}} = [-1, 1] \quad \textit{almost surely.} \tag{1.3}$$

The random normalization in Theorem 1.1 reflects the small-prime environment. For each fixed x, $Q(x)$ is the predictable quadratic variation of the martingale obtained by revealing the large-prime signs in (1.2). Normalizing by this variance recovers the constants 1 and -1 of the classical LIL. This is consistent with the role of self-normalization in martingale limit theory, see [6, 12]. For the Steinhaus model, deterministically normalized restricted and full sums converge in distribution to Gaussian scale mixtures [9, 14]. Theorem 1.1 provides an almost sure analogue of this principle for the restricted Rademacher sum, with the random variance entering the normalization and the limit constants remaining deterministic.

Theorem 1.1 can also be viewed as an arithmetic counterpart of Stout's LIL for martingales normalized by their conditional variances [25] and of self-normalized LILs, see, for instance, [6]. Although $(M^{>}(x))_{x\geq 1}$ is not a martingale with respect to the filtration $(\mathcal{F}_x)_{x\geq 1}$, a martingale structure emerges when x is fixed. Indeed, conditionally on $\mathcal{F}_{x^{1/2}}$, the coefficients $M(x/p)$ in (1.2) are fixed, and the partial sums obtained by successively revealing the signs $f(p)$ for $x^{1/2} < p \leq x$, in increasing order of p, form a martingale with terminal value $M^{>}(x)$ and terminal predictable quadratic variation $Q(x)$. This structure depends on x, since both the coefficients and the range of summation vary with x, and summands may enter or leave the sum, see (3.14). Thus, the classical martingale LIL does not directly yield the desired a.s. asymptotics as $x \to \infty$. To pass from this fixed-x structure to an LIL, we need the interpolation argument of Section 3 for the upper bound and the analysis of conditional covariances via random Euler products in Sections 6 and 7 for the lower bound.

In the accompanying paper [4] we prove the counterpart of Theorem 1.1 for the full sum $M(x)$. The normalization there uses its full predictable quadratic variation

$$\sum_{p\leq x}\Big(\sum_{\substack{m\leq x/p\\ P(m)<p}} f(m)\Big)^2$$

with respect to the filtration obtained by revealing the prime signs. We also show that this full quadratic variation has no deterministic asymptotic equivalent, even in probability along a deterministic subsequence. Furthermore, we prove in [4] that the a.s. upper limit of

$$\frac{|M(n)|}{n^{1/2}(\log\log n)^{1/4}}$$

cannot be a finite deterministic constant. The restricted sum theorem isolates the conditional independence and covariance analysis needed for this approach. The full sum lower bound requires an additional analysis of the contributions from smaller prime factors, since its quadratic variation is larger than $Q(x)$. We expect that the main ingredients of our proof, namely, the uniform interpolation estimate of Proposition 2.6, which relies on fourth-moment bounds for the entering and leaving summands, and the selection of weakly correlated sums by means of high moments of integrals of random Euler products, can be adapted to other random sums possessing a similar conditionally independent structure, for instance, to restricted sums of Steinhaus random multiplicative functions.

The remainder of the paper is organized as follows. Section 2 is concerned with the upper bound for lim sup in Theorem 1.1. The proof rests heavily upon the uniform interpolation estimate stated in Proposition 2.6 and proved in Section 3. Its proof uses several auxiliary results, most notably the fourth-moment bound of Lemma 3.5, obtained using hypercontractivity [3, 22]. The lower bound for lim sup in Theorem 1.1 is obtained in Section 4, and Corollary 1.2 is proved in Section 5. Finally, the conditional covariance estimate (4.3), which is the core of the argument for the lower bound, is proved in Section 7. The auxiliary results needed for its proof, including numerous number-theoretic estimates, are established in Section 6.

2 Proof of Theorem 1.1 – the upper bound

The first part of the paper is concerned with the upper bound for lim sup.

Proposition 2.1. *We have*

$$\limsup_{x\to\infty} \frac{|M^{>}(x)|}{(2Q(x)\log\log Q(x))^{1/2}} \le 1 \quad \text{a.s.} \tag{2.1}$$

We begin with the main idea of the proof of Proposition 2.1. We work on logarithmic blocks $[\mathrm{e}^T, \mathrm{e}^{2T}]$, where $T = 2^r$ for $r \in \mathbb{N} := \{1, 2, \ldots\}$. We also put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In what follows, we refer to such T as dyadic T. On each block we choose a geometric grid with logarithmic spacing $T^{-\alpha}$. At a fixed grid point z, conditioning on $\mathcal{F}_{\sqrt{z}}$ makes the coefficients $M(z/p)$ in (1.2) fixed, while the signs $f(p)$ for $p > z^{1/2}$ remain independent. The conditional exponential bound of Lemma 2.4, followed by a union bound over the grid, gives the desired upper estimate there, see Proposition 2.2.

The main difficulty is extending this estimate to each real x in the block. We choose a nearby grid point $z \le x$ and control both $M^{>}(x) - M^{>}(z)$ and the distance between their coefficient vectors. For $p \in \mathcal{P}$ and $x \ge 2$, put $a_p(x) := M(x/p)\mathbb{1}_{\{p>x^{1/2}\}}$. Then

$$M^{>}(x) = \sum_p f(p)a_p(x) \quad \text{and} \quad Q(x) = \sum_p (a_p(x))^2.$$

Further, we put

$$\mathbf{a}(x) := (a_p(x))_{p\in\mathcal{P}} \quad \text{and} \quad d(x,z) := \|\mathbf{a}(x) - \mathbf{a}(z)\|_2 = \Big(\sum_p (a_p(x) - a_p(z))^2\Big)^{1/2} \tag{2.2}$$

for $x, z \ge 2$. For fixed x, the vector $\mathbf{a}(x)$ has a finite support, since $a_p(x) = 0$ when $p > x$. The large prime decomposition also gives

$$\mathbf{a}(x) = \sum_{\substack{2\le n\le x\\ n \text{ squarefree}\\ P(n)>x^{1/2}}} f\Big(\frac{n}{P(n)}\Big)\, e_{P(n)}, \tag{2.3}$$

where e_p is the standard unit vector with 1 in the coordinate indexed by the prime p and 0 in all other coordinates. In particular, $Q(x) = \|\mathbf{a}(x)\|_2^2$, and an application of the triangle inequality yields $|(Q(x))^{1/2} - (Q(z))^{1/2}| \le d(x,z)$. A finer auxiliary grid then makes the control uniform in x, see Proposition 2.6. Finally, the lower bound for $Q(x)$ in Lemma 2.5 shows that the interpolation errors are negligible on the LIL scale.

Fix $0 < \beta < 1/2$ and put $\alpha = \beta/2$. For each dyadic T, define

$$C_T := \{\mathrm{e}^{T+kT^{-\alpha}} :\ 0 \le k \le \lfloor T^{1+\alpha}\rfloor\} \cup \{\mathrm{e}^{2T}\}. \tag{2.4}$$

Thus $|C_T| = O(T^{1+\alpha})$, and each $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$ has a grid point below it at logarithmic distance at most $T^{-\alpha}$. Here and hereafter, for a finite discrete set A, we write interchangeably $|A|$ or $\#A$ for the number of its elements.

Proposition 2.2. *For each fixed $0 < \beta < 1/2$, almost surely, for all sufficiently large dyadic T and each $z \in C_T$,*

$$|M^{>}(z)| \le (1+\beta)(2Q(z)\log\log Q(z))^{1/2}.$$

For a proof of Proposition 2.2, we need an observation regarding orthogonality and two auxiliary results. The sequence $(f(n))_{n\geq 1}$ satisfies

$$\mathbb{E}[f(n)f(m)] = (\mu(n))^2 \mathbb{1}_{\{n=m\}}. \tag{2.5}$$

This shows that the variables $f(1)$, $f(2)$, $f(3), \ldots$ are orthogonal. Hence

$$\mathbb{E}[(M(x))^2] = \sum_{1\leq n\leq x} \mathbb{E}[(f(n))^2] = \sum_{1\leq n\leq x} (\mu(n))^2 \leq x, \quad x \geq 0 \tag{2.6}$$

and thereupon

$$\mathbb{E}[Q(x)] = \sum_{x^{1/2}<p\leq x} \sum_{m\leq x/p} (\mu(m))^2 = \#\{n \leq x : \ n \text{ squarefree}, \ P(n) > x^{1/2}\} \leq x, \quad x \geq 0.$$

Here, the second equality follows from the unique representation $n = P(n)m$. Indeed, if $x^{1/2} < p \leq x$ and $m \leq x/p$, then $m < x^{1/2} < p$. Thus p does not divide m, and pm is squarefree if, and only if, m is squarefree. Moreover, $P(pm) = p$.

For later use, we note that, by the fundamental maximal inequality for partial sums of orthogonal random variables (see, for instance, Theorem 2.3.1 on p. 18 in [26]),

$$\mathbb{E}\big[\max_{u\leq v}(M(u))^2\big] = \mathbb{E}\big[\max_{1\leq u\leq \lfloor v\rfloor}(M(u))^2\big] \leq v\Big(\frac{\log(4v)}{\log 2}\Big)^2.$$

Furthermore, by Markov's inequality,

$$\mathbb{P}\{\max_{u\leq b_r} |M(u)| > b_r^{2/3}\} = O(b_r^{-1/3}(\log b_r)^2) = o(1), \quad r \to \infty \tag{2.7}$$

for any sequence $(b_r)_{r\geq 1}$ diverging to $+\infty$.

Recall that, with $y(s) > 0$ for all sufficiently large s, $x(s) = \Theta(y(s))$ as $s \to \infty$ means that $|x(s)|/y(s)$ is bounded and bounded away from zero for all sufficiently large s.

Lemma 2.3. *For all sufficiently large x,*

$$\frac{x}{3\log x} \leq Q(x) \leq 2x^{3/2}. \tag{2.8}$$

Consequently, uniformly in $\mathrm{e}^T \leq x \leq \mathrm{e}^{2T}$,

$$\log Q(x) = \Theta(T) \quad \textit{and} \quad \log\log Q(x) = \log T + O(1) = \log\log x + O(1). \tag{2.9}$$

Proof. For $x > 4$, we have $(x/2, x] \subset (x^{1/2}, x]$, whence $Q(x) \geq \sum_{x/2<p\leq x}(M(\lfloor x/p\rfloor))^2$. Under the last sum, $\lfloor x/p\rfloor = 1$ and consequently $(M(\lfloor x/p\rfloor))^2 = (f(1))^2 = 1$. With $\pi(x)$ denoting the number of prime numbers that do not exceed x, this proves that

$$Q(x) \geq \pi(x) - \pi(x/2) \geq x/(3\log x)$$

for all sufficiently large x. Here, the last inequality is secured by the prime number theorem. Since $|M(u)| \leq u$ for $u \geq 1$,

$$Q(x) \leq x^2 \sum_{p>x^{1/2}} p^{-2} \leq x^2 \sum_{n>x^{1/2}} n^{-2} \leq 2x^{3/2}$$

for all sufficiently large x. The last part of the lemma follows from (2.8). □

For each prime p, put $\mathcal{F}_{p^-} := \sigma(f(q) : \ q \in \mathcal{P}, \ q < p)$.

Lemma 2.4. *For each fixed sufficiently large x and any positive $\mathcal{F}_{x^{1/2}}$-measurable random variable v,*

$$\mathbb{P}_{x^{1/2}}\{|M^{>}(x)| \geq v\} \leq 2\exp\big(-v^2/(2Q(x))\big) \quad \textit{almost surely.} \tag{2.10}$$

Also, if $(b_p)_p$ is a family of real random variables such that $b_p = 0$ for all primes p outside a deterministic finite set $\mathcal{P}_1$, with b_p measurable with respect to $\mathcal{F}_{p^-}$ for $p \in \mathcal{P}_1$, then, for deterministic $t, v > 0$,

$$\mathbb{P}\Big\{\Big|\sum_p f(p)b_p\Big| \geq t,\ \sum_p b_p^2 \leq v\Big\} \leq 2\exp\big(-t^2/(2v)\big). \tag{2.11}$$

Proof. Although (2.10) follows from Hoeffding's inequality, see, for instance, Lemma 2.3 in [1], and (2.11) can be obtained by the standard supermartingale argument underlying Hoeffding's inequality, we provide details.

Noting that, conditionally on $\mathcal{F}_{x^{1/2}}$, the variables $f(p)$ for $p > x^{1/2}$ are independent and that v and the coefficients $M(x/p)$ are $\mathcal{F}_{x^{1/2}}$-measurable, hence fixed upon conditioning, an application of the Hoeffding inequality yields

$$\mathbb{P}_{x^{1/2}}\{M^{>}(x) \geq v\} \leq \exp\big(-v^2/(2Q(x))\big) \quad \text{a.s.}$$

Applying the same argument to $-M^{>}(x)$ and then adding the two conditional bounds we arrive at (2.10).

To prove (2.11), fix $\theta > 0$. Since b_p is $\mathcal{F}_{p^-}$-measurable and $f(p)$ is independent of $\mathcal{F}_{p^-}$, we have

$$\mathbb{E}_{p^-}\big[\exp\left(\theta f(p)b_p - \theta^2 b_p^2/2\right)\big] = \exp(-\theta^2 b_p^2/2)\cosh(\theta b_p) \leq 1, \quad p \in \mathcal{P}_1.$$

Here, $\mathbb{E}_{p^-}$ denotes the conditional expectation given $\mathcal{F}_{p^-}$. With this at hand, successive conditioning gives

$$\mathbb{E}\Big[\exp\Big(\theta\sum_p f(p)b_p - \frac{\theta^2}{2}\sum_p b_p^2\Big)\Big] \leq 1.$$

Consequently, by Markov's inequality,

$$\begin{aligned}\mathbb{P}\Big\{\sum_p f(p)b_p \geq t,\ \sum_p b_p^2 \leq v\Big\} &\leq \mathbb{P}\Big\{\exp\Big(\theta\sum_p f(p)b_p - (\theta^2/2)\sum_p b_p^2\Big) \geq \exp(\theta t - \theta^2 v/2)\Big\}\\ &\leq \exp(-\theta t + \theta^2 v/2)\mathbb{E}\Big[\exp\Big(\theta\sum_p f(p)b_p - (\theta^2/2)\sum_p b_p^2\Big)\Big] \leq \exp(-\theta t + \theta^2 v/2).\end{aligned}$$

Choosing $\theta = t/v$ we obtain the upper bound $\exp(-t^2/(2v))$. Applying the same argument to $(-b_p)_p$ and adding the resulting bounds proves (2.11). □

Proof of Proposition 2.2. For all sufficiently large T and each $z \in C_T$, the variable $(1+\beta)(2Q(z)\log\log Q(z))^{1/2}$ is positive and $\mathcal{F}_{z^{1/2}}$-measurable. Using (2.10) for the first inequality and (2.9) for the second, we infer

$$\mathbb{P}_{z^{1/2}}\{|M^{>}(z)| > (1+\beta)(2Q(z)\log\log Q(z))^{1/2}\} \leq 2(\log Q(z))^{-(1+\beta)^2} \leq C_\beta T^{-(1+\beta)^2}$$

uniformly for $z \in C_T$, for a deterministic constant $C_\beta > 0$. Taking expectations and then a union bound over the deterministic grid C_T yields

$$\mathbb{P}\{\exists z \in C_T:\ |M^{>}(z)| > (1+\beta)(2Q(z)\log\log Q(z))^{1/2}\} \leq C'_\beta T^{1+\alpha-(1+\beta)^2}.$$

The exponent is negative since $\alpha = \beta/2 < (1+\beta)^2 - 1$. For $T = 2^r$, these probabilities are summable in r. Invoking the direct part of the Borel-Cantelli lemma completes the proof. □

Lemma 2.5 bounds Q uniformly from below. Its proof is given, together with that of formula (4.2), in Section 4.2.

Lemma 2.5. *For each fixed $b > 0$, almost surely, for all sufficiently large dyadic T, simultaneously for all $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$,*

$$Q(x) \geq xT^{-b}.$$

The following proposition provides the uniform interpolation estimate. We postpone its proof to Section 3.

Proposition 2.6. *Fix $\alpha \in (0, 1/4)$. Almost surely, for all sufficiently large dyadic T and each $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$, there is a $z = z_T(x) \in C_T$ such that $z \leq x$, $\log(x/z) \leq 2T^{-\alpha}$, and*

$$|M^>(x) - M^>(z)| + d(x, z) \leq 2x^{1/2}\, T^{-\alpha/12}.$$

We are ready to prove the main result of the first part of the paper.

Proof of Proposition 2.1. Fix $\alpha \in (0, 1/4)$ and put $\beta = 2\alpha$. Given $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$, let $z = z_T(x)$ be as in Proposition 2.6. Throughout the remainder of the proof, T ranges over dyadic values. All inequalities hold a.s. for all sufficiently large T, and all asymptotic relations hold a.s. as $T \to \infty$, uniformly in $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$.

Using the inequality which has already appeared in the paragraph following (2.3) we obtain

$$\bigl|(Q(x))^{1/2} - (Q(z))^{1/2}\bigr| \leq d(x, z) \leq 2x^{1/2}T^{-\alpha/12}.$$

Applying Lemma 2.5 with $b = \alpha/12$ and Proposition 2.6, we infer

$$\Bigl|\Bigl(\frac{Q(z)}{Q(x)}\Bigr)^{1/2} - 1\Bigr| = \frac{\bigl|(Q(z))^{1/2} - (Q(x))^{1/2}\bigr|}{(Q(x))^{1/2}} \leq \frac{d(x, z)}{(Q(x))^{1/2}} \leq \frac{2x^{1/2}T^{-\alpha/12}}{x^{1/2}T^{-\alpha/24}} = 2T^{-\alpha/24} \;\to\; 0.$$

By (2.9), $\log\log Q(z)/\log\log Q(x) = 1 + o(1)$. By Lemma 2.5, with $b = \alpha/12$, we have $Q(x) \geq xT^{-\alpha/12}$. In particular, $\log Q(x) \geq \log x - (\alpha/12)\log T \geq T/2$, whence $\log\log Q(x) \geq \log(T/2) \geq (\log T)/2$ for all sufficiently large T. Using these inequalities to bound the denominator and Proposition 2.6 to bound the numerator, we obtain

$$\frac{|M^>(x) - M^>(z)|}{(2Q(x)\log\log Q(x))^{1/2}} \leq \frac{2x^{1/2}T^{-\alpha/12}}{x^{1/2}T^{-\alpha/24}(\log T)^{1/2}} = 2\frac{T^{-\alpha/24}}{(\log T)^{1/2}} \;\to\; 0.$$

Finally, writing, for $x \in [\mathrm{e}^T, \mathrm{e}^{2T}]$,

$$\frac{|M^>(x)|}{(2Q(x)\log\log Q(x))^{1/2}} \leq \frac{|M^>(z)|}{(2Q(z)\log\log Q(z))^{1/2}}\Bigl(\frac{Q(z)\log\log Q(z)}{Q(x)\log\log Q(x)}\Bigr)^{1/2} + \frac{|M^>(x) - M^>(z)|}{(2Q(x)\log\log Q(x))^{1/2}}$$

and applying the preceding asymptotic relations and Proposition 2.2 yields

$$\limsup_{r\to\infty} \sup_{x\in[\mathrm{e}^{2^r}, \mathrm{e}^{2^{r+1}}]} \frac{|M^>(x)|}{(2Q(x)\log\log Q(x))^{1/2}} \leq 1 + \beta \quad \text{a.s.}$$

Since the intervals $\bigl[\mathrm{e}^{2^r}, \mathrm{e}^{2^{r+1}}\bigr]$ cover all sufficiently large x, we obtain

$$\limsup_{x\to\infty} \frac{|M^>(x)|}{(2Q(x)\log\log Q(x))^{1/2}} \leq 1 + \beta \quad \text{a.s.}$$

Taking a countable sequence $\alpha \to 0+$, with $\beta = 2\alpha$, completes the proof. □

3 Proof of Proposition 2.6

We start with a number of auxiliary results needed for the proof of Proposition 2.6.

Throughout this section $T = 2^r$ tends to ∞ through dyadic values and $0 < \alpha < 1/4$ is fixed. Put $N = \lfloor \mathrm{e}^{2T} \rfloor$ and $\delta = T^{-250}$.

For $u \geq 0$ and $s \geq 0$, put

$$M_N(u) = \sum_{n \leq \min(u,N)} f(n) \quad \text{and} \quad A_N(s) = \int_0^\infty \frac{(M_N(\mathrm{e}^s u) - M_N(u))^2}{u^2}\,\mathrm{d}u.$$

Also, for $z \in [\mathrm{e}^T, \mathrm{e}^{2T}]$, put

$$R_T(z) = \sum_{p > z^{1/2}} \frac{1}{\delta p} \int_p^{(1+\delta)p} (M(z/p) - M(z/t))^2\,\mathrm{d}t.$$

This sum is finite because $M(z/p) = M(z/t) = 0$ for $t > p > z$.

Put $\pi(x) := \#\{p :\ p \leq x\}$ for $x \geq 0$. As a preparation for the proof of Lemma 3.1 given below, we note that, by formula (6.14) on p. 179 in [21],

$$\pi(x) = \int_2^x \frac{\mathrm{d}y}{\log y} + R(x), \quad x \geq 2, \tag{3.1}$$

where, for some positive constants c and c_1,

$$|R(x)| \leq c_1 x \mathrm{e}^{-c(\log x)^{1/2}}, \quad x \geq 2. \tag{3.2}$$

Lemma 3.1. *Almost surely, for all sufficiently large dyadic T, and each $\mathrm{e}^T \leq x \leq y \leq \mathrm{e}^{2T}$ with $\theta = \log(y/x) \leq T^{-\alpha}$,*

$$\frac{(d(x,y))^2}{x} \leq 10\frac{A_N(\theta)}{T} + 4\frac{R_T(x)}{x} + 8\frac{R_T(y)}{y} + C(\omega)\frac{(\log T)^6}{T^{1+\alpha}}. \tag{3.3}$$

Here, $C(\omega)$ is an almost surely finite random constant independent of x, y and T.

Proof. Fix $\lambda > 0$. We claim that, for all sufficiently large T, all $v \in [\mathrm{e}^{T/2}/2, 2\mathrm{e}^{2T}]$ and all $\ell \in [0, vT^{-\lambda}]$,

$$\#\{p :\ v \leq p \leq v + \ell\} \leq 5vT^{-\lambda-1}. \tag{3.4}$$

Indeed, for all sufficiently large T, we have $\log v \geq T/3$ and $v + \ell \leq 2v$. Hence, by (3.2),

$$\begin{aligned}\#\{p :\ v < p \leq v + \ell\} = \pi(v+\ell) - \pi(v) &= \int_v^{v+\ell} \frac{\mathrm{d}y}{\log y} + R(v+\ell) - R(v) \\ &\leq \frac{\ell}{\log v} + c_1(v+\ell)\mathrm{e}^{-c(\log(v+\ell))^{1/2}} + c_1 v \mathrm{e}^{-c(\log v)^{1/2}} \\ &\leq \frac{3\ell}{T} + 3c_1 v \mathrm{e}^{-c'T^{1/2}} \leq 3vT^{-\lambda-1} + 3c_1 v \mathrm{e}^{-c'T^{1/2}},\end{aligned}$$

where $c' := 3^{-1/2}c$. Since $\lim_{T\to\infty} T^{\lambda+1}\mathrm{e}^{-c'T^{1/2}} = 0$, the last expression is at most $4vT^{-\lambda-1}$ for all sufficiently large T. Including the possible prime at the left endpoint v adds at most one. As $vT^{-\lambda-1} \geq \mathrm{e}^{T/2}T^{-\lambda-1}/2 \to \infty$ as $T \to \infty$, this proves (3.4).

Write, for $2 \leq x \leq y$,

$$(d(x,y))^2 = A(x,y) + B(x,y),$$

where

$$A(x,y)=\sum_{p>y^{1/2}}|M(y/p)-M(x/p)|^2 \quad \text{and} \quad B(x,y)=\sum_{x^{1/2}<p\le y^{1/2}}|M(x/p)|^2.$$

We first estimate $A(x,y)$. Using twice the inequality $(a+b)^2\le 2(a^2+b^2)$, $a,b\in\mathbb{R}$, we obtain, for $t\in[p,(1+\delta)p]$,

$$(M(y/p)-M(x/p))^2\le 2(M(y/t)-M(x/t))^2+4(M(y/p)-M(y/t))^2+4(M(x/p)-M(x/t))^2.$$

Now we divide the latter inequality by δp, integrate it over $t\in[p,(1+\delta)p]$ and then sum over $p>y^{1/2}$. This gives

$$\begin{aligned}A(x,y)&\le 2\sum_{p>y^{1/2}}(\delta p)^{-1}\int_p^{(1+\delta)p}(M(y/t)-M(x/t))^2\,\mathrm{d}t+4R_T(y)+4R_T(x)\\&\le 2\int_{y^{1/2}}^{y}(M(y/t)-M(x/t))^2\sum_{t/(1+\delta)\le p\le t}(\delta p)^{-1}\,\mathrm{d}t+4R_T(y)+4R_T(x)\end{aligned}$$

having utilized for the last inequality Fubini's theorem, the fact that the inequalities $p>y^{1/2}$ and $t\in[p,(1+\delta)p]$ entail $t>y^{1/2}$ and $p\in[t/(1+\delta),t]$, and also $M(y/t)-M(x/t)=0$ for $t>y$. Further, for $t\in[y^{1/2},y]$,

$$\sum_{t/(1+\delta)\le p\le t}(\delta p)^{-1}\le(1+\delta)(\delta t)^{-1}\#\{p:\ t/(1+\delta)\le p\le t\}\le 5/T.$$

The last inequality is justified by (3.4), with $\lambda=250$, $v=t/(1+\delta)$ and $\ell=\delta t/(1+\delta)=\delta v=vT^{-250}$. The formula applies because $v\in[\mathrm{e}^{T/2}/2,\mathrm{e}^{2T}]$ for all sufficiently large T. Substituting this estimate into the preceding bound for $A(x,y)$ gives

$$A(x,y)\le 10T^{-1}\int_{y^{1/2}}^{y}(M(y/t)-M(x/t))^2\,\mathrm{d}t+4R_T(x)+4R_T(y).$$

Both arguments of M in this integral are at most $y^{1/2}\le\mathrm{e}^T<N$. Hence, M may be replaced by M_N. Changing the variable $u=x/t$ and enlarging the domain of integration, we obtain

$$\begin{aligned}\int_{y^{1/2}}^{y}(M(y/t)-M(x/t))^2\,\mathrm{d}t&=x\int_{x/y}^{x/y^{1/2}}\frac{(M_N(\mathrm{e}^\theta u)-M_N(u))^2}{u^2}\,\mathrm{d}u\\&\le x\int_0^\infty\frac{(M_N(\mathrm{e}^\theta u)-M_N(u))^2}{u^2}\,\mathrm{d}u=xA_N(\theta),\end{aligned}$$

where $\theta=\log(y/x)$. Thus,

$$A(x,y)\le 10T^{-1}xA_N(\theta)+4R_T(x)+4R_T(y). \tag{3.5}$$

Now we work towards estimating $B(x,y)$. Relation (1.1), with $\varepsilon=1$, implies that, a.s., after enlarging a finite random constant if necessary, we have

$$|M(u)|\le C_1(\omega)u^{1/2}\,(1+\log\log(u+3))^3,\quad u\ge 1. \tag{3.6}$$

The inequality $x/p\le x^{1/2}\le\mathrm{e}^T$ holds whenever $x^{1/2}<p\le y^{1/2}$. Consequently, there exists an absolute constant $c_1>0$ such that, for all sufficiently large T,

$$1+\log\log(x/p+3)\le 1+\log\log(\mathrm{e}^T+3)\le c_1\log T.$$

Observe that $x/p \geq 1$ as a consequence of $p \leq y^{1/2} \leq \mathrm{e}^T \leq x$. Using now (3.6) we infer $(M(x/p))^2 \leq C_2(\omega)p^{-1}x(\log T)^6$ a.s., where $C_2(\omega) := c_1^6(C_1(\omega))^2$ and thereupon

$$B(x,y) \leq C_2(\omega)x(\log T)^6 \sum_{x^{1/2}<p\leq y^{1/2}} p^{-1}. \tag{3.7}$$

Further,

$$\sum_{x^{1/2}<p\leq y^{1/2}} p^{-1} \leq \sum_{x^{1/2}<p\leq \mathrm{e}^{T^{-\alpha}/2}x^{1/2}} p^{-1} \leq x^{-1/2}\#\{p:\ x^{1/2} < p \leq \mathrm{e}^{T^{-\alpha}/2}x^{1/2}\} \leq 5T^{-1-\alpha}.$$

Here, the first inequality holds because $y/x \leq \exp(T^{-\alpha})$ entails $y^{1/2} \leq \exp(T^{-\alpha}/2)x^{1/2}$. The last inequality is justified by (3.4), with $\lambda = \alpha$, $v = x^{1/2}$ and $\ell = x^{1/2}(\exp(T^{-\alpha}/2) - 1)$. The formula applies because $v \in [\mathrm{e}^{T/2}, \mathrm{e}^T]$ and, for all sufficiently large T, $\exp(T^{-\alpha}/2) - 1 \leq T^{-\alpha}$, whence $\ell \leq vT^{-\alpha}$. Substituting the estimate for the prime sum into (3.7), we obtain

$$B(x,y) \leq C(\omega)xT^{-1-\alpha}(\log T)^6, \tag{3.8}$$

where $C(\omega) := 5C_2(\omega)$. Finally, combining (3.5), (3.8) and

$$\frac{R_T(y)}{x} = \frac{y}{x}\frac{R_T(y)}{y} \leq \mathrm{e}^{T^{-\alpha}}\frac{R_T(y)}{y} \leq 2\frac{R_T(y)}{y}$$

for sufficiently large T, we arrive at (3.3). □

Lemma 3.2. *Almost surely, for all sufficiently large dyadic T,*

$$\sup_{0\leq\theta\leq T^{-\alpha}} \frac{A_N(\theta)}{T} \leq T^{-\alpha/2}, \tag{3.9}$$

where $\alpha \in (0, 1/4)$ is fixed.

Proof. For $u, s \geq 0$, put $D_s(u) := M_N(\mathrm{e}^s u) - M_N(u)$. If $0 \leq \theta \leq T^{-\alpha}$ and $T^{-\alpha} \leq t \leq 2T^{-\alpha}$, then $D_\theta(u) = D_t(u) - D_{t-\theta}(\mathrm{e}^\theta u)$. The inequality $(a-b)^2 \leq 2a^2 + 2b^2$, $a, b \in \mathbb{R}$, in combination with the change of variable $v = \mathrm{e}^\theta u$, gives

$$A_N(\theta) \leq 2A_N(t) + 2\mathrm{e}^\theta A_N(t-\theta).$$

Integrating this inequality over $t \in [T^{-\alpha}, 2T^{-\alpha}]$ and then multiplying both sides by T^α, we obtain, simultaneously for all $\theta \in [0, T^{-\alpha}]$,

$$A_N(\theta) \leq 2T^\alpha \int_{T^{-\alpha}}^{2T^{-\alpha}} A_N(t)\,\mathrm{d}t + 2\mathrm{e}^\theta T^\alpha \int_{T^{-\alpha}-\theta}^{2T^{-\alpha}-\theta} A_N(s)\,\mathrm{d}s \leq c_* T^\alpha \int_0^{2T^{-\alpha}} A_N(s)\,\mathrm{d}s$$

and thereupon

$$\mathbb{E}\big[\sup_{0\leq\theta\leq T^{-\alpha}} A_N(\theta)\big] \leq c_* T^\alpha \int_0^{2T^{-\alpha}} \mathbb{E}[A_N(s)]\,\mathrm{d}s \tag{3.10}$$

for all sufficiently large T, where $c_* := 2(1 + \mathrm{e}^{1/2})$. We have used that both $[T^{-\alpha}, 2T^{-\alpha}]$ and $[T^{-\alpha} - \theta, 2T^{-\alpha} - \theta]$ are contained in $[0, 2T^{-\alpha}]$, and that $\mathrm{e}^\theta \leq \mathrm{e}^{1/2}$ for all sufficiently large T.

To estimate the expectation, we start with a representation

$$D_s(u) = M_N(\mathrm{e}^s u) - M_N(u) = \sum_{n\leq N} f(n)\mathbb{1}_{\{u<n\leq \mathrm{e}^s u\}} = \sum_{n\leq N} f(n)\mathbb{1}_{\{n\mathrm{e}^{-s}\leq u<n\}}.$$

Invoking (2.5) we then obtain

$$\mathbb{E}[(D_s(u))^2] = \sum_{n,m\le N} \mathbb{E}[f(n)f(m)]\mathbb{1}_{\{ne^{-s}\le u<n\}}\mathbb{1}_{\{me^{-s}\le u<m\}} = \sum_{n\le N} (\mu(n))^2\mathbb{1}_{\{ne^{-s}\le u<n\}}.$$

In view of Fubini's theorem and

$$\sum_{n=1}^{m} n^{-1} = 1 + \sum_{n=2}^{m} n^{-1} \le 1 + \int_1^m u^{-1}\,\mathrm{d}u = 1 + \log m, \quad m \in \mathbb{N},$$

we conclude that

$$\mathbb{E}[A_N(s)] = \int_0^\infty \frac{\mathbb{E}[D_s(u)^2]}{u^2}\,\mathrm{d}u = \sum_{n\le N}(\mu(n))^2 \int_{ne^{-s}}^{n} \frac{\mathrm{d}u}{u^2} = (\mathrm{e}^s-1)\sum_{n\le N}\frac{\mu^2(n)}{n}$$
$$\le (1+\log N)(\mathrm{e}^s-1) \le 3T(\mathrm{e}^s-1),$$

where the last inequality follows from $N = \lfloor \mathrm{e}^{2T}\rfloor$ and $1+\log N \le 1+2T \le 3T = 3\cdot 2^r$ for $r\in\mathbb{N}$. Substituting this bound into (3.10) yields

$$T^{-1}\mathbb{E}\Big[\sup_{0\le\theta\le T^{-\alpha}} A_N(\theta)\Big] \le 3c_*T^\alpha \int_0^{2T^{-\alpha}} (\mathrm{e}^s-1)\,\mathrm{d}s = 3c_*T^\alpha\big(\mathrm{e}^{2T^{-\alpha}} - 1 - 2T^{-\alpha}\big) \le 9c_*T^{-\alpha}$$

for all sufficiently large T. By Markov's inequality,

$$\mathbb{P}\Big\{T^{-1}\sup_{0\le\theta\le T^{-\alpha}} A_N(\theta) > T^{-\alpha/2}\Big\} \le 9c_*T^{-\alpha/2}.$$

Since $T = 2^r$, these probabilities are summable in r. An application of the direct part of the Borel-Cantelli lemma completes the proof of (3.9). □

Lemma 3.3. *Let $G_T \subset [\mathrm{e}^T, \mathrm{e}^{2T}]$ be a deterministic set such that $|G_T| = O(T^{122})$ as $T\to\infty$. Then, almost surely, for all sufficiently large dyadic T,*

$$\max_{z\in G_T}\frac{R_T(z)}{z} \le T^{-\alpha/2}, \tag{3.11}$$

where $\alpha\in(0,1/4)$ is fixed.

Proof. Fix a deterministic $z\in[\mathrm{e}^T,\mathrm{e}^{2T}]$. For $p\le t\le(1+\delta)p$, we obtain with the help of (2.5)

$$\mathbb{E}[(M(z/p)-M(z/t))^2] = \sum_{z/t<n\le z/p}(\mu(n))^2 \le \#\{n:\ ((1+\delta)p)^{-1}z < n \le p^{-1}z\}.$$

This, in combination with Fubini's theorem, gives

$$\mathbb{E}[R_T(z)] = \sum_{p>z^{1/2}}(\delta p)^{-1}\int_p^{(1+\delta)p}\mathbb{E}\big[(M(z/p)-M(z/t))^2\big]\mathrm{d}t$$
$$\le \sum_{p>z^{1/2}} \#\{n:\ (1+\delta)^{-1}z < np \le z\} = \#\{m:\ (1+\delta)^{-1}z < m \le z,\ P(m) > z^{1/2}\}$$
$$\le \#\{m:\ (1+\delta)^{-1}z < m \le z\} \le \delta z + 1,$$

whence $z^{-1}\mathbb{E}[R_T(z)] \le \delta + \mathrm{e}^{-T}$. To explain the last equality, observe that each pair (p,n) counted in the prime sum gives an integer $m = pn$ satisfying $(1+\delta)^{-1}z < m \le z$. Moreover, $p > z^{1/2}$ and $n \le z/p < z^{1/2} < p$ ensure that p is the unique prime divisor of m exceeding $z^{1/2}$.

Thus distinct counted pairs give distinct integers m. Conversely, each integer m counted on the right-hand side of the equality gives the pair $(p,n) = (P(m), m/P(m))$ counted on the left.

Since G_T is deterministic, Markov's inequality and a union bound yield

$$\mathbb{P}\{\max_{z\in G_T}(z^{-1}R_T(z)) > T^{-\alpha/2}\} \le \sum_{z\in G_T}\mathbb{P}\{z^{-1}R_T(z) > T^{-\alpha/2}\} \le (T^{-250}+\mathrm{e}^{-T})|G_T|T^{\alpha/2}$$
$$= (T^{-250}+\mathrm{e}^{-T})O(T^{122+\alpha/2}).$$

These bounds are summable over $T = 2^r$. An application of the direct part of the Borel-Cantelli lemma completes the proof of (3.11). □

Lemma 3.4 is a special case of Lemma 2 on pp. 130–131 in [19], with $\nu = 4$ and $\gamma = 2$. Although their result is stated for real random variables, the same argument applies to random vectors, with the absolute value replaced by a norm.

Lemma 3.4. *For $m \in \mathbb{N}$, let $Y_1, \dots, Y_m$ be random scalars or random vectors and let $u_1, \dots, u_m$ be nonnegative deterministic numbers. Suppose that, for each collection of consecutive indices $I = \{a, a+1, \dots, b\} \subseteq \{1, \dots, m\}$,*

$$\mathbb{E}\Big[\Big\|\sum_{j\in I} Y_j\Big\|^4\Big] \le \Big(\sum_{j\in I} u_j\Big)^2.$$

Then

$$\mathbb{E}\Big[\max_{0\le k\le m}\Big\|\sum_{j\le k} Y_j\Big\|^4\Big] \le C\Big(\sum_{j=1}^{m} u_j\Big)^2 \tag{3.12}$$

for an absolute constant $C > 0$.

The proof of Lemma 3.5 uses hypercontractive moment bounds, as in Lemmas 2.2 and 2.3 of [18], where the full sum M is considered, together with the maximal inequality (3.12). A related argument for full weighted Steinhaus sums appears in Lemma 2 of [13]. Here, we treat $M^{>}$ and simultaneously control the coefficient distance d, taking account of both the appearance and the removal of summands.

Lemma 3.5. *There is an absolute constant $C > 0$ such that, for all sufficiently large u and $u + u^{1/2} \le v \le 3u/2$,*

$$\mathbb{E}\Big[\sup_{u\le x\le v}\big(|M^{>}(x) - M^{>}(u)| + d(u,x)\big)^4\Big] \le Cu^{2/3}(v-u)^{4/3}(\log u)^{52/3}. \tag{3.13}$$

Proof. We divide the proof into five steps.

Step 1. Recall that, for an integer $n \ge 2$, $P(n)$ denotes its largest prime factor. Define

$$\mathcal{A} := \big\{n \ge 2 :\ n \text{ is squarefree and } n < (P(n))^2\big\}.$$

The set $\mathcal{A}$ consists precisely of the squarefree integers which can occur in (that is, give a nonzero contribution to) $M^{>}(x)$ for some x. Indeed, recalling that $f(n) = 0$ unless n is squarefree, the integer n is present in $M^{>}(x)$ if, and only if, $n \le x < (P(n))^2$. Consequently,

$$M^{>}(x) = \sum_{n\in\mathcal{A}} f(n)\,\mathbb{1}_{\{n\le x<(P(n))^2\}}.$$

For $n \in \mathcal{A}$ and $u \le x$,

$$\mathbb{1}_{\{n\le x<(P(n))^2\}} - \mathbb{1}_{\{n\le u<(P(n))^2\}} = \mathbb{1}_{\{u<n\le x\}} - \mathbb{1}_{\{u<(P(n))^2\le x\}}$$

and thereupon

$$M^{>}(x) - M^{>}(u) = E(u,x) - L(u,x), \tag{3.14}$$

where

$$E(u,x) := \sum_{\substack{n\in\mathcal{A}\\ u<n\le x}} f(n) \quad \text{and} \quad L(u,x) := \sum_{\substack{n\in\mathcal{A}\\ u<(P(n))^2\le x}} f(n).$$

The coefficient vector $\mathbf{a}(x)$ admits the analogous representation, see (2.2) for the definition and (2.3) for the alternative representation:

$$\mathbf{a}(x) = \sum_{n\in\mathcal{A}} f\Big(\frac{n}{P(n)}\Big) e_{P(n)} \mathbb{1}_{\{n\le x<(P(n))^2\}}. \tag{3.15}$$

Indeed, for fixed prime p, the integers $n\in\mathcal{A}$ satisfying $P(n)=p$ have the form $n=pm$ with squarefree $m<p$. Their contribution to the coordinate indexed by p of the right-hand side of (3.15) is

$$\mathbb{1}_{\{x<p^2\}} \sum_{m\le x/p} f(m) = \mathbb{1}_{\{p>x^{1/2}\}} M(x/p) = a_p(x).$$

This proves (3.15).

Finally,

$$\mathbf{a}(x) - \mathbf{a}(u) = E_{\mathbf{a}}(u,x) - L_{\mathbf{a}}(u,x), \tag{3.16}$$

where

$$E_{\mathbf{a}}(u,x) := \sum_{\substack{n\in\mathcal{A}\\ u<n\le x}} f\Big(\frac{n}{P(n)}\Big) e_{P(n)} \quad \text{and} \quad L_{\mathbf{a}}(u,x) := \sum_{\substack{n\in\mathcal{A}\\ u<(P(n))^2\le x}} f\Big(\frac{n}{P(n)}\Big) e_{P(n)}.$$

Step 2. By formula (2.2) in [18], applied with $m=4$, for each finitely supported family of real coefficients (c_n) indexed by squarefree integers,

$$\mathbb{E}\Big[\Big|\sum_n c_n f(n)\Big|^4\Big] \le \Big(\sum_n 3^{\omega(n)} c_n^2\Big)^2, \tag{3.17}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. With some extra work, this inequality can also be derived from Theorem 3 on p. 376 in [3], applied with $p=2$, $q=4$ and $r=3^{-1/2}$, as well as from the $(2,4)$-Hypercontractivity Theorem in Section 9.2 of [22].

We shall also need a vector-valued version of (3.17). For each finitely supported family of real coefficients c_n indexed by squarefree integers $n\in\mathcal{A}$,

$$\mathbb{E}\Big[\Big\|\sum_n c_n f\Big(\frac{n}{P(n)}\Big) e_{P(n)}\Big\|_2^4\Big] \le \Big(\sum_n 3^{\omega(n)} c_n^2\Big)^2. \tag{3.18}$$

To prove this, for each prime p, let V_p denote the coordinate indexed by p of the random vector on the left-hand side of (3.18), that is, $\sum_n c_n f(n/P(n)) e_{P(n)} = \sum_p V_p e_p$. Explicitly,

$$V_p = \sum_{n:\ P(n)=p} c_n f(n/p).$$

Applying (3.17) to the coefficients c_n appearing in the latter sum gives

$$\big(\mathbb{E}\big[|V_p|^4\big]\big)^{1/2} \le \sum_{n:\ P(n)=p} 3^{\omega(n/p)} c_n^2 = \sum_{n:\ P(n)=p} 3^{\omega(n)-1} c_n^2. \tag{3.19}$$

We have used for the equality that, for a squarefree integer $n \geq 2$, $\omega(n/P(n)) = \omega(n) - 1$. Combining the Minkowski inequality with (3.19), we obtain

$$\Big(\mathbb{E}\Big\| \sum_n c_n f\Big(\frac{n}{P(n)}\Big) e_{P(n)}\Big\|_2^4\Big)^{1/2} = \Big(\mathbb{E}\Big[\Big(\sum_p |V_p|^2\Big)^2\Big]\Big)^{1/2}$$
$$\leq \sum_p \big(\mathbb{E}[|V_p|^4]\big)^{1/2} \leq \sum_p \sum_{n:\ P(n)=p} 3^{\omega(n)-1} c_n^2 = \sum_n 3^{\omega(n)-1} c_n^2.$$

Squaring both sides and using $1/3 < 1$ proves (3.18).

Step 3. Now we prove that

$$\mathbb{E}\big[\sup_{u\leq x\leq v} |E(u,x)|^4\big] + \mathbb{E}\big[\sup_{u\leq x\leq v} \|E_{\mathbf{a}}(u,x)\|_2^4\big] \leq C_1 u^{2/3}(v-u)^{4/3}(\log u)^{52/3} \tag{3.20}$$

for an absolute constant $C_1 > 0$.

Let $d_k(n)$ denote the k-fold divisor function, that is,

$$d_k(n) := \#\{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_1 \cdot \ldots \cdot a_k = n\}.$$

If n is squarefree, then $d_k(n) = k^{\omega(n)}$ because each prime factor of n can be assigned independently to one of the k factors. For each fixed integer $k \geq 2$, according to Exercise 18(c) on p. 43 in [21],

$$\sum_{n\leq z} d_k(n) \leq c_0 z(\log z)^{k-1}, \quad z \geq 2 \tag{3.21}$$

for a constant $c_0 = c_0(k) > 0$.

Define $\mathcal{E} := \{n \in \mathcal{A} :\ u < n \leq v\}$ and observe that $|\mathcal{E}| \leq v - u + 1 \leq 2(v-u)$. Combining Hölder's inequality with (3.21), we then obtain

$$\sum_{n\in\mathcal{E}} 3^{\omega(n)} \leq |\mathcal{E}|^{2/3}\Big(\sum_{\substack{n\leq v\\ n \text{ squarefree}}} 27^{\omega(n)}\Big)^{1/3} \leq |\mathcal{E}|^{2/3}\Big(\sum_{n\leq v} d_{27}(n)\Big)^{1/3}$$
$$\leq c_1 u^{1/3}(v-u)^{2/3}(\log u)^{26/3}, \tag{3.22}$$

where $c_1 := 2^9 3^{1/3}(c_0(27))^{1/3}$, having utilized $\log(3/2) + \log u \leq 2\log u$ for $u \geq 3/2$.

Write $\mathcal{E} = \{n_1, \ldots, n_m\}$, where $n_1 < \ldots < n_m$. For $x \in [u, v]$, put

$$k(x) := \#\{j \in \{1, \ldots, m\} :\ n_j \leq x\}.$$

Then

$$E(u,x) = \sum_{j=1}^{k(x)} f(n_j) \text{ and } E_{\mathbf{a}}(u,x) = \sum_{j=1}^{k(x)} f(n_j/P(n_j)) e_{P(n_j)},$$

where an empty sum is understood to be zero. Consequently,

$$\sup_{u\leq x\leq v} |E(u,x)| = \max_{0\leq k\leq m} \Big|\sum_{j=1}^{k} f(n_j)\Big|,$$
$$\sup_{u\leq x\leq v} \|E_{\mathbf{a}}(u,x)\|_2 = \max_{0\leq k\leq m} \Big\|\sum_{j=1}^{k} f(n_j/P(n_j)) e_{P(n_j)}\Big\|_2.$$

Assume that $m \geq 1$. For $1 \leq j \leq m$, put $Y_j := f(n_j)$ and $u_j := 3^{\omega(n_j)}$. By (3.17), for all $1 \leq a \leq b \leq m$,

$$\mathbb{E}\Big[\Big|\sum_{j=a}^{b} Y_j\Big|^4\Big] \leq \Big(\sum_{j=a}^{b} u_j\Big)^2.$$

Hence, (3.12) and (3.22) give

$$\mathbb{E}\Big[\sup_{u\le x\le v}|E(u,x)|^4\Big]=\mathbb{E}\Big[\max_{0\le k\le m}\Big|\sum_{j=1}^{k}Y_j\Big|^4\Big]\le C\Big(\sum_{j=1}^{m}u_j\Big)^2=C\Big(\sum_{n\in\mathcal{E}}3^{\omega(n)}\Big)^2$$
$$\le Cc_1^2u^{2/3}(v-u)^{4/3}(\log u)^{52/3}.$$

Applying (3.12) to $(f(n_j/P(n_j))e_{P(n_j)})_{1\le j\le m}$, with the same weights u_j, and using (3.18), gives the same bound for the second expectation in (3.20). Thus, (3.20) holds with $C_1:=2Cc_1^2$. If $m=0$, both $x\mapsto E(u,x)$ and $x\mapsto E_{\mathbf{a}}(u,x)$ vanish on $[u,v]$, and (3.20) holds trivially.

Step 4. We next prove that

$$\mathbb{E}[\sup_{u\le x\le v}|L(u,x)|^4]+\mathbb{E}[\sup_{u\le x\le v}\|L_{\mathbf{a}}(u,x)\|_2^4]\le C_2(v-u)^2(\log u)^4 \tag{3.23}$$

for an absolute constant $C_2>0$.

Define $\mathcal{L}:=\{n\in\mathcal{A}:\ u<(P(n))^2\le v\}$. If $n\in\mathcal{L}$ and $p=P(n)$, then $u^{1/2}<p\le v^{1/2}$. Conversely, once such a prime p is fixed, the relevant integers have the form $n=pm$, with m squarefree and $m<p$. Hence

$$\sum_{n\in\mathcal{L}}3^{\omega(n)}=3\sum_{u^{1/2}<p\le v^{1/2}}\ \sum_{\substack{m<p\\ m\ \text{squarefree}}}3^{\omega(m)}.$$

Trivially,

$$\#\{p:\ u^{1/2}<p\le v^{1/2}\}\le v^{1/2}-u^{1/2}+1\le u^{-1/2}(v-u)/2+1. \tag{3.24}$$

For each such prime p, using (3.21) with $k=3$ and $p\le(3/2)^{1/2}u^{1/2}$, we infer

$$\sum_{\substack{m<p\\ m\ \text{squarefree}}}3^{\omega(m)}\le\sum_{m<p}d_3(m)\le c_0(3)p(\log p)^2\le c_3u^{1/2}(\log u)^2,$$

where $c_3=(3/2)^{5/4}c_0(3)$. The last inequality holds for $u\ge(3/2)^{1/2}$ and, in particular, under the standing assumptions which imply that $u\ge4$. Combining this with (3.24) and using $u^{1/2}\le v-u$, we obtain

$$\sum_{n\in\mathcal{L}}3^{\omega(n)}\le3c_3(v-u+u^{1/2})(\log u)^2\le6c_3(v-u)(\log u)^2. \tag{3.25}$$

Write the primes in $(u^{1/2},v^{1/2}]$ as $p_1<\ldots<p_s$ and put

$$Y_j:=\sum_{\substack{n\in\mathcal{L}\\ P(n)=p_j}}f(n),\quad \boldsymbol{Y}_j:=\sum_{\substack{n\in\mathcal{L}\\ P(n)=p_j}}f(n/p_j)e_{p_j}\ \text{and}\ u_j:=\sum_{\substack{n\in\mathcal{L}\\ P(n)=p_j}}3^{\omega(n)}.$$

Assume that $s\ge1$. Then $L(u,x)$ and $L_{\mathbf{a}}(u,x)$ are the partial sums of (Y_j) and $(\boldsymbol{Y}_j)$, respectively, taken over those indices j for which $p_j^2\le x$. The bounds (3.17) and (3.18) apply to every consecutive collection of these groups. Invoking (3.12), with weights u_j, for these two sequences and then using (3.25), we obtain (3.23), with $C_2:=72Cc_3^2$, where C is the constant in (3.12). If $s=0$, both sums vanish, and the conclusion holds trivially.

Step 5. Recall that $d(u,x)=\|\mathbf{a}(x)-\mathbf{a}(u)\|_2$. It follows from (3.14) and (3.16) that

$$|M^{>}(x)-M^{>}(u)|+\|\mathbf{a}(x)-\mathbf{a}(u)\|_2\le|E(u,x)|+|L(u,x)|+\|E_{\mathbf{a}}(u,x)\|_2+\|L_{\mathbf{a}}(u,x)\|_2.$$

Using the inequality $(a_1+\ldots+a_4)^4\le4^3\sum_{j=1}^4a_j^4$ and then (3.20) and the consequence of (3.23)

$$\mathbb{E}[\sup_{u\le x\le v}|L(u,x)|^4]+\mathbb{E}[\sup_{u\le x\le v}\|L_{\mathbf{a}}(u,x)\|_2^4]\le C_2(v-u)^2(\log u)^4$$
$$\le2^{-2/3}C_2u^{2/3}(v-u)^{4/3}(\log u)^{52/3},$$

we arrive at (3.13), with $C:=4^3(C_1+2^{-2/3}C_2)$. The proof of Lemma 3.5 is complete. □

Proof of Proposition 2.6. We introduce a very fine deterministic grid inside $[\mathrm{e}^T, \mathrm{e}^{2T}]$ given by

$$D_T = \{\mathrm{e}^{T+jT^{-120}} : \ j = 0, 1, \dots, T^{121}\}.$$

The grid contains both endpoints of $[\mathrm{e}^T, \mathrm{e}^{2T}]$ and splits this interval into T^{121} closed intervals with disjoint interiors and equal logarithmic length T^{-120}, which we call cells.

We first compare arbitrary points with the endpoints of their cells. To this end, we show that each cell $[u, v]$ satisfies the standing assumptions of Lemma 3.5. Indeed,

$$u^{-1/2}(v-u) = u^{1/2}(\mathrm{e}^{T^{-120}} - 1) \geq \mathrm{e}^{T/2}T^{-120} \ \to \ \infty, \quad T \to \infty.$$

Moreover, for all sufficiently large T, $v/u = \mathrm{e}^{T^{-120}} \leq 3/2$ and thereupon $u + u^{1/2} \leq v \leq 3u/2$. Hence, Lemma 3.5 applies and yields, in combination with Markov's inequality

$$\mathbb{P}\{ \sup_{u\leq x\leq v} (|M^{>}(x) - M^{>}(u)| + d(u,x)) > u^{1/2}\, T^{-2}\}$$
$$\leq 2^{52/3} C \frac{u^{2/3}(u(\mathrm{e}^{T^{-120}} - 1))^{4/3} T^{52/3}}{u^2 T^{-8}} = O(T^{-134}).$$

A union bound then gives

$$\mathbb{P}\{ \max_{[u,v] \text{ a cell}} u^{-1/2} \sup_{u\leq x\leq v} (|M^{>}(x) - M^{>}(u)| + d(u,x)) > T^{-2}\} = T^{121} O(T^{-134}) = O(T^{-13}).$$

Since the right-hand side is summable over dyadic T, we conclude, by the direct part of the Borel-Cantelli lemma, that, a.s., for all sufficiently large T and simultaneously for all cells,

$$\sup_{u\leq x\leq v} (|M^{>}(x) - M^{>}(u)| + d(u,x)) \leq u^{1/2}\, T^{-2}. \tag{3.26}$$

For $y \in D_T$, let $c = c_T(y)$ be the largest point of C_T, see (2.4) for the definition, with $c \leq y$. The deterministic pair (c, y) satisfies $\log(y/c) \leq T^{-\alpha}$. Indeed, if $y = \mathrm{e}^{2T}$, then $c = \mathrm{e}^{2T}$. If $y = \exp(T + jT^{-120})$ for some $j \in \mathbb{N}_0$, $j \leq T^{121} - 1$, then $c = \exp(T + \lfloor jT^{-120+\alpha}\rfloor T^{-\alpha})$, so that $\log(y/c) = jT^{-120} - \lfloor jT^{-120+\alpha}\rfloor T^{-\alpha} \leq T^{-\alpha}$. Combining Lemmas 3.1, 3.2 and Lemma 3.3 (applied to $G_T = C_T \cup D_T$), we obtain, a.s., for all sufficiently large T, simultaneously for all $y \in D_T$,

$$\frac{(d(c,y))^2}{c} \leq 22T^{-\alpha/2} + C(\omega)T^{-1-\alpha}(\log T)^6 \leq T^{-\alpha/3}. \tag{3.27}$$

For each fixed pair (c, y), put $b_p := a_p(y) - a_p(c)$. Whenever $a_p(y)$ is nonzero, we have $p > y^{1/2}$ and $y/p < p$, so $a_p(y)$ is $\mathcal{F}_{p^-}$-measurable. The same argument applies to $a_p(c)$. Hence, b_p is $\mathcal{F}_{p^-}$-measurable for each prime p. Moreover, $b_p = 0$ outside the deterministic finite set $\mathcal{P}_1 := \mathcal{P} \cap (c^{1/2}, y]$. Applying (2.11), with the deterministic choices $t = c^{1/2}\, T^{-\alpha/12}$ and $v = cT^{-\alpha/3}$, we obtain

$$\mathbb{P}\{|M^{>}(y) - M^{>}(c)| > c^{1/2}\, T^{-\alpha/12}, \ (d(c,y))^2 \leq cT^{-\alpha/3}\} \leq 2\exp\big(-T^{\alpha/6}/2\big).$$

From this and a union bound it follows that

$$\mathbb{P}\Big\{ \bigcup_{y\in D_T} \{|M^{>}(y) - M^{>}(c_T(y))| > c_T(y)^{1/2} T^{-\alpha/12}, \ d(c_T(y), y)^2 \leq c_T(y) T^{-\alpha/3}\}\Big\}$$
$$\leq 2(T^{121} + 1)\exp\big(-T^{\alpha/6}/2\big).$$

These bounds are summable over $T = 2^r$. By the direct part of the Borel–Cantelli lemma, the events on the left only occur finitely often a.s. Recalling (3.27), we conclude that a.s., for all sufficiently large T and all $y \in D_T$,

$$|M^{>}(y) - M^{>}(c_T(y))| \leq (c_T(y))^{1/2} T^{-\alpha/12} \quad \text{and} \quad d(c_T(y), y) \leq (c_T(y))^{1/2} T^{-\alpha/6}. \tag{3.28}$$

Here, the second inequality follows from (3.27), whereas the first inequality follows from the second and the preceding Borel-Cantelli argument.

Finally, for $x \in [\mathrm{e}^T, \mathrm{e}^{2T})$, choose the largest $y \in D_T$ with $y \le x$ and put $z = c_T(y)$. Then $z \le y \le x$, $\log(x/z) \le 2T^{-\alpha}$ and the points x and y lie in the same cell of D_T. By (3.26), (3.28), and the triangle inequality,

$$\begin{aligned}|M^>(x)-M^>(z)|+d(x,z) &\le |M^>(x)-M^>(y)|+d(x,y)+|M^>(y)-M^>(c_T(y))|+d(y,c_T(y))\\ &\le y^{1/2}T^{-2} + (c_T(y))^{1/2}\big(T^{-\alpha/12}+T^{-\alpha/6}\big) \le 2x^{1/2}T^{-\alpha/12}.\end{aligned}$$

If $x = \mathrm{e}^{2T}$, then $z = y = x$, and the latter inequality holds trivially. The proof of Proposition 2.6 is complete. □

4 Proof of Theorem 1.1 – the lower bound

4.1 Our strategy

The second part of the paper is concerned with the lower bound for lim sup and the upper bound for lim inf.

Proposition 4.1. *We have*

$$\limsup_{x\to\infty} \frac{M^>(x)}{(2Q(x)\log\log Q(x))^{1/2}} \ge 1 \quad \text{a.s.} \tag{4.1}$$

and

$$\liminf_{x\to\infty} \frac{M^>(x)}{(2Q(x)\log\log Q(x))^{1/2}} \le -1 \quad \text{a.s.}$$

Theorem 1.1 is an immediate consequence of Propositions 2.1 and 4.1.

Fix $\lambda \in (0,1)$ and, for $r \in \mathbb{N}$, put

$$T_r := 4^r, \quad Y_r := \exp(T_r + T_r^{1-\lambda}), \quad a_r := Y_r^{1/2} \quad \text{and} \quad \mathcal{H}_r := \mathcal{F}_{a_r}.$$

Define the deterministic set

$$\mathcal{X}_r := \{\exp(T_r + k) : \ k \in \mathbb{Z},\ 0 \le k \le \lfloor T_r^{1-\lambda} \rfloor\}.$$

For $x \in \mathcal{X}_r$, put

$$\begin{aligned} U_r(x) &:= \sum_{x^{1/2}<p\le a_r} f(p)M(x/p), & u_r(x) &:= \sum_{x^{1/2}<p\le a_r} (M(x/p))^2,\\ V_r(x) &:= \sum_{a_r<p\le x} f(p)M(x/p), & v_r(x) &:= \sum_{a_r<p\le x} (M(x/p))^2. \end{aligned}$$

Then $M^>(x) = U_r(x) + V_r(x)$ and $Q(x) = u_r(x) + v_r(x)$. Each $x \in \mathcal{X}_r$ satisfies $x \le Y_r = a_r^2$. For $p > a_r$, this entails $x/p < a_r$. Hence $v_r(x)$ and all the coefficients $M(x/p)$ appearing in $V_r(x)$ are $\mathcal{H}_r$-measurable. Conditionally on $\mathcal{H}_r$, the variables $(f(p))_{a_r<p\le Y_r}$ remain independent Rademacher variables. In particular,

$$\mathbb{E}_{a_r}[V_r(x)] = 0 \quad \text{and} \quad \mathbb{E}_{a_r}[(V_r(x))^2] = v_r(x).$$

For the primes appearing in $U_r(x)$ and $u_r(x)$, we have $p \le a_r$ and $x/p < x^{1/2} \le a_r$. Hence $U_r(x)$ and $u_r(x)$ are also $\mathcal{H}_r$-measurable.

The lower bound for the lim sup is driven by the principal term V_r, comprising all primes exceeding a_r. Formula (4.3) in Proposition 4.2 is the main novel ingredient of a number-theoretic nature and the most technically demanding part of the paper. With probability tending to one, it controls, for each $x \in \mathcal{X}_r$, the number of points $y \neq x$ for which $|\mathbb{E}_{a_r}[V_r(x)V_r(y)]|$ is large relative to $(xy)^{1/2}$. Lemma 4.3 provides a large $\mathcal{H}_r$-measurable subset $\mathcal{G}_r \subseteq \mathcal{X}_r$ on which $U_r(x)$ is negligible relative to the intended normalization and $Q(x)/v_r(x)$ is close to one, on suitable events of probability tending to one. A greedy selection then gives a sufficiently large $\mathcal{H}_r$-measurable subset $\mathcal{S}_r \subseteq \mathcal{G}_r$ whose normalized sums have small pairwise conditional covariances. Lemma 4.4 shows that the maximum of n weighted Rademacher sums with unit variances, sufficiently small coefficients and vanishing covariance upper bounds exceeds $(1-d)(2\log n)^{1/2}$ with probability tending to one, for each fixed $d \in (0,1)$. Applying this lemma conditionally on $\mathcal{H}_r$ directly to the sums $V_r(x)$ indexed by $\mathcal{S}_r$, and using the bound for $U_r(x)$, gives the required conditional exceedance estimate for $M^{>}$. The conditional Borel-Cantelli lemma then yields infinitely many exceedances a.s. The assertion for the lim inf follows by reversing the signs of both terms.

4.2 Auxiliary results

We start with a number of preparatory results, needed for the proof of Proposition 4.1.

For $x, y \in \mathcal{X}_r$, put

$$C_r(x,y) := \mathbb{E}_{a_r}[V_r(x)V_r(y)] = \sum_{a_r < p \le \min(x,y)} M(x/p)M(y/p).$$

Proposition 4.2. *There is $c_0 > 0$ such that*

$$\lim_{r\to\infty} \mathbb{P}\Big\{ \inf_{x\in\mathcal{X}_r} \frac{v_r(x)}{x} \ge \frac{c_0}{(\log\log a_r)^{11/20}} \Big\} = 1. \tag{4.2}$$

Also, for each fixed $\kappa \in (0,1)$,

$$\lim_{r\to\infty} \mathbb{P}\Big\{ \max_{x\in\mathcal{X}_r} \#\Big\{ y \in \mathcal{X}_r \setminus \{x\} : \ |C_r(x,y)| > (\log\log a_r)^{-3/5}(xy)^{1/2} \Big\} \le (\log a_r)^{\kappa} \Big\} = 1. \tag{4.3}$$

Here, we only give a proof of (4.2), for it rests on a result already available in the literature. Since the proof of (4.3) requires a novel argument, it is postponed until Section 7. The auxiliary estimates are developed in Section 6.

Proof of (4.2) *and Lemma 2.5.* According to Proposition 1 in [16], there exist absolute constants $c_H, C_H > 0$ such that, for all sufficiently large X,

$$\frac{1}{z}\sum_{X<p\le z} (M(z/p))^2 \ge \frac{c_H}{\log X}\int_1^{X^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u, \quad X^{8/7} \le z \le X^{4/3} \tag{4.4}$$

and, for $1 \le W \le (\log\log X)^{1/100}$,

$$\mathbb{P}\left\{ \frac{1}{\log X}\int_1^{X^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u \ge \frac{\mathrm{e}^{-2.2W}}{(\log\log X)^{1/2}} \right\} \ge 1 - C_H \mathrm{e}^{-0.1W}. \tag{4.5}$$

We first treat (4.2). For all sufficiently large r, each $x \in \mathcal{X}_r$ satisfies

$$x^{3/4} > a_r, \quad x^{3/28} \ge a_r^{1/7} \quad \text{and} \quad \log(x^{3/4}) \le (3/2)\log a_r.$$

Indeed, the first two inequalities follow, respectively, from $(3/4)T_r > (1/2)(T_r + T_r^{1-\lambda})$ and $(3/28)T_r \geq (1/14)(T_r + T_r^{1-\lambda})$, both of which hold eventually. The third is a consequence of $x \leq a_r^2$, which is secured by $x \in \mathcal{X}_r$.

Applying (4.4) with $X = x^{3/4}$ and $z = x$, which is admissible because $X^{8/7} = x^{6/7} \leq x = X^{4/3}$, we obtain

$$\frac{v_r(x)}{x} = \frac{1}{x} \sum_{a_r < p \leq x} (M(x/p))^2 \geq \frac{1}{x} \sum_{x^{3/4} < p \leq x} (M(x/p))^2 \geq \frac{c_H}{\log(x^{3/4})} \int_1^{x^{3/28}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u$$
$$\geq \frac{2c_H}{3\log a_r} \int_1^{a_r^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u.$$

This inequality holds simultaneously for all $x \in \mathcal{X}_r$, and its right-hand side does not depend on x.

Next we apply (4.5) with $X = a_r$ and $W = (\log\log\log a_r)/44$. For all sufficiently large r, we have $1 \leq W \leq (\log\log a_r)^{1/100}$ and thereupon

$$\mathbb{P}\Big\{\frac{1}{\log a_r} \int_1^{a_r^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u \geq (\log\log a_r)^{-11/20}\Big\} \geq 1 - C_H(\log\log a_r)^{-1/440}.$$

Consequently, choosing $c_0 := 2c_H/3$, we obtain

$$\mathbb{P}\left\{\inf_{x \in \mathcal{X}_r} \frac{v_r(x)}{x} \geq \frac{c_0}{(\log\log a_r)^{11/20}}\right\} \geq 1 - C_H(\log\log a_r)^{-1/440} \;\to\; 1, \quad r \to \infty,$$

thereby proving (4.2).

Now we pass to the proof of Lemma 2.5. Put $q := 7/6$ and then $J := \lceil \log 2/\log q \rceil = 5$. For $0 \leq j < J$, define $z_j := \mathrm{e}^{Tq^j}$ and $X_j := z_j^{7/8}$. If $x \in [z_j, z_j^{7/6}]$, then $X_j^{8/7} = z_j \leq x \leq z_j^{7/6} = X_j^{4/3}$ and $x^{1/2} \leq z_j^{7/12} < z_j^{7/8} = X_j$. Hence, each prime satisfying $X_j < p \leq x$ also satisfies $x^{1/2} < p \leq x$. Using (4.4), with $X = X_j$ and $z = x$, we obtain

$$\frac{Q(x)}{x} = \frac{1}{x} \sum_{x^{1/2} < p \leq x} (M(x/p))^2 \geq \frac{1}{x} \sum_{X_j < p \leq x} (M(x/p))^2 \geq \frac{c_H}{\log X_j} \int_1^{X_j^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u.$$

Next, we apply (4.5) to each of the $J = 5$ integrals, with $W_j = (\log\log X_j)^{1/100}/2$. For all sufficiently large T, these choices satisfy $1 \leq W_j \leq (\log\log X_j)^{1/100}$ for each $0 \leq j < J$. Put

$$\mathcal{E}_T := \bigcap_{j=0}^{J-1} \Big\{\frac{1}{\log X_j} \int_1^{X_j^{1/7}} \frac{(M(u))^2}{u^2}\,\mathrm{d}u \geq \frac{\mathrm{e}^{-1.1\,(\log\log X_j)^{1/100}}}{(\log\log X_j)^{1/2}}\Big\}.$$

Since $\log\log X_j = \log T + \log(7/8) + j\log(7/6)$, we have, for all sufficiently large T and all $0 \leq j < J$, $(\log T)/2 \leq \log\log X_j \leq 2\log T$. Consequently, the union bound and (4.5) ensure that

$$\mathbb{P}(\mathcal{E}_T^c) \leq C_H \sum_{j=0}^{J-1} \mathrm{e}^{-(\log\log X_j)^{1/100}/20} \leq 5C_H \mathrm{e}^{-c_1(\log T)^{1/100}},$$

where $c_1 := 2^{-1/100}/20$. For dyadic $T = 2^r$, we have, for $r_0 \in \mathbb{N}$ large enough,

$$\sum_{r \geq r_0} \mathbb{P}(\mathcal{E}_{2^r}^c) \leq 5C_H \sum_{r \geq r_0} \mathrm{e}^{-c_1(r\log 2)^{1/100}} < \infty.$$

By the direct part of the Borel-Cantelli lemma, $\mathcal{E}_{2^r}$ occurs eventually a.s.

Observe that $\bigcup_{j=0}^{J-1}[z_j, z_j^{7/6}] = [\mathrm{e}^T, \mathrm{e}^{Tq^J}] \supseteq [\mathrm{e}^T, \mathrm{e}^{2T}]$ because $q^J > 2$. On $\mathcal{E}_T$, this fact, together with the preceding deterministic comparison, entails

$$\inf_{\mathrm{e}^T \le x \le \mathrm{e}^{2T}} \frac{Q(x)}{x} \ge c_H \min_{0 \le j < J} \frac{\mathrm{e}^{-1.1(\log\log X_j)^{1/100}}}{(\log\log X_j)^{1/2}} \ge \frac{c_2 \mathrm{e}^{-C_2(\log T)^{1/100}}}{(\log T)^{1/2}},$$

where $c_2 := 2^{-1/2} c_H$ and $C_2 := 1.1 \times 2^{1/100}$. Finally,

$$\lim_{T\to\infty} \frac{C_2(\log T)^{1/100} + (\log\log T)/2 - \log c_2}{\log T} = 0.$$

Thus, for each fixed $b > 0$ and all sufficiently large T, $C_2(\log T)^{1/100} + (\log\log T)/2 - \log c_2 \le b \log T$, or, equivalently,

$$\frac{c_2 \mathrm{e}^{-C_2(\log T)^{1/100}}}{(\log T)^{1/2}} \ge \mathrm{e}^{-b\log T} = T^{-b}.$$

Combining these observations completes the proof of Lemma 2.5. □

For each realization ω and each $x \in \mathcal{X}_r$, define

$$B_r(x;\omega) := \{y \in \mathcal{X}_r \setminus \{x\} : \ |C_r(x,y;\omega)| > (\log\log a_r)^{-3/5}(xy)^{1/2}\}.$$

For distinct $x, y \in \mathcal{X}_r$, we call (x,y) a bad pair if $y \in B_r(x;\omega)$, and a good pair otherwise. We also call y a bad or good neighbor of x, respectively.

Formula (4.3) gives, for each fixed $\kappa \in (0,1)$,

$$\lim_{r\to\infty} \mathbb{P}\{\max_{x\in\mathcal{X}_r} |B_r(x)| \le (\log a_r)^\kappa\} = 1. \tag{4.6}$$

By definition, each good pair (x,y) satisfies $|C_r(x,y)| \le (\log\log a_r)^{-3/5}(xy)^{1/2}$.

On the events appearing in (4.2), we have $(v_r(x)v_r(y))^{1/2} \ge c_0(\log\log a_r)^{-11/20}(xy)^{1/2}$. Consequently, for each good pair (x,y),

$$\begin{aligned}\frac{|C_r(x,y)|}{(v_r(x)v_r(y))^{1/2}} = \frac{|C_r(x,y)|}{(xy)^{1/2}} \frac{(xy)^{1/2}}{(v_r(x)v_r(y))^{1/2}} &\le \frac{1}{(\log\log a_r)^{3/5}} \frac{(\log\log a_r)^{11/20}}{c_0} \\ &= \frac{1}{c_0(\log\log a_r)^{1/20}} = o(1), \quad r\to\infty. \end{aligned} \tag{4.7}$$

For later use, we note that by Mertens' theorem (see, for instance, Theorem 2.7 (d) in [21]),

$$\sum_{p\le z} \frac{1}{p} = \log\log z + b + O\Big(\frac{1}{\log z}\Big), \quad z\to\infty, \tag{4.8}$$

where $b = \gamma - \sum_p \sum_{k\ge2}(1/(kp^k))$ and γ is the Euler-Mascheroni constant.

With c_0 as in Proposition 4.2, put

$$\mathcal{G}_r := \left\{ x \in \mathcal{X}_r : \ (U_r(x))^2 + u_r(x) \le \frac{c_0 x}{2(\log\log a_r)^{31/20}} \right\}.$$

At this point the set $\mathcal{G}_r$ may be empty. Recall the notation $\mathcal{H}_r = \mathcal{F}_{a_r}$.

Lemma 4.3. *The random set $\mathcal{G}_r$ is $\mathcal{H}_r$-measurable and $\lim_{r\to\infty} \mathbb{P}\{|\mathcal{G}_r| \ge |\mathcal{X}_r|/2\} = 1$.*

On the events appearing in (4.2), for each $x \in \mathcal{G}_r$,

$$\frac{Q(x)}{v_r(x)} = 1 + O\big((\log\log a_r)^{-1}\big) \quad \textit{and} \quad |U_r(x)| \le \Big(\frac{Q(x)}{\log\log a_r}\Big)^{1/2} \tag{4.9}$$

where the O-term is uniform in $x \in \mathcal{G}_r$.

Proof. Measurability follows because, for each $x \in \mathcal{X}_r$, $U_r(x)$ and $u_r(x)$ are $\mathcal{H}_r$-measurable.

Uniformly in $x \in \mathcal{X}_r$,

$$\mathbb{E}\big[(U_r(x))^2\big] = \mathbb{E}[u_r(x)] \le x \sum_{x^{1/2}<p\le a_r} p^{-1} \le x \sum_{\exp(T_r/2)<p\le a_r} p^{-1} = O(xT_r^{-1}) + O(xT_r^{-\lambda})$$
$$= O(xT_r^{-\lambda}), \quad r \to \infty.$$

Here, the first equality is a consequence of $\mathbb{E}_{x^{1/2}}\big[(U_r(x))^2\big] = u_r(x)$, the first inequality is secured by (2.6), the second equality follows from (4.8) and

$$\log\log a_r - \log\log\exp(T_r/2) = \log(T_r + T_r^{1-\lambda}) - \log T_r = O(T_r^{-\lambda}), \quad r \to \infty,$$

and the last equality holds because $\lambda \in (0,1)$.

Hence, by Markov's inequality, uniformly in $x \in \mathcal{X}_r$,

$$\mathbb{P}\{x \notin \mathcal{G}_r\} \le \frac{2(\log\log a_r)^{31/20}}{c_0 x}\mathbb{E}\big[(U_r(x))^2 + u_r(x)\big] = O((\log\log a_r)^{31/20}T_r^{-\lambda}) = o(1)$$

as $r \to \infty$. The last equality is ensured by $\log\log a_r \sim \log T_r$ as $r \to \infty$. Summing over the deterministic set $\mathcal{X}_r$ we obtain $\mathbb{E}|\mathcal{X}_r \setminus \mathcal{G}_r| = o(|\mathcal{X}_r|)$ as $r \to \infty$. With this at hand, another application of Markov's inequality gives

$$\mathbb{P}\{|\mathcal{G}_r| \le |\mathcal{X}_r|/2\} = \mathbb{P}\{|\mathcal{X}_r \setminus \mathcal{G}_r| \ge |\mathcal{X}_r|/2\} \le \frac{2}{|\mathcal{X}_r|}\mathbb{E}\big[|\mathcal{X}_r \setminus \mathcal{G}_r|\big] \;\to\; 0, \quad r \to \infty.$$

On the events appearing in (4.2), for each $x \in \mathcal{G}_r$,

$$0 \le \frac{u_r(x)}{v_r(x)} \le \frac{1}{2\log\log a_r}, \quad \frac{Q(x)}{v_r(x)} = 1 + \frac{u_r(x)}{v_r(x)}.$$

This proves the first assertion of (4.9), uniformly over these points and realizations. Furthermore,

$$|U_r(x)| \le \Big(\frac{c_0 x}{2(\log\log a_r)^{31/20}}\Big)^{1/2} \le \Big(\frac{v_r(x)}{2\log\log a_r}\Big)^{1/2} \le \Big(\frac{Q(x)}{\log\log a_r}\Big)^{1/2},$$

which proves the second assertion. □

Fix $\kappa \in (0, 1-\lambda)$ and let c_0 be as in Proposition 4.2. Denote by $\mathcal{E}_r$ the intersection of the events

$$\max_{u\le a_r}|M(u)| \le a_r^{2/3}, \qquad \min_{x\in\mathcal{X}_r}\frac{v_r(x)}{x} \ge \frac{c_0}{(\log\log a_r)^{11/20}},$$
$$\max_{x\in\mathcal{X}_r}|B_r(x)| \le (\log a_r)^{\kappa}, \qquad |\mathcal{G}_r| \ge |\mathcal{X}_r|/2.$$

Each event is $\mathcal{H}_r$-measurable, hence so is $\mathcal{E}_r$. By (2.7), with $b_r = a_r$, (4.2), (4.6) and Lemma 4.3, we have $\lim_{r\to\infty}\mathbb{P}(\mathcal{E}_r) = 1$.

Fix a realization and consider the finite graph whose vertex set is $\mathcal{G}_r$, with two distinct vertices joined by an edge if they form a bad pair. Since $C_r(x,y) = C_r(y,x)$, this graph is undirected. An independent set in this graph, being a set containing no two vertices joined by an edge, is a subset of $\mathcal{G}_r$ containing no bad pair.

We construct such a set recursively. Initially, all points of $\mathcal{G}_r$ are available. If there is an available point, select the smallest one, include it in $\mathcal{S}_r$, and remove both that point and all its available bad neighbors from the available set. Repeat until no available points remain. If $\mathcal{G}_r$ is empty, put $\mathcal{S}_r = \varnothing$. Every two distinct selected points form a good pair. Indeed, when a point is selected, all its bad neighbors that could otherwise be selected later are removed. Thus

$\mathcal{S}_r$ is an independent set. The construction is $\mathcal{H}_r$-measurable. Indeed, the set $\mathcal{X}_r$ is finite and deterministic, membership in $\mathcal{G}_r$ and each bad-neighbor relation are $\mathcal{H}_r$-measurable, and at each step the choice is determined by the usual order on the real line.

On $\mathcal{E}_r$, we have $|\mathcal{G}_r| \geq |\mathcal{X}_r|/2$ by the definition of $\mathcal{E}_r$, and each selection removes at most $1 + (\log a_r)^\kappa$ points, namely, the selected point x and at most $|B_r(x)| \leq (\log a_r)^\kappa$ of its bad neighbors. Since the number of selections is equal to $|\mathcal{S}_r|$, each point of $\mathcal{G}_r$ is removed at most once, and all points of $\mathcal{G}_r$ have been removed when the procedure terminates, we infer $|\mathcal{G}_r| \leq |\mathcal{S}_r|(1 + (\log a_r)^\kappa)$. Hence,

$$|\mathcal{S}_r| \geq \frac{|\mathcal{X}_r|}{2(1 + (\log a_r)^\kappa)}.$$

In view of $\log |\mathcal{X}_r| = \log\left((\lfloor T_r^{1-\lambda} \rfloor) + 1\right) = (1-\lambda)\log T_r + o(1)$ and $\log\log a_r = \log T_r + O(1)$ as $r \to \infty$, we obtain, on $\mathcal{E}_r$,

$$\log |\mathcal{S}_r| \geq (1 - \lambda - \kappa)\log T_r - O(1), \quad r \to \infty. \tag{4.10}$$

Here, the constant implicit in $O(1)$ is deterministic and independent of the realization in $\mathcal{E}_r$.

We proceed with a lemma regarding maxima of weighted Rademacher sums. This is the last ingredient needed for the proof of Proposition 4.1.

Lemma 4.4. *For each $n \in \mathbb{N}$, let I_n be a finite set and let $(\xi_{n,j})_{j \in I_n}$ be independent copies of a random variable ξ with $\mathbb{P}\{\xi = \pm 1\} = 1/2$. Put*

$$Z_{n,i} := \sum_{j \in I_n} c_{n,j,i}\xi_{n,j}, \quad 1 \leq i \leq n,$$

where the coefficients are real and deterministic, and $\sum_{j \in I_n} c_{n,j,i}^2 = 1$ for $1 \leq i \leq n$. Assume that $\beta_n := \max_{j \in I_n,\ 1 \leq i \leq n} |c_{n,j,i}|$ satisfies $\lim_{n\to\infty} \beta_n \log n = 0$ and that, for some deterministic sequence $(\rho_n)_{n \geq 1}$ satisfying $\lim_{n\to\infty} \rho_n = 0$,

$$\max_{1 \leq i < k \leq n} \sum_{j \in I_n} c_{n,j,i}c_{n,j,k} \leq \rho_n$$

for all sufficiently large n. Then, for each fixed $\delta \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}\Big\{ \max_{1 \leq i \leq n} Z_{n,i} > (1-\delta)(2\log n)^{1/2} \Big\} = 1. \tag{4.11}$$

With $n_0 \geq 2$ fixed independently of the coefficient array, the convergence is uniform over coefficient arrays satisfying, for all $n \geq n_0$, $\beta_n \leq b_n$ and

$$\max_{1 \leq i < k \leq n} \sum_{j \in I_n} c_{n,j,i}c_{n,j,k} \leq r_n,$$

where $(b_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ are given deterministic nonnegative sequences satisfying $\lim_{n\to\infty} b_n \log n = 0$ and $\lim_{n\to\infty} r_n = 0$.

Proof. Replacing ρ_n by $\max(\rho_n, 0)$ if needed, we can and do assume that $\rho_n \in [0,1)$ for all sufficiently large n.

Fix $\delta \in (0,1)$ and choose $\alpha \in (1-\delta, 1)$. Put $t_n := \alpha(2\log n)^{1/2}$ and $h_n := t_n + 4$ for $n \in \mathbb{N}$. We first obtain a uniform lower bound for the individual exceedance probabilities. For $1 \leq i \leq n$ and $h \in \mathbb{R}$, put

$$K_{n,i}(h) := \log \mathbb{E}\big[\mathrm{e}^{hZ_{n,i}}\big] = \sum_{j \in I_n} \log\cosh(hc_{n,j,i}).$$

In view of

$$\sum_{j\in I_n} c_{n,j,i}^4 \le \beta_n^2 \sum_{j\in I_n} c_{n,j,i}^2 = \beta_n^2,$$

the expansions $\log\cosh z = z^2/2 + O(z^4)$ and $\tanh z = z + O(z^3)$ as $z \to 0$ show that, uniformly in $1 \le i \le n$,

$$K_{n,i}(h_n) = h_n^2/2 + O(\beta_n^2 h_n^4) = h_n^2/2 + o(1) \quad \text{and} \quad K'_{n,i}(h_n) = \sum_{j\in I_n} c_{n,j,i} \tanh(h_n c_{n,j,i})$$
$$= h_n + O(\beta_n^2 h_n^3) = h_n + o(1), \quad n \to \infty. \quad (4.12)$$

Here, we have used $\lim_{n\to\infty} \beta_n \log n = 0$. Moreover, in view of $\cosh z \ge 1$ for $z \in \mathbb{R}$, we conclude that, for each $h \in \mathbb{R}$,

$$K''_{n,i}(h) = \sum_{j\in I_n} \frac{c_{n,j,i}^2}{\cosh^2(h c_{n,j,i})} \le 1. \quad (4.13)$$

Introduce the exponentially tilted probability measure $\mathbb{P}_{n,i}^{(h_n)}$ by

$$\frac{\mathrm{d}\mathbb{P}_{n,i}^{(h_n)}}{\mathrm{d}\mathbb{P}} = \exp(h_n Z_{n,i} - K_{n,i}(h_n)), \quad 1 \le i \le n,\ n \ge 2.$$

Under $\mathbb{P}_{n,i}^{(h_n)}$, the variable $Z_{n,i}$ has mean $K'_{n,i}(h_n)$ and variance $K''_{n,i}(h_n) \le 1$. Hence, by Chebyshev's inequality, $\mathbb{P}_{n,i}^{(h_n)}\{|Z_{n,i} - K'_{n,i}(h_n)| \le 2\} \ge 3/4$. By the second part of (4.12), for all sufficiently large n, uniformly in $1 \le i \le n$,

$$\{|Z_{n,i} - K'_{n,i}(h_n)| \le 2\} \subseteq \{t_n < Z_{n,i} \le t_n + 7\}.$$

Consequently,

$$\mathbb{P}\{Z_{n,i} > t_n\} = \mathbb{E}_{n,i}^{(h_n)}\left[\mathrm{e}^{-h_n Z_{n,i} + K_{n,i}(h_n)} \mathbb{1}_{\{Z_{n,i} > t_n\}}\right] \ge \mathbb{E}_{n,i}^{(h_n)}\left[\mathrm{e}^{-h_n Z_{n,i} + K_{n,i}(h_n)} \mathbb{1}_{\{t_n < Z_{n,i} \le t_n + 7\}}\right]$$
$$\ge (3/4)\exp(K_{n,i}(h_n) - h_n(t_n + 7)).$$

Recalling the first part of (4.12) and noting that $h_n^2/2 - h_n(t_n + 7) = -t_n^2/2 - 7t_n - 20$, we conclude that there exists an absolute constant $C > 0$ such that, for all sufficiently large n,

$$\min_{1\le i\le n} \mathbb{P}\{Z_{n,i} > t_n\} \ge \exp(-t_n^2/2 - 7t_n - C) =: q_n. \quad (4.14)$$

Now we estimate the joint exceedance probabilities. Put

$$r_{n,i,k} := \sum_{j\in I_n} c_{n,j,i} c_{n,j,k}, \quad 1 \le i, k \le n,\ n \ge 2.$$

For $1 \le i, k \le n$, $i \ne k$ and $h > 0$, the inequality $\log\cosh z \le z^2/2$ for $z \in \mathbb{R}$ entails

$$\log \mathbb{E}\big[\mathrm{e}^{h(Z_{n,i} + Z_{n,k})}\big] \le (h^2/2) \sum_{j\in I_n} (c_{n,j,i} + c_{n,j,k})^2 = h^2(1 + r_{n,i,k}) \le h^2(1 + \rho_n).$$

An application of Markov's inequality, with $h = t_n/(1 + \rho_n)$, then yields

$$\mathbb{P}\{Z_{n,i} > t_n,\ Z_{n,k} > t_n\} \le \mathbb{P}\{Z_{n,i} + Z_{n,k} > 2t_n\} \le \exp(-t_n^2/(1 + \rho_n)). \quad (4.15)$$

For $n \in \mathbb{N}$, put $N_n := \sum_{i=1}^n \mathbb{1}_{\{Z_{n,i} > t_n\}}$ and $H_n := \max_{1\le i\le n} Z_{n,i}$. By (4.14),

$$\mathbb{E}[N_n] \ge n q_n = \exp\{(1 - \alpha^2)\log n - O((\log n)^{1/2})\} \ \to\ \infty, \quad n \to \infty.$$

Combining (4.14) and (4.15), we obtain

$$\frac{\mathbb{E}[N_n^2]}{(\mathbb{E}[N_n])^2} = \frac{1}{\mathbb{E}[N_n]} + \frac{\sum_{1\le i\ne k\le n}\mathbb{P}\{Z_{n,i} > t_n,\ Z_{n,k} > t_n\}}{(\mathbb{E}[N_n])^2}$$
$$\le \frac{1}{nq_n} + \frac{\exp(-t_n^2/(1+\rho_n))}{q_n^2} = \frac{1}{nq_n} + \exp\left(\frac{\rho_n t_n^2}{1+\rho_n} + 14t_n + 2C\right) =: R_n.$$

Hence, by the second moment method,

$$\mathbb{P}\{H_n > t_n\} = \mathbb{P}\{N_n > 0\} \ge \frac{(\mathbb{E}[N_n])^2}{\mathbb{E}[N_n^2]} \ge R_n^{-1}. \tag{4.16}$$

For later use, we note that $R_n \ge \mathrm{e}^{2C} \ge 1$. Since $t_n^2 = 2\alpha^2 \log n$, $\lim_{n\to\infty} nq_n = \infty$ and $\lim_{n\to\infty}\rho_n = 0$, we further have

$$\log R_n = \frac{2\alpha^2\rho_n}{1+\rho_n}\log n + O((\log n)^{1/2}) = o(\log n), \quad n\to\infty.$$

Now we work towards strengthening the lower bound in (4.16). The function F_n defined by $F_n(z) := \max_{1\le i\le n}\sum_{j\in I_n} c_{n,j,i}(2z_j - 1)$ for $z \in \mathbb{R}^{|I_n|}$ is convex. It is also 2-Lipschitz with respect to the Euclidean norm, because, by the Cauchy-Schwarz inequality,

$$|F_n(z) - F_n(z')| \le 2\max_{1\le i\le n}\left|\sum_{j\in I_n} c_{n,j,i}(z_j - z_j')\right| \le 2\max_{1\le i\le n}\Big(\sum_{j\in I_n} c_{n,j,i}^2\Big)^{1/2}\Big(\sum_{j\in I_n}(z_j - z_j')^2\Big)^{1/2}$$
$$= 2\Big(\sum_{j\in I_n}(z_j - z_j')^2\Big)^{1/2}.$$

Let μ_n be a median of the distribution of H_n, that is, a not necessarily unique number satisfying $\mathbb{P}\{H_n \le \mu_n\} \ge 1/2$ and $\mathbb{P}\{H_n \ge \mu_n\} \ge 1/2$. Since $H_n = F_n(\eta^{(n)})$, where $\eta^{(n)}$ is the vector with coordinates $(\xi_{n,j}+1)/2$, $j\in I_n$, we obtain with the help of Talagrand's convex concentration inequality as given in Corollary 4 of [17] with $p = 2$,

$$\mathbb{P}\{|H_n - \mu_n| > s\} \le 4\exp(-s^2/16), \quad s > 0. \tag{4.17}$$

If $\mu_n < t_n$, then (4.16) and (4.17) imply $R_n^{-1} \le 4\exp(-(t_n-\mu_n)^2/16)$. As a consequence,

$$\mu_n \ge t_n - 4(\log(4R_n))^{1/2} = \alpha(2\log n)^{1/2} - o((\log n)^{1/2}), \quad n\to\infty$$

because $\log R_n = o(\log n)$ as $n\to\infty$. If $\mu_n \ge t_n$, then the latter estimate holds trivially. Since $\alpha > 1-\delta$, we conclude that $s_n := \mu_n - (1-\delta)(2\log n)^{1/2} \to \infty$ as $n\to\infty$. In particular, $s_n > 0$ for all sufficiently large n. For such n, an application of (4.17) with $s = s_n/2$ yields

$$\mathbb{P}\{H_n \le (1-\delta)(2\log n)^{1/2}\} = \mathbb{P}\{H_n - \mu_n \le -s_n\}$$
$$\le \mathbb{P}\{|H_n - \mu_n| > s_n/2\} \le 4\mathrm{e}^{-s_n^2/64} \ \to 0$$

as $n\to\infty$. This proves (4.11).

For the claimed uniformity, repeat the preceding argument with ρ_n replaced by r_n and β_n bounded by b_n. All remainder estimates and all thresholds for sufficiently large n can then be chosen independently of the coefficient array. □

4.3 Proof of Proposition 4.1

Fix $q \in (0,1)$. Choose $\lambda \in (0,1)$, $\kappa \in (0, 1-\lambda)$ and $d \in (0,1)$ satisfying

$$(1-d)(1-\lambda-\kappa)^{1/2} > q. \tag{4.18}$$

The constructions of Sections 4.1 and 4.2 are carried out with these λ and κ. Recall that $\mathcal{S}_r \subseteq \mathcal{G}_r$ is the $\mathcal{H}_r$-measurable set obtained by the greedy selection in Subsection 4.2, so that any two distinct points of $\mathcal{S}_r$ form a good pair. Put $t_r(x) := (1-d)(2v_r(x)\log(\max(|\mathcal{S}_r|,2)))^{1/2}$ for $x \in \mathcal{X}_r$, and define

$$A_r := \{\exists x \in \mathcal{S}_r : \ V_r(x) > t_r(x)\}.$$

By (4.10), there is a deterministic sequence $(n_r)_{r\geq 1}$ with $\lim_{r\to\infty} n_r = \infty$ such that, on $\mathcal{E}_r$, $|\mathcal{S}_r| \geq n_r$ for all sufficiently large r. In particular, the maximum in the definition of $t_r(x)$ only matters outside $\mathcal{E}_r$. We first estimate $\mathbb{P}_{a_r}(A_r)$ from below. To this end, we apply Lemma 4.4 conditionally on $\mathcal{H}_r$. On $\mathcal{E}_r$, for $x \in \mathcal{S}_r$, we use a representation

$$\frac{V_r(x)}{(v_r(x))^{1/2}} = \sum_{a_r < p \leq Y_r} c_{r,p,x} f(p),$$

where $c_{r,p,x} := \mathbb{1}_{\{p\leq x\}} M(x/p)(v_r(x))^{-1/2}$. As explained in Subsection 4.1, all the coefficients $c_{r,p,x}$ are $\mathcal{H}_r$-measurable, and conditionally on $\mathcal{H}_r$, the variables $(f(p))_{a_r<p\leq Y_r}$ remain independent Rademacher variables. Also, $\sum_{a_r<p\leq Y_r} c_{r,p,x}^2 = 1$. On $\mathcal{E}_r$, the bounds $\max_{u\leq a_r}|M(u)| \leq a_r^{2/3}$ and $v_r(x) \geq c_0 x/(\log\log a_r)^{11/20}$ for $x \in \mathcal{X}_r$ give

$$\begin{aligned}\beta_r := \max_{x\in\mathcal{S}_r, a_r<p\leq Y_r} |c_{r,p,x}| &\leq c_0^{-1/2}(\log\log a_r)^{11/40} a_r^{2/3} \mathrm{e}^{-T_r/2} \\ &= c_0^{-1/2}(\log\log a_r)^{11/40} \exp\left(-T_r/6 + T_r^{1-\lambda}/3\right) \leq \mathrm{e}^{-T_r/8} \end{aligned} \tag{4.19}$$

for all sufficiently large r, uniformly over the realizations in $\mathcal{E}_r$, having utilized the bound $x \geq \mathrm{e}^{T_r}$ secured by $x \in \mathcal{X}_r$. Since $\log|\mathcal{S}_r| \leq \log|\mathcal{X}_r| \sim (1-\lambda)\log T_r$ as $r\to\infty$, inequality (4.19) entails $\lim_{r\to\infty}\beta_r \log|\mathcal{S}_r| = 0$ uniformly over the realizations in $\mathcal{E}_r$.

Every two distinct points of $\mathcal{S}_r$ form a good pair. Consequently, uniformly over the realizations in $\mathcal{E}_r$, for $x, y \in \mathcal{S}_r$, $x \neq y$,

$$\Big|\sum_{a_r<p\leq Y_r} c_{r,p,x}c_{r,p,y}\Big| = \frac{|C_r(x,y)|}{(v_r(x)v_r(y))^{1/2}} \leq \frac{1}{c_0(\log\log a_r)^{1/20}} =: \rho_r \ \to\ 0, \quad r\to\infty.$$

The estimates in the proof of Lemma 4.4 depend only on the number of sums, the maximal coefficient, and the upper bound on the pairwise covariances. By (4.10), (4.19) and the bound on ρ_r, we have $|\mathcal{S}_r| \geq n_r$, $\lim_{r\to\infty}\beta_r\log|\mathcal{S}_r| = 0$ and $\lim_{r\to\infty}\rho_r = 0$, with deterministic bounds uniform over the realizations in $\mathcal{E}_r$. More precisely, on $\mathcal{E}_r$, $n_r \leq |\mathcal{S}_r| \leq |\mathcal{X}_r| \leq T_r + 1$ for all sufficiently large r. Hence, on $\mathcal{E}_r$, $\beta_r \leq \mathrm{e}^{-T_r/8} \leq b_{|\mathcal{S}_r|}$ and $\rho_r \leq r_{|\mathcal{S}_r|}$, where $b_n := \mathrm{e}^{-(n-1)/8}$ and $r_n := c_0^{-1}(\log((n-1)/2))^{-1/20}$ for $n \geq 4$. These deterministic sequences satisfy $\lim_{n\to\infty} b_n \log n = 0$ and $\lim_{n\to\infty} r_n = 0$, so that the uniform version of Lemma 4.4 applies, with $n = |\mathcal{S}_r|$ and $\delta = d$. If $\eta_n(d)$ denotes the resulting deterministic upper bound for the probability of the complement of the event in (4.11), then one can take $\theta_r(d) := \sup_{n\geq n_r}\eta_n(d)$ below. Applying the lemma conditionally on $\mathcal{H}_r$, we conclude that, for the chosen $d \in (0,1)$, there exists a *deterministic* sequence $(\theta_r(d))_{r\geq 1}$ with $\lim_{r\to\infty}\theta_r(d) = 0$ such that, for all sufficiently large r, a.s. on $\mathcal{E}_r$,

$$\mathbb{P}_{a_r}(A_r) \geq 1 - \theta_r(d). \tag{4.20}$$

Lemma 2.3 implies that, uniformly in $x \in \mathcal{X}_r$,

$$\log\log Q(x) = \log T_r + O(1) \tag{4.21}$$

with a deterministic $O(1)$-constant.

We claim that, on $\mathcal{E}_r$, a point $x \in \mathcal{S}_r$ satisfying $V_r(x) > (1-d)(2v_r(x)\log|\mathcal{S}_r|)^{1/2}$ also satisfies $M^>(x) = U_r(x) + V_r(x) > q(2Q(x)\log\log Q(x))^{1/2}$ for all sufficiently large r. Indeed, since $\mathcal{S}_r \subseteq \mathcal{G}_r$, such a point satisfies

$$\frac{U_r(x)+V_r(x)}{(2Q(x)\log\log Q(x))^{1/2}} > (1-d)\left(\frac{v_r(x)}{Q(x)}\frac{\log|\mathcal{S}_r|}{\log\log Q(x)}\right)^{1/2} - \frac{1}{(2\log\log a_r\log\log Q(x))^{1/2}}$$
$$\geq (1-d)(1-\lambda-\kappa)^{1/2} - o(1) > q$$

for all sufficiently large r. Here we have used the second part of (4.9) for the first inequality, the first part of (4.9), (4.10) and (4.21) for the second inequality, with all error estimates being uniform over $x \in \mathcal{S}_r$ and the realizations in $\mathcal{E}_r$. The last inequality follows from the fact that (4.18) is a strict inequality.

Define the events

$$B_r^+ := \{\exists x \in \mathcal{X}_r :\ M^>(x) > q(2Q(x)\log\log Q(x))^{1/2}\}, \quad r \in \mathbb{N}.$$

By the preceding implication, for all sufficiently large r, a.s. on $\mathcal{E}_r$,

$$\mathbb{P}_{a_r}(B_r^+) \geq \mathbb{P}_{a_r}(A_r) \geq 1 - \theta_r(d).$$

Consequently, for all sufficiently large r, a.s., $\mathbb{P}_{a_r}(B_r^+) \geq \mathbb{1}_{\mathcal{E}_r}/2$. Since $\lim_{r\to\infty}\mathbb{P}(\mathcal{E}_r) = 1$ entails $\mathbb{P}(\mathcal{E}_r \text{ i.o.}) = 1$, we conclude that

$$\sum_{r\geq 1}\mathbb{P}_{a_r}(B_r^+) = \infty \quad \text{a.s.} \tag{4.22}$$

Now we verify the filtration requirements for the conditional Borel-Cantelli lemma, see, for instance, Theorem 5.3.2 on p. 240 in [8]. The sigma-fields $(\mathcal{H}_r)_{r\geq 0}$ form a filtration. Moreover, B_r^+ is $\mathcal{F}_{Y_r}$-measurable. Since $\log Y_r = T_r + T_r^{1-\lambda}$ and $\log a_{r+1} = 2T_r + (4T_r)^{1-\lambda}/2$, we have $Y_r < a_{r+1}$ for all sufficiently large r. Hence $B_r^+ \in \mathcal{H}_{r+1}$. With this at hand, the conditional Borel-Cantelli lemma applied to (4.22) yields $\mathbb{P}(B_r^+ \text{ i.o.}) = \mathbb{P}\{\sum_r \mathbb{P}_{a_r}(B_r^+) = \infty\} = 1$. Taking a countable sequence $q \to 1-$, with the corresponding choices $\lambda = \lambda_q$, $\kappa = \kappa_q$ and $d = d_q$ satisfying (4.18), and intersecting the resulting events of probability one, we arrive at (4.1).

The second assertion of Proposition 4.1 follows by symmetry. Indeed, conditionally on $\mathcal{H}_r$, the vector $(f(p))_{a_r<p\leq Y_r}$ has the same distribution as $(-f(p))_{a_r<p\leq Y_r}$, and this sign reversal changes $V_r(x)$ into $-V_r(x)$, whereas the $\mathcal{H}_r$-measurable quantities $U_r(x)$, $v_r(x)$, $Q(x)$, $\mathcal{S}_r$ and $\mathcal{E}_r$ remain unchanged. Hence, the preceding argument applies to $-M^>(x) = -U_r(x) - V_r(x)$.

5 Proof of Corollary 1.2

Since $M^>(x) = M^>(\lfloor x\rfloor)$ and $Q(x) = Q(\lfloor x\rfloor)$ for $x \geq 1$ (all changes occur at integers pm or prime squares p^2), the function whose cluster set appears in (1.3) is constant on each interval $[n, n+1)$, $n \in \mathbb{N}$. Hence, it suffices to prove (1.3) with x replaced by $n \in \mathbb{N}$.

For each prime p and $n \in \mathbb{N}_0$, put $v_p(n) := M(n/p)\mathbb{1}_{\{p>n^{1/2}\}}$, so that $Q(n) = \sum_{p\leq n}(v_p(n))^2$. Also, put

$$h_n := \Big(\sum_{p\leq n}(v_p(n) - v_p(n-1))^2\Big)^{1/2}, \quad n \in \mathbb{N}.$$

In the notation of Section 2, $v_p(n) = a_p(n)$ and $h_n = d(n-1,n)$ for $n \geq 3$. If n is not the square of a prime, then, for $p \leq n^{1/2}$, $v_p(n) = v_p(n-1) = 0$, whereas, for $p > n^{1/2}$,

$$v_p(n) - v_p(n-1) = M(n/p) - M((n-1)/p) = \begin{cases} f(n/p), & \text{if } p|n, \\ 0, & \text{if } p\nmid n. \end{cases}$$

There is at most one prime divisor $p > n^{1/2}$, since two such divisors would have product greater than n. Consequently,

$$M^{>}(n) - M^{>}(n-1) = \begin{cases} f(p)f(n/p), & \text{if there exists a prime } \ p > n^{1/2} \text{ dividing } n, \\ 0, & \text{otherwise.} \end{cases}$$

If $n = p^2$ for a prime p, then $v_p(n) - v_p(n-1) = -M(p-1)$, whereas $v_q(n) - v_q(n-1) = 0$ for each prime $q \neq p$. Hence,

$$M^{>}(n) - M^{>}(n-1) = -f(p)M(p-1).$$

Summarizing,

$$|M^{>}(n) - M^{>}(n-1)| \le h_n \le \begin{cases} 1, & n \text{ is not the square of a prime,} \\ |M(p-1)|, & n = p^2, \ p \text{ prime.} \end{cases}$$

Using (3.6) yields

$$h_n = O\big(n^{1/4}(\log\log n)^3\big), \quad n \to \infty \quad \text{a.s.}$$

Together with the deterministic lower bound $Q(n) \ge n/(3\log n)$ for all sufficiently large n (see (2.8)), this implies

$$\lim_{n\to\infty} \frac{h_n}{(Q(n))^{1/2}} = 0 \quad \text{a.s.} \tag{5.1}$$

The triangle inequality in ℓ^2 (the space of square summable sequences) yields $\big|(Q(n))^{1/2} - (Q(n-1))^{1/2}\big| \le h_n$. Thus (5.1) and $\lim_{n\to\infty} Q(n) = \infty$ a.s. give $\lim_{n\to\infty}(Q(n-1)/Q(n)) = 1$ a.s. and thereupon, with $D(x) := (2Q(x)\log\log Q(x))^{1/2}$ for all sufficiently large x, $\lim_{n\to\infty}(D(n-1)/D(n)) = 1$ a.s. because $\log\log Q(x) \sim \log\log x$ as $x \to \infty$ (see Lemma 2.3). Writing $Z_n = M^{>}(n)/D(n)$, we have

$$Z_n - Z_{n-1} = \frac{M^{>}(n) - M^{>}(n-1)}{D(n)} + Z_{n-1}\Big(\frac{D(n-1)}{D(n)} - 1\Big).$$

The sequence (Z_n) is eventually bounded by Theorem 1.1. Therefore,

$$\lim_{n\to\infty}(Z_n - Z_{n-1}) = 0 \quad \text{a.s.} \tag{5.2}$$

Fix a sample point at which both limit relations of Theorem 1.1 and (5.2) hold. For any $z \in (-1,1)$, the sequence (Z_n) crosses z upwards i.o. Hence, there exists an increasing sequence (n_k), depending on the chosen sample point, such that $Z_{n_k-1} \le z < Z_{n_k}$. Therefore,

$$0 < Z_{n_k} - z \le |Z_{n_k} - Z_{n_k-1}| \ \to \ 0, \quad k \to \infty,$$

so that z is a subsequential limit. The endpoints -1 and 1 are subsequential limits by Theorem 1.1, and the same relation excludes subsequential limits outside $[-1,1]$. We conclude that (1.3) holds.

6 Preparation for the proof of (4.3)

We start with an outline of the proof of (4.3). Lemma 6.4 represents $C_r(x,y)(xy)^{-1/2}$, up to a negligible error, as a double integral involving the random Euler product F_{a_r}. Lemma 6.8 shows that, with probability tending to one, this double integral can be replaced by the integral $I_r(x,y)$ over pairs of nearby arguments at which the partial Euler products stay below the so-called strong barriers. The integrals $I_r(x,y)$ are controlled simultaneously for all $x, y \in \mathcal{X}_r$ in Lemma 6.11 with the help of high moments, using the moment bound of Lemma 6.10 for the Euler products subject to the strong barriers. The barrier estimates are obtained in Section 6.2 and in the three subsequent sections by means of a change of measure and a comparison with a Gaussian random walk. Finally, (4.3) is deduced from Lemmas 6.8 and 6.11 in Section 7.

6.1 Standard analytic estimates

We start with an estimate for a sum over primes which is secured by the prime number theorem with a remainder.

Lemma 6.1. *There exist absolute constants $c', C' > 0$ such that, for all $u > s \geq 3$ and $v \in \mathbb{R}$,*

$$\left| \sum_{s<p\leq u} p^{-1-\mathrm{i}v} - \int_{\log s}^{\log u} \frac{\mathrm{e}^{-\mathrm{i}vy}}{y}\, \mathrm{d}y \right| \leq C'(1+|v|)\mathrm{e}^{-c'(\log s)^{1/2}}. \tag{6.1}$$

Proof. Recall the notation $\pi(x) = \#\{p : \ p \leq x\}$ for $x \geq 0$ for the prime counting function. Representing the sum as a Lebesgue-Stieltjes integral with respect to π and using (3.1), we obtain

$$\sum_{s<p\leq u} p^{-1-\mathrm{i}v} = \int_s^u \frac{z^{-1-\mathrm{i}v}}{\log z}\, \mathrm{d}z + \int_{(s,u]} z^{-1-\mathrm{i}v}\, \mathrm{d}R(z) = \int_{\log s}^{\log u} \frac{\mathrm{e}^{-\mathrm{i}vy}}{y}\, \mathrm{d}y + \int_{(s,u]} z^{-1-\mathrm{i}v}\, \mathrm{d}R(z).$$

Integration by parts, in combination with (3.2), gives

$$\begin{aligned}\left| \int_{(s,u]} z^{-1-\mathrm{i}v}\, \mathrm{d}R(z) \right| &= \left| u^{-1-\mathrm{i}v}R(u) - s^{-1-\mathrm{i}v}R(s) + (1+\mathrm{i}v)\int_s^u R(z) z^{-2-\mathrm{i}v}\, \mathrm{d}z \right| \\ &\leq 2c_1\mathrm{e}^{-c(\log s)^{1/2}} + c_1(1+|v|)\int_s^\infty z^{-1}\mathrm{e}^{-c(\log z)^{1/2}}\mathrm{d}z\end{aligned}$$

for all s, u satisfying $u > s \geq 3$ and all $v \in \mathbb{R}$. Noting that

$$\int_s^\infty z^{-1}\mathrm{e}^{-c(\log z)^{1/2}}\mathrm{d}z = 2c^{-2}(1+c(\log s)^{1/2})\mathrm{e}^{-c(\log s)^{1/2}} \leq C_1'\mathrm{e}^{-c'(\log s)^{1/2}}$$

for any fixed $c' \in (0, c)$ and all $s \geq 3$, with an appropriate constant $C_1' > 0$ independent of s, we arrive at (6.1). □

For $v \neq 0$, integration by parts yields

$$\left| \int_{\log s}^{\log u} \frac{\mathrm{e}^{-\mathrm{i}vy}}{y}\, \mathrm{d}y \right| \leq \frac{3}{|v| \log s}. \tag{6.2}$$

The asymptotic relations below are standard consequences of Mertens' estimates.

Lemma 6.2. *As $q \to \infty$,*

$$\prod_{p\leq \mathrm{e}^q} (1+p^{-1}) = 6\pi^{-2}\mathrm{e}^\gamma q + O(1), \quad \sum_{p\leq \mathrm{e}^q} \frac{\log p}{p} = q + O(1), \quad \sum_{p\leq \mathrm{e}^q} \frac{(\log p)^2}{p} = O(q^2), \tag{6.3}$$

where γ is the Euler-Mascheroni constant.

Proof. By Theorem 2.7(e) in [21], $\prod_{p\leq z}(1-p^{-1})^{-1} = \mathrm{e}^\gamma \log z + O(1)$. It is a standard fact that

$$\prod_{p\leq z}(1-p^{-2}) = 6\pi^{-2} + O\left(\sum_{p>z} p^{-2}\right) = 6\pi^{-2} + O(z^{-1}).$$

The first asymptotic relation in (6.3) follows from these and a representation

$$\prod_{p\leq z}(1+p^{-1}) = \prod_{p\leq z}(1-p^{-2}) \prod_{p\leq z}(1-p^{-1})^{-1}.$$

The second asymptotic relation is just Theorem 2.7 (b) in [21]. The third follows from the second with the help of $(\log p)^2 \leq q \log p$ for $p \leq \mathrm{e}^q$. □

For each $z > \mathrm{e}^{\mathrm{e}}$ and $j \in \mathbb{N}_0$, put $J_*(z) := \lfloor \log\log z - 1 \rfloor$ and $q_{j,z} = \mathrm{e}^{-j}\log z$. Define now the finite products

$$F_{j,z}(t) = \prod_{p \le \exp(q_{j,z})} (1 + f(p)p^{-1/2-\mathrm{i}t}), \quad j \in \mathbb{Z},\ 0 \le j \le J_*(z),\ t \in \mathbb{R}. \tag{6.4}$$

Each product is nonzero because $p^{-1/2} < 1$. Recalling that $P(n)$ denotes the largest prime factor of an integer $n \ge 2$ and that $P(1) = 1$, we expand the product to obtain

$$F_{j,z}(t) = \sum_{\substack{n \ge 1:\ n \text{ squarefree} \\ P(n) \le \mathrm{e}^{q_{j,z}}}} \frac{f(n)}{n^{1/2+\mathrm{i}t}}. \tag{6.5}$$

Using orthogonality of $f(2),\ f(3), \ldots$ for the first equality and the first part of (6.3) for the asymptotic relation we obtain

$$\mathbb{E}\big[|F_{j,z}(t)|^2\big] = \sum_{\substack{n \ge 1:\ n \text{ squarefree} \\ P(n) \le \mathrm{e}^{q_{j,z}}}} n^{-1} = \prod_{p \le \mathrm{e}^{q_{j,z}}} (1 + p^{-1}) = 6\pi^{-2}\mathrm{e}^{\gamma} q_{j,z} + O(1) \tag{6.6}$$

with an absolute $O(1)$-constant.

We shall write for simplicity F_z for $F_{0,z}$ and state explicitly that

$$F_z(t) = \prod_{p \le z} \left(1 + f(p)p^{-1/2-\mathrm{i}t}\right).$$

Lemma 6.3. *There exists an absolute constant $K > 0$ such that, for all $z > \mathrm{e}^{\mathrm{e}}$ and $t, w \in \mathbb{R}$,*

$$\mathbb{E}\big[|F_z(t)F_z(w)|\big] \le K(\log z)^{1/2}\Big(\min\left(\log z, |t-w|^{-1}\right)^{1/2} + \min\left(\log z, |t+w|^{-1}\right)^{1/2} + \log(2 + |t| + |w|)\Big), \tag{6.7}$$

where $1/0 := \infty$.

Proof. Put $R := 2 + |t| + |w|$. For $z \ge 200$, apply the second part of Euler Product Result 2 on pp. 2107–2108 in [16] with $x = 200$, $y = z$, $\sigma = 0$, $t_1 = t$ and $t_2 = w$. Since

$$\prod_{p \le 200} |1 + f(p)p^{-1/2-\mathrm{i}t}||1 + f(p)p^{-1/2-\mathrm{i}w}| \le \prod_{p \le 200} (1 + p^{-1/2})^2,$$

that result implies the existence of an absolute constant $K_1 > 0$ such that

$$\mathbb{E}\big[|F_z(t)F_z(w)|\big] \le K_1(\log z)^{1/2}\left(1 + \min\left(\log z,\ |t-w|^{-1} + |t+w|^{-1} + (\log R)^2\right)\right)^{1/2}.$$

Using the inequality $\min(L, a+b+c) \le \min(L,a) + \min(L,b) + c$ which holds for $L > 0$ and $a, b, c \ge 0$, together with subadditivity of $x \mapsto x^{1/2}$ on $[0,\infty)$ we infer

$$\begin{aligned}\Big(1 + \min\Big(\log z,\ |t-w|^{-1} + |t+w|^{-1} + (\log R)^2\Big)\Big)^{1/2} &\le 1 + \min(\log z,\ |t-w|^{-1})^{1/2} + \min(\log z, |t+w|^{-1})^{1/2} + \log R \\ &\le (1 + 1/\log 2)\left(\min(\log z, |t-w|^{-1})^{1/2} + \min(\log z, |t+w|^{-1})^{1/2} + \log R\right).\end{aligned}$$

We have used $R \ge 2$ for the last inequality. These inequalities remain valid when a or b is infinite, with the convention in the statement. This proves (6.7) for $z \ge 200$ with $K = K_1(1 + 1/\log 2)$.

For $\mathrm{e}^{\mathrm{e}} < z < 200$, $\mathbb{E}[|F_z(t)F_z(w)|] \le B := \prod_{p\le 200}(1+p^{-1/2})^2$, whereas $(\log z)^{1/2}\log R \ge \mathrm{e}^{1/2}\log 2$. Thus choosing

$$K := \max\left(K_1\left(1+\frac{1}{\log 2}\right),\ \frac{B}{\mathrm{e}^{1/2}\log 2}\right)$$

establishes (6.7) throughout the range $z > \mathrm{e}^{\mathrm{e}}$ and $t, w \in \mathbb{R}$. □

For $r \in \mathbb{N}$, put $H_r := T_r^8$, $\omega_r := (\log\log a_r)^4/T_r$ and $K_r(v) := \sum_{a_r<p\le Y_r} p^{-1-\mathrm{i}v}$ for $v \in \mathbb{R}$. Recall that $Y_r = a_r^2$ and $\log a_r = (T_r + T_r^{1-\lambda})/2$. We claim that

$$|K_r(v)| \le C\min\left(1, \frac{1}{T_r|v|}\right), \quad |v| \le 2H_r \tag{6.8}$$

and

$$\int_{-\omega_r}^{\omega_r} |K_r(v)|\,\mathrm{d}v \le 10C\frac{\log\log T_r}{T_r} \tag{6.9}$$

for an absolute constant $C > 0$ and all sufficiently large r, with the minimum interpreted as 1 at $v = 0$. Indeed, the integral appearing in (6.1), with $s = a_r$ and $u = Y_r$, is bounded in absolute value by $\log\log Y_r - \log\log a_r = \log 2$ and, for $v \ne 0$, by $3/(|v|\log a_r)$, see (6.2). The right-hand side of (6.1), uniformly in $|v| \le 2H_r$, is

$$O\left(T_r^8\exp(-c'(\log a_r)^{1/2})\right) = o(T_r^{-9}), \quad r \to \infty.$$

Therefore, it can be absorbed into the bound in (6.8). Integrating this bound separately over the regions $|v| \le T_r^{-1}$ and $T_r^{-1} < |v| \le \omega_r$ proves (6.9). For instance,

$$\int_{T_r^{-1}<|v|\le\omega_r} |K_r(v)|\,\mathrm{d}v \le \frac{2C}{T_r}\int_{T_r^{-1}}^{\omega_r}\frac{\mathrm{d}v}{v} = \frac{8C}{T_r}\log\log((T_r+T_r^{1-\lambda})/2) \le 8C\frac{\log\log T_r}{T_r}.$$

Lemma 6.4 represents the conditional covariances by double integrals involving random Euler products, with an error uniform over $\mathcal{X}_r$. Related representations are derived in Section 3.2 of [16] and Lemma 5.2 of [10]. Since those results do not directly cover our setting, we provide a proof for $\mathcal{X}_r$, using orthogonality to control the quadratic truncation error directly on the line $\Re s = 1/2$.

Recall the notation

$$C_r(x,y) = \sum_{a_r<p\le\min(x,y)} M(x/p)M(y/p), \quad x, y \in \mathcal{X}_r.$$

Lemma 6.4. *There exist $\mathcal{H}_r$-measurable events $\mathcal{P}_r$ with $\lim_{r\to\infty}\mathbb{P}(\mathcal{P}_r) = 1$ such that, on $\mathcal{P}_r$, simultaneously for all $x, y \in \mathcal{X}_r$,*

$$\frac{C_r(x,y)}{(xy)^{1/2}} = \frac{1}{4\pi^2}\int_{-H_r}^{H_r}\int_{-H_r}^{H_r}\frac{F_{a_r}(v)\overline{F_{a_r}(t)}x^{\mathrm{i}v}y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)}K_r(v-t)\,\mathrm{d}v\,\mathrm{d}t + O(T_r^{-1}), \tag{6.10}$$

where $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$. The implicit constant is deterministic and independent of x, y and the realization in $\mathcal{P}_r$.

Proof. For $h \ge 1$ and $z > 0$, put

$$P_h(z) := \frac{1}{2\pi}\int_{-h}^{h}\frac{z^{1/2+\mathrm{i}t}}{1/2+\mathrm{i}t}\,\mathrm{d}t \quad \text{and} \quad \Delta_h(z) := \mathbb{1}_{[1,\infty)}(z) - P_h(z).$$

We claim that there exists an absolute constant $C > 0$ such that

$$|\Delta_h(z)| \le C z^{1/2} \min\left(1, \frac{1}{h|\log z|}\right), \quad h \ge 1, \quad z > 0, \tag{6.11}$$

where the minimum is interpreted as 1 at $z = 1$.

Proof of (6.11). Assume first that $h|\log z| > 1$, so that $z \ne 1$. An application of Theorem 5.1 on p. 138 in [21], with $a_1 = 1$, $a_n = 0$ for $n \ge 2$, $\sigma_0 = 1/2$ and $x = z$ yields

$$\mathbb{1}_{[1,\infty)}(z) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \frac{z^{1/2+\mathrm{i}t}}{1/2+\mathrm{i}t}\, \mathrm{d}t.$$

Here, the half-weight convention present in the theorem has no effect because $z \ne 1$ and all coefficients other than a_1 vanish. The preceding equality entails

$$\Delta_h(z) = \frac{z^{1/2}}{2\pi} \lim_{R\to\infty} \int_{h\le|t|\le R} \frac{\mathrm{e}^{\mathrm{i}t\log z}}{1/2+\mathrm{i}t}\, \mathrm{d}t.$$

For $R > h$, integration by parts gives

$$\left| \int_h^R \frac{\mathrm{e}^{\mathrm{i}t\log z}}{1/2+\mathrm{i}t}\, \mathrm{d}t \right| \le \frac{1}{|\log z|} \left(\frac{1}{R} + \frac{1}{h} + \int_h^R \frac{\mathrm{d}t}{t^2} \right) = \frac{2}{h|\log z|}.$$

The same bound holds for the integral over $[-R, -h]$. Hence,

$$|\Delta_h(z)| \le \frac{2z^{1/2}}{\pi h|\log z|},$$

which proves (6.11) in this case.

Now assume that $h|\log z| \le 1$. After simple manipulations, we obtain

$$P_h(z) = \frac{z^{1/2}}{\pi} \int_0^h \frac{\frac{1}{2}\cos(t\log z) + t\sin(t\log z)}{1/4+t^2}\, \mathrm{d}t.$$

Using $|\sin(t\log z)| \le t|\log z|$ for $t \ge 0$, we infer

$$\begin{aligned} |P_h(z)| &\le \frac{z^{1/2}}{\pi} \left(\frac{1}{2} \int_0^h \frac{\mathrm{d}t}{1/4+t^2} + |\log z| \int_0^h \frac{t^2}{1/4+t^2}\, \mathrm{d}t \right) \\ &\le \frac{z^{1/2}}{\pi} \left(\frac{\pi}{2} + h|\log z| \right) \le \left(\frac{1}{2} + \frac{1}{\pi} \right) z^{1/2}. \end{aligned}$$

Since $\mathbb{1}_{[1,\infty)}(z) \le z^{1/2}$ for all $z > 0$, this entails $|\Delta_h(z)| \le (3/2 + 1/\pi) z^{1/2}$ and proves (6.11) also in the second case, including $z = 1$.

For $x \in \mathcal{X}_r$ and $a_r < p \le Y_r$, put

$$A_{r,p}(x) := x^{-1/2} M(x/p) \quad \text{and} \quad B_{r,p}(x) := \frac{1}{2\pi p^{1/2}} \int_{-H_r}^{H_r} \frac{F_{a_r}(t)(x/p)^{\mathrm{i}t}}{1/2+\mathrm{i}t}\, \mathrm{d}t$$

and denote the corresponding vectors by $\boldsymbol{A}_r(x) := (A_{r,p}(x))_{a_r<p\le Y_r}$ and $\boldsymbol{B}_r(x) := (B_{r,p}(x))_{a_r<p\le Y_r}$. Using (6.5) with $j = 0$ and $z = a_r$, we conclude that

$$B_{r,p}(x) = x^{-1/2} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} f(n) P_{H_r}\left(x/(pn)\right).$$

Since $x \le Y_r = a_r^2$ and $p > a_r$, we have $x/p < a_r$. Hence,

$$A_{r,p}(x) = x^{-1/2} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} f(n) \mathbb{1}_{[1,\infty)}(x/(pn))$$

and thereupon

$$A_{r,p}(x) - B_{r,p}(x) = x^{-1/2} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} f(n) \Delta_{H_r}\left(x/(pn)\right).$$

All sums in these representations are finite.

By orthogonality of $f(2)$, $f(3), \dots$ and (6.11),

$$\begin{aligned}\mathbb{E}\left[|A_{r,p}(x) - B_{r,p}(x)|^2\right] &= \frac{1}{x} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} \left|\Delta_{H_r}\left(\frac{x}{pn}\right)\right|^2 \\ &\le C^2 \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} \frac{1}{pn} \min\left(1, \frac{1}{H_r^2 |\log(x/(pn))|^2}\right).\end{aligned}$$

Summing over p and writing $\|\cdot\|_2$ for the Euclidean norm, we obtain

$$\mathbb{E}\left[\|\boldsymbol{A}_r(x) - \boldsymbol{B}_r(x)\|_2^2\right] \le C^2 \mathcal{R}_r(x), \tag{6.12}$$

where

$$\mathcal{R}_r(x) := \sum_{a_r < p \le Y_r} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r}} \frac{1}{pn} \min\left(1, \frac{1}{H_r^2 |\log(x/(pn))|^2}\right).$$

Here and hereafter, minimum involving an expression with a vanishing denominator is interpreted as 1.

We next prove that, uniformly in $x \in \mathcal{X}_r$,

$$\mathcal{R}_r(x) = O(H_r^{-1}), \quad r \to \infty. \tag{6.13}$$

To this end, we treat separately the ranges $x/\mathrm{e} \le pn \le \mathrm{e}x$ and $pn < x/\mathrm{e}$ or $pn > \mathrm{e}x$. In the first range, p is the unique prime factor of the integer pn exceeding a_r. Thus,

$$\begin{aligned}&\sum_{a_r < p \le Y_r} \sum_{\substack{n\ge 1:\ n \text{ squarefree} \\ P(n)\le a_r,\ x/\mathrm{e}\le pn\le \mathrm{e}x}} \frac{1}{pn} \min\left(1, \frac{1}{H_r^2 |\log(x/(pn))|^2}\right) \\ &\quad\le \sum_{\substack{m\in\mathbb{N} \\ x/\mathrm{e}\le m\le \mathrm{e}x}} \frac{1}{m} \min\left(1, \frac{1}{H_r^2 |\log(x/m)|^2}\right) \le \frac{\mathrm{e}}{x} \sum_{\substack{m\in\mathbb{N} \\ x/\mathrm{e}\le m\le \mathrm{e}x}} \min\left(1, \frac{\mathrm{e}^2 x^2}{H_r^2 |x-m|^2}\right) \\ &\quad\le \frac{2\mathrm{e}}{x}\left(1 + \sum_{j\ge 1} \min\left(1, \frac{\mathrm{e}^2 x^2}{H_r^2 j^2}\right)\right) \\ &\quad\le \frac{2\mathrm{e}^3}{x}\left(1 + \sum_{j\ge 1} \min\left(1, \frac{x^2}{H_r^2 j^2}\right)\right).\end{aligned}$$

Here, the second inequality follows from $m^{-1} \le \mathrm{e}/x$ and $|\log(x/m)| \ge |x-m|/(\mathrm{e}x)$, which is justified by the mean value theorem for differentiable functions. For the third inequality, we have used

$$\#\{m \in \mathbb{N} : |m - x| < 1\} \le 2, \qquad \#\{m \in \mathbb{N} : j \le |m-x| < j+1\} \le 2, \quad j \in \mathbb{N}.$$

The inequality $x \geq \mathrm{e}^{T_r}$, secured by $x \in \mathcal{X}_r$, implies that $\lim_{r\to\infty}(x/H_r) = \infty$ uniformly in $x \in \mathcal{X}_r$. Splitting the last sum at $j = \lfloor x/H_r \rfloor$ gives

$$\sum_{j\geq 1} \min\left(1, \frac{x^2}{H_r^2 j^2}\right) \leq \frac{x}{H_r} + \frac{x^2}{H_r^2} \sum_{j > x/H_r} j^{-2} = O(x/H_r).$$

Thus the contribution from the first range is $O(H_r^{-1})$, uniformly in $x \in \mathcal{X}_r$.

In the second range, $|\log(x/(pn))| > 1$. This ensures that the contribution of the second range to $\mathcal{R}_r(x)$ does not exceed

$$H_r^{-2} \sum_{a_r < p \leq Y_r} p^{-1} \sum_{\substack{n\geq 1:\ n \text{ squarefree} \\ P(n)\leq a_r}} n^{-1}$$
$$= H_r^{-2} \sum_{a_r < p \leq a_r^2} p^{-1} \prod_{q \leq a_r} (1 + q^{-1}) = O(T_r H_r^{-2}) = o(H_r^{-1}), \quad r \to \infty.$$

Here, we have used (4.8), (6.3) and $\log a_r = O(T_r)$. This proves (6.13). Now (6.13), in combination with (6.12), secures

$$\mathbb{E}\left[\|\boldsymbol{A}_r(x) - \boldsymbol{B}_r(x)\|_2^2\right] = O(H_r^{-1}) \tag{6.14}$$

uniformly in $x \in \mathcal{X}_r$.

Since $|\mathcal{X}_r| = \lfloor T_r^{1-\lambda} \rfloor + 1 \leq 2T_r$, Markov's inequality and the union bound entail

$$\mathbb{P}\left\{\max_{x\in\mathcal{X}_r} \|\boldsymbol{A}_r(x) - \boldsymbol{B}_r(x)\|_2 > T_r^{-2}\right\}$$
$$\leq T_r^4 \sum_{x\in\mathcal{X}_r} \mathbb{E}\left[\|\boldsymbol{A}_r(x) - \boldsymbol{B}_r(x)\|_2^2\right] = O(T_r^5/H_r) = O(T_r^{-3}). \tag{6.15}$$

Furthermore, by (2.6) and (4.8),

$$\mathbb{E}\left[\|\boldsymbol{A}_r(x)\|_2^2\right] = x^{-1} \sum_{a_r < p \leq Y_r} \mathbb{E}[(M(x/p))^2] \leq \sum_{a_r < p \leq Y_r} p^{-1} = O(1)$$

uniformly in $x \in \mathcal{X}_r$. Another application of Markov's inequality and the union bound gives

$$\mathbb{P}\left\{\max_{x\in\mathcal{X}_r} \|\boldsymbol{A}_r(x)\|_2 > T_r\right\} = O(T_r^{-1}). \tag{6.16}$$

For $r \in \mathbb{N}$, define the events

$$\mathcal{P}_r := \left\{\max_{x\in\mathcal{X}_r} \|\boldsymbol{A}_r(x) - \boldsymbol{B}_r(x)\|_2 \leq T_r^{-2}, \quad \max_{x\in\mathcal{X}_r} \|\boldsymbol{A}_r(x)\|_2 \leq T_r\right\}.$$

All coordinates of the vectors involved are $\mathcal{H}_r$-measurable, hence so is $\mathcal{P}_r$. It follows from (6.15) and (6.16) that $\mathbb{P}(\mathcal{P}_r^c) = O(T_r^{-1})$ as $r \to \infty$.

For complex vectors, we shall use the notation $\langle a, b\rangle = \sum_p a_p \overline{b_p}$. On $\mathcal{P}_r$, the Cauchy-Schwarz inequality ensures that, simultaneously for all $x, y \in \mathcal{X}_r$,

$$\begin{aligned}
&|\langle \boldsymbol{B}_r(x), \boldsymbol{B}_r(y)\rangle - \langle \boldsymbol{A}_r(x), \boldsymbol{A}_r(y)\rangle| \\
&\leq |\langle \boldsymbol{B}_r(x) - \boldsymbol{A}_r(x), \boldsymbol{A}_r(y)\rangle| + |\langle \boldsymbol{A}_r(x), \boldsymbol{B}_r(y) - \boldsymbol{A}_r(y)\rangle| + |\langle \boldsymbol{B}_r(x) - \boldsymbol{A}_r(x), \boldsymbol{B}_r(y) - \boldsymbol{A}_r(y)\rangle| \\
&\leq \|\boldsymbol{B}_r(x) - \boldsymbol{A}_r(x)\|_2 \|\boldsymbol{A}_r(y)\|_2 + \|\boldsymbol{A}_r(x)\|_2 \|\boldsymbol{B}_r(y) - \boldsymbol{A}_r(y)\|_2 + \|\boldsymbol{B}_r(x) - \boldsymbol{A}_r(x)\|_2 \|\boldsymbol{B}_r(y) - \boldsymbol{A}_r(y)\|_2 \\
&\leq 2T_r^{-1} + T_r^{-4}.
\end{aligned}$$

Since $M(u) = 0$ for $u < 1$,

$$\langle \boldsymbol{A}_r(x), \boldsymbol{A}_r(y) \rangle = \frac{C_r(x,y)}{(xy)^{1/2}}.$$

On the other hand, substituting the definitions of the coordinates of $\boldsymbol{B}_r(x)$ and $\boldsymbol{B}_r(y)$, interchanging the finite prime sum with the integrals and using $\overline{\frac{y^{\mathrm{i}t}}{1/2+\mathrm{i}t}} = \frac{y^{-\mathrm{i}t}}{1/2-\mathrm{i}t}$ we finally obtain

$$\langle \boldsymbol{B}_r(x), \boldsymbol{B}_r(y) \rangle = \frac{1}{4\pi^2} \int_{-H_r}^{H_r} \int_{-H_r}^{H_r} \frac{F_{a_r}(v)\overline{F_{a_r}(t)} x^{\mathrm{i}v} y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)} \left(\sum_{a_r < p \le Y_r} p^{-1-\mathrm{i}(v-t)} \right) \mathrm{d}v \, \mathrm{d}t.$$

The sum in parentheses is $K_r(v-t)$. Combining the last three displays proves (6.10). □

6.2 Barriers for the partial Euler products

Recall the notation $J_*(a_r) = \lfloor \log\log a_r - 1 \rfloor$, $q_{j,a_r} = \mathrm{e}^{-j} \log a_r$ for $j \in \mathbb{N}_0$ and

$$F_{j,a_r}(t) = \prod_{p \le \exp(q_{j,a_r})} \left(1 + f(p) p^{-1/2-\mathrm{i}t}\right), \quad 0 \le j \le J_*(a_r), \ t \in \mathbb{R}.$$

In particular, $F_{0,a_r}(t) = F_{a_r}(t)$.

In this section we impose upper bounds, also called *barriers*, on the moduli of the partial Euler products F_{j,a_r} associated with the Euler product F_{a_r} appearing in (6.10). We shall show that the part of the integral in (6.10) where the modulus of at least one partial Euler product exceeds its barrier makes a negligible contribution to $C_r(x,y)(xy)^{-1/2}$. Related barrier estimates are given in Multiplicative Chaos Results 1 and 2 in Section 2.4 of [16] and used in Section 3.3 of that paper. We provide proofs for the particular barriers needed here.

Fix $B \ge 1$ and $b \in (0,1)$. For each $t \in \mathbb{R}$, define the $\mathcal{H}_r$-measurable events

$$\mathcal{D}_r(t) := \bigcap_{0 \le j \le J_*(a_r)} \left\{ |F_{j,a_r}(t)| \le q_{j,a_r} (\log\log a_r)^2 \right\} \tag{6.17}$$

and

$$\mathcal{A}_r(t) := \mathcal{D}_r(t) \cap \bigcap_{0 \le j \le \lfloor (1-b)\log\log a_r \rfloor} \left\{ |F_{j,a_r}(t)| \le q_{j,a_r} (\log\log a_r)^{-B} \right\}. \tag{6.18}$$

The second event is defined for all sufficiently large r, satisfying $\lfloor (1-b)\log\log a_r \rfloor \le J_*(a_r)$. We refer to the quantities $q_{j,a_r}(\log\log a_r)^2$ and $q_{j,a_r}(\log\log a_r)^{-B}$ as the *weak* and *strong barriers*, respectively. The constants B and b will be chosen later.

Lemma 6.5, proved using a derivative estimate, shows that, with probability tending to one, the modulus of each partial Euler product remains below its weak barrier for all arguments belonging to the prescribed interval. Lemma 6.7 uses a change of measure and a comparison of barrier probabilities with those of the Gaussian random walk, together with Lemma 6.6, to bound the expectation of $|F_{a_r}(t)|^2$ over realizations satisfying the weak barriers but violating at least one strong barrier.

Lemma 6.5. *There exists an absolute constant $C > 0$ such that, for all sufficiently large r,*

$$\mathbb{P}\{|F_{j,a_r}(t)| \le q_{j,a_r}(\log\log a_r)^2 \text{ for all } 0 \le j \le J_*(a_r) \text{ and } |t| \le (\log\log a_r)^2 + 2\} \ge 1 - C/\log\log a_r.$$

Proof. Fix $0 \le j \le J_*(a_r)$ and put $q := q_{j,a_r}$. Differentiating in t the finite expansion (6.5), we obtain

$$F'_{j,a_r}(t) = -\mathrm{i} \sum_{\substack{n \ge 1:\ n \text{ squarefree} \\ P(n) \le \mathrm{e}^q}} \frac{f(n)\log n}{n^{1/2+\mathrm{i}t}}.$$

By orthogonality of $f(2)$, $f(3)$, …,

$$\mathbb{E}[|F'_{j,a_r}(t)|^2] = \sum_{\substack{n\ge 1:\ n \text{ squarefree}\\ P(n)\le \mathrm{e}^q}} \frac{(\log n)^2}{n}.$$

Since n is squarefree,

$$(\log n)^2 = \sum_{p\mid n} (\log p)^2 + 2 \sum_{\substack{p<q'\\ p,q'\mid n}} \log p \log q'.$$

Here and hereafter, p, q' and ℓ denote primes. Hence,

$$\sum_{\substack{n\ge 1:\ n \text{ squarefree}\\ P(n)\le \mathrm{e}^q}} \frac{(\log n)^2}{n} = \sum_{p\le \mathrm{e}^q} (\log p)^2 \sum_{\substack{n \text{ squarefree}\\ P(n)\le \mathrm{e}^q,\ p\mid n}} \frac{1}{n} + 2 \sum_{p<q'\le \mathrm{e}^q} \log p \log q' \sum_{\substack{n \text{ squarefree}\\ P(n)\le \mathrm{e}^q,\ p,q'\mid n}} \frac{1}{n}.$$

For a fixed prime $p \le \mathrm{e}^q$,

$$\sum_{\substack{n \text{ squarefree}\\ P(n)\le \mathrm{e}^q,\ p\mid n}} \frac{1}{n} = \frac{1}{p} \prod_{\substack{\ell\le \mathrm{e}^q\\ \ell\ne p}} (1+\ell^{-1}) = \frac{1}{p+1} \prod_{\ell\le \mathrm{e}^q} (1+\ell^{-1}).$$

Similarly, requiring both $p \mid n$ and $q' \mid n$ gives the factor $1/((p+1)(q'+1))$. Summarizing,

$$\begin{aligned}\mathbb{E}[|F'_{j,a_r}(t)|^2] &= \prod_{p\le \mathrm{e}^q} (1+p^{-1}) \Big(\sum_{\ell\le \mathrm{e}^q} \frac{(\log \ell)^2}{\ell+1} + 2 \sum_{\ell<q'\le \mathrm{e}^q} \frac{\log \ell \log q'}{(\ell+1)(q'+1)} \Big) \\ &\le \prod_{p\le \mathrm{e}^q} (1+p^{-1}) \Big(\sum_{\ell\le \mathrm{e}^q} \frac{(\log \ell)^2}{\ell} + \Big(\sum_{\ell\le \mathrm{e}^q} \frac{\log \ell}{\ell} \Big)^2 \Big).\end{aligned}$$

Invoking (6.3), we conclude that

$$\mathbb{E}[|F'_{j,a_r}(t)|^2] \le C_1 q^3, \tag{6.19}$$

where $C_1 > 0$ is a constant independent of r, j and t.

Let I be a closed interval of length one. For each realization, there exists $s \in I$ such that

$$|F_{j,a_r}(s)|^2 \le \int_I |F_{j,a_r}(u)|^2 \, \mathrm{d}u.$$

According to the fundamental theorem of calculus, for each $t \in I$,

$$|F_{j,a_r}(t)|^2 = |F_{j,a_r}(s)|^2 + \int_s^t \frac{\mathrm{d}}{\mathrm{d}u} |F_{j,a_r}(u)|^2 \, \mathrm{d}u.$$

This, together with

$$\Big| \frac{\mathrm{d}}{\mathrm{d}u} |F_{j,a_r}(u)|^2 \Big| = 2 \Big| \mathrm{Re}\big(F'_{j,a_r}(u) \overline{F_{j,a_r}(u)}\big) \Big| \le 2 |F_{j,a_r}(u)| |F'_{j,a_r}(u)|,$$

entails

$$\sup_{t\in I} |F_{j,a_r}(t)|^2 \le \int_I |F_{j,a_r}(u)|^2 \, \mathrm{d}u + 2 \int_I |F_{j,a_r}(u)| |F'_{j,a_r}(u)| \, \mathrm{d}u.$$

Taking expectations and using the Cauchy–Schwarz inequality, (6.6) and (6.19), we obtain

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in I} |F_{j,a_r}(t)|^2\big] &\le \int_I \mathbb{E}[|F_{j,a_r}(u)|^2] \, \mathrm{d}u + 2 \int_I \big(\mathbb{E}[|F_{j,a_r}(u)|^2]\big)^{1/2} \big(\mathbb{E}[|F'_{j,a_r}(u)|^2]\big)^{1/2} \, \mathrm{d}u \\ &\le C_2(q+q^2) \le 2C_2 q^2\end{aligned}$$

having utilized $q \geq \mathrm{e}$, where $C_2 > 0$ is a constant independent of r, j and t.

The interval $[-(\log\log a_r)^2 - 2, (\log\log a_r)^2 + 2]$ can be covered by $\lfloor 2(\log\log a_r)^2 \rfloor + 5$ closed intervals of length one. For each such interval I, Markov's inequality yields

$$\mathbb{P}\{\sup_{t\in I} |F_{j,a_r}(t)| > q_{j,a_r}(\log\log a_r)^2\} \leq 2C_2(\log\log a_r)^{-4}.$$

Since there are $J_*(a_r) + 1$ possible values of j and $J_*(a_r) + 1 \leq \log\log a_r$, an application of the union bound yields, for all sufficiently large r, with $C = 16C_2$,

$$\mathbb{P}\{\exists\ |t| \leq (\log\log a_r)^2 + 2,\ \exists\ 0 \leq j \leq J_*(a_r) :\ |F_{j,a_r}(t)| > q_{j,a_r}(\log\log a_r)^2\} \leq C(\log\log a_r)^{-1},$$

thereby completing the proof. □

6.3 Two estimates for a Gaussian random walk

We shall use the following estimates to control the probability that a standard random walk with Gaussian increments stays below a barrier and, additionally, approaches that barrier at a sufficiently late time. Estimates of this type are classical. In fact, (6.20) holds in the stronger form $\mathbb{P}\{\max_{0\leq k\leq n} S_k \leq h\} \leq C(h+1)n^{-1/2}$, see, for instance, Lemma 3.3 in [23] and Probability Result 1 in [15], and related ballot-type estimates with a prescribed endpoint can be found, for instance, in Probability Result 2 of [15]. Since Lemma 6.6 in the present form suffices for our purposes and we have not found (6.21) in the literature in the form needed, we give a short self-contained proof based on the reflection principle. The terms involving d_L and L^{-19} are harmless in our applications.

Lemma 6.6. *Let $G_1, G_2, \ldots$ be independent random variables with the normal distribution of mean zero and variance $1/2$ and put $S_0 := 0$ and $S_k := G_1 + \cdots + G_k$ for $k \in \mathbb{N}$. For integer $L \geq 3$, put $d_L := (20\log L)^{1/2}$. There exist absolute positive constants C and C' such that, for integers $1 \leq n \leq L$ and $h \geq 0$,*

$$\mathbb{P}\{\max_{0\leq k\leq n} S_k \leq h\} \leq C\Big(\frac{h + d_L}{n^{1/2}} + L^{-19}\Big). \tag{6.20}$$

Furthermore, for integers $1 \leq m < n \leq L$ and $h, a \geq 0$,

$$\mathbb{P}\{\max_{0\leq k\leq n} S_k \leq h,\ \max_{m\leq k\leq n} S_k \geq -a\} \leq C'\Big(\frac{(h + d_L)(h + a + d_L)}{(m(n-m))^{1/2}} + L^{-19}\Big). \tag{6.21}$$

Proof. Since $(S_k)_{k\in\mathbb{N}_0}$ has the same distribution as $(B(k))_{k\in\mathbb{N}_0}$, where $B = (B(s))_{s\geq 0}$ is a Brownian motion with $\mathbb{E}[B(s)^2] = s/2$, we can and do work with $(B(k))_{k\in\mathbb{N}_0}$. By the reflection principle (see, for instance, Example 8.4.1 on p. 372 in [8]),

$$\mathbb{P}\{\sup_{0\leq s\leq u} B(s) \leq z\} = 2\Phi\big(z(2/u)^{1/2}\big) - 1, \quad u > 0,\ z \geq 0, \tag{6.22}$$

where Φ denotes the standard normal distribution function.

For each $k \in \mathbb{N}_0$, another application of the reflection principle gives

$$\begin{aligned}\mathbb{P}\{\sup_{0\leq s\leq 1} |B(k+s) - B(k)| > d_L\} &= \mathbb{P}\{\sup_{0\leq s\leq 1} |B(s)| > d_L\} \leq 4\big(1 - \Phi\big((2)^{1/2} d_L\big)\big) \\ &= 4\pi^{-1/2}\int_{d_L}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y \leq 2(\pi^{1/2}d_L)^{-1}\mathrm{e}^{-d_L^2} \leq (5\pi)^{-1/2}\mathrm{e}^{-d_L^2}\end{aligned}$$

having utilized $d_L \geq 20^{1/2}$. This ensures that the event

$$\mathcal{Q}_{n,L} := \{\max_{0\leq k<n} \sup_{0\leq s\leq 1} |B(k+s) - B(k)| \leq d_L\}$$

satisfies $\mathbb{P}(\mathcal{Q}_{n,L}^c) \le (5\pi)^{-1/2} n \mathrm{e}^{-d_L^2} \le (5\pi)^{-1/2} L^{-19}$. For each $s \in [0,n]$, choosing $k \in \{0,\dots,n-1\}$ such that $s \in [k,k+1]$ we obtain

$$B(s) \le B(k) + |B(s) - B(k)| \le \max_{0\le j\le n} B(j) + \max_{0\le j<n} \sup_{0\le u\le 1} |B(j+u) - B(j)|.$$

In particular, the last inequality holds with $\sup_{0\le s\le n} B(s)$ replacing $B(s)$ on the left-hand side. Hence, on $\mathcal{Q}_{n,L}$, the inequality $\max_{0\le k\le n} B(k) \le h$ implies $\sup_{0\le s\le n} B(s) \le h + d_L$. Thus, using (6.22) and the trivial estimate

$$2\Phi(u) - 1 = \frac{2}{(2\pi)^{1/2}} \int_0^u \mathrm{e}^{-v^2/2}\, \mathrm{d}v \le (2/\pi)^{1/2}\, u, \quad u \ge 0,$$

we obtain

$$\mathbb{P}\{\max_{0\le k\le n} B(k) \le h\} \le \mathbb{P}\{\sup_{0\le s\le n} B(s) \le h + d_L\} + \mathbb{P}(\mathcal{Q}_{n,L}^c) \le \frac{2(h+d_L)}{(\pi n)^{1/2}} + (5\pi)^{-1/2} L^{-19}.$$

This proves (6.20) with $C = 2\pi^{-1/2}$.

To prove (6.21), observe that, on $\mathcal{Q}_{n,L}$, the event on the left-hand side of (6.21), with $B(k)$ replacing S_k, entails

$$\sup_{0\le s\le m} B(s) \le h + d_L,\ -a \le \sup_{m\le s\le n} B(s) \le h + d_L.$$

Conditionally on the Brownian path up to time m, the latter maximum has the same distribution as $B(m) + \sup_{0\le s\le n-m} B^*(s)$, where B^* is an independent copy of B. It follows that

$$\begin{aligned}
&\mathbb{P}\{\sup_{0\le s\le m} B(s) \le h + d_L,\ -a \le \sup_{m\le s\le n} B(s) \le h + d_L\} \\
&\qquad = \mathbb{E}\left[\begin{array}{l}\mathbb{P}\{-a - B(m) \le \sup\limits_{0\le s\le n-m} B^*(s) \le h + d_L - B(m) \mid (B(s))_{s\in[0,m]}\} \\ \qquad \times \mathbb{1}_{\{\sup_{0\le s\le m} B(s) \le h+d_L,\ B(m)\le h+d_L\}}\end{array}\right] \\
&\qquad \le \frac{2}{\pi^{1/2}} \frac{h+a+d_L}{(n-m)^{1/2}} \mathbb{P}\{\sup_{0\le s\le m} B(s) \le h + d_L\} \le \frac{4}{\pi} \frac{(h+d_L)(h+a+d_L)}{(m(n-m))^{1/2}}.
\end{aligned}$$

Here, we have used that, according to (6.22), the random variable $\sup_{0\le s\le n-m} B^*(s)$ has density

$$z \mapsto \frac{2}{(\pi(n-m))^{1/2}} \exp\left(-\frac{z^2}{n-m}\right) \mathbb{1}_{(0,\infty)}(z), \quad z \in \mathbb{R},$$

which is bounded by $2(\pi(n-m))^{-1/2}$. Consequently, conditionally on the Brownian path up to time m, the probability that this random variable belongs to the interval $[-a - B(m), h + d_L - B(m)]$, whose length is $h+a+d_L$, does not exceed $2(h+a+d_L)(\pi(n-m))^{-1/2}$. This inequality remains valid if the interval intersects, or is entirely contained in, the negative half-line, since the density vanishes there. Now (6.21), with $C' = 4/\pi$, follows from

$$\begin{aligned}
&\mathbb{P}\{\max_{0\le k\le n} B(k) \le h,\ \max_{m\le k\le n} B(k) \ge -a\} \\
&\qquad\qquad \le \mathbb{P}\{\sup_{0\le s\le m} B(s) \le h + d_L,\ -a \le \sup_{m\le s\le n} B(s) \le h + d_L\} + \mathbb{P}(\mathcal{Q}_{n,L}^c).
\end{aligned}$$

□

6.4 A change of measure and comparison of barrier probabilities

In this section we compare, under a tilted measure, the probabilities that partial sums of logarithmic Euler-product blocks satisfy prescribed barriers with the corresponding probabilities for the Gaussian random walk in Lemma 6.6.

For fixed $r \in \mathbb{N}$ and $t \in \mathbb{R}$, introduce the probability measure $\mathbb{P}_{r,t}$ by

$$\frac{\mathrm{d}\mathbb{P}_{r,t}}{\mathrm{d}\mathbb{P}} = \frac{|F_{a_r}(t)|^2}{\mathbb{E}[|F_{a_r}(t)|^2]} \tag{6.23}$$

and denote the corresponding expectation and variance by $\mathbb{E}_{r,t}$ and $\mathrm{Var}_{r,t}$, respectively. The density in (6.23) is a product of factors depending on separate prime signs. Hence, the variables $(f(p))_{p\le a_r}$ remain independent under $\mathbb{P}_{r,t}$. Furthermore, noting that

$$|1 + f(p)p^{-1/2-\mathrm{i}t}|^2 = 1 + p^{-1} + 2f(p)p^{-1/2}\cos(t\log p),$$

we infer

$$\mathbb{E}_{r,t}[f(p)] = \frac{\mathbb{E}[f(p)|1 + f(p)p^{-1/2-\mathrm{i}t}|^2]}{\mathbb{E}[|1 + f(p)p^{-1/2-\mathrm{i}t}|^2]} = \frac{2p^{-1/2}\cos(t\log p)}{1 + p^{-1}}, \quad p \le a_r.$$

For $p \in \mathcal{P}$ and $t \in \mathbb{R}$, put $\xi_p(t) := \log|1 + f(p)p^{-1/2-\mathrm{i}t}|$. Since $|f(p)p^{-1/2-\mathrm{i}t}| = p^{-1/2} < 1$, we use an expansion

$$\begin{aligned}\log|1 + f(p)p^{-1/2-\mathrm{i}t}| = \mathrm{Re}\log(1 + f(p)p^{-1/2-\mathrm{i}t}) &= \mathrm{Re}\sum_{k\ge1}\frac{(-1)^{k+1}(f(p)p^{-1/2-\mathrm{i}t})^k}{k}\\ &= f(p)p^{-1/2}\cos(t\log p) - (2p)^{-1}\cos(2t\log p) + g_{f,t}(p),\end{aligned}$$

where in the first equality log denotes the principal branch of the logarithm. Here, $g_{f,t}(p) := \mathrm{Re}\sum_{k\ge3}\frac{(-1)^{k+1}(f(p)p^{-1/2-\mathrm{i}t})^k}{k}$, and using $1/k \le 1/3$ and $p \ge 2$, we infer

$$|g_{f,t}(p)| \le \sum_{k\ge3}\frac{p^{-k/2}}{k} \le \frac{1}{3}\sum_{k\ge3}p^{-k/2} = \frac{p^{-3/2}}{3(1-p^{-1/2})} \le \frac{p^{-3/2}}{3(1-2^{-1/2})} =: Cp^{-3/2}.$$

For $p \le a_r$, using the preceding formula for $\mathbb{E}_{r,t}[f(p)]$ and the equality $2\cos^2\theta - \cos(2\theta)/2 = 1 + \cos(2\theta)/2$, we obtain

$$\mathbb{E}_{r,t}[\xi_p(t)] = \frac{1}{p} + \frac{1}{2p}\cos(2t\log p) + O(p^{-3/2}). \tag{6.24}$$

Observe that $(\log|1 + f(p)p^{-1/2-\mathrm{i}t}|)^2 = (2p)^{-1}(1 + \cos(2t\log p)) + O(p^{-3/2})$. This, together with (6.24), shows that

$$\mathrm{Var}_{r,t}[\xi_p(t)] = (2p)^{-1}(1 + \cos(2t\log p)) + O(p^{-3/2}). \tag{6.25}$$

In the three preceding formulas, all implicit constants are absolute. Also, in view of

$$|\xi_p(t)| \le -\log(1 - p^{-1/2}) \le \frac{p^{-1/2}}{1 - p^{-1/2}} \le \frac{p^{-1/2}}{1 - 2^{-1/2}},$$

we infer

$$\mathbb{E}_{r,t}\left[|\xi_p(t) - \mathbb{E}_{r,t}[\xi_p(t)]|^3\right] \le C_1 p^{-3/2}, \tag{6.26}$$

where $C_1 := 8(1 - 2^{-1/2})^{-3}$.

For all sufficiently large r, put $N_r := \lfloor \log\log a_r - (\log\log\log a_r)^2 \rfloor$ and

$$X_{r,j}(t) := \log\left|\frac{F_{j,a_r}(t)}{F_{j+1,a_r}(t)}\right| = \sum_{\exp(q_{j+1,a_r})<p\le\exp(q_{j,a_r})} \xi_p(t), \quad 0 \le j < N_r,\ t \in \mathbb{R}. \tag{6.27}$$

For each fixed $r \in \mathbb{N}$ and $0 \le j < N_r$, the variable $X_{r,j}(t)$ is defined by primes inside $(\mathrm{e}^{q_{j+1,a_r}}, \mathrm{e}^{q_{j,a_r}}]$. For different j these intervals are disjoint, which ensures that the variables $X_{r,j}(t)$ are independent under $\mathbb{P}_{r,t}$.

We claim that, uniformly in $(\log\log a_r)^{-2} \le |t| \le (\log\log a_r)^2 + 2$ and $0 \le j < N_r$, $j \in \mathbb{Z}$, for each fixed $A > 0$,

$$\mathbb{E}_{r,t}[X_{r,j}(t)] = 1 + O\big((\log\log a_r)^{-A}\big), \quad \mathrm{Var}_{r,t}[X_{r,j}(t)] = \frac{1}{2} + O\big((\log\log a_r)^{-A}\big), \tag{6.28}$$

where the implicit constants depend on A and do not depend on j, nor t, and that, for all sufficiently large r,

$$\sum_{\exp(q_{j+1,a_r})<p\le\exp(q_{j,a_r})} \mathbb{E}_{r,t}\big[|\xi_p(t) - \mathbb{E}_{r,t}[\xi_p(t)]|^3\big] \le 3C_1 \exp(-q_{N_r,a_r}/2). \tag{6.29}$$

Indeed, by (4.8),

$$\sum_{\exp(q_{j+1,a_r})<p\le\exp(q_{j,a_r})} p^{-1} = 1 + O(q_{N_r,a_r}^{-1}) = 1 + O\big((\log\log a_r)^{-A}\big).$$

The last equality is a consequence of $q_{N_r,a_r} \ge \exp((\log\log\log a_r)^2)$. Furthermore, by Lemma 6.1 and (6.2), for $t \ne 0$,

$$\begin{aligned}\left|\sum_{\exp(q_{j+1,a_r})<p\le\exp(q_{j,a_r})} \frac{\cos(2t\log p)}{p}\right| &\le \frac{3}{2|t|q_{N_r,a_r}} + C'(1+2|t|)\exp\big(-c'(q_{N_r,a_r})^{1/2}\big)\\ &= O\big((\log\log a_r)^{-A}\big).\end{aligned}$$

Finally, for all sufficiently large r,

$$\sum_{p>\exp(q_{N_r,a_r})} p^{-3/2} \le \sum_{n>\exp(q_{N_r,a_r})} n^{-3/2} \le 3\exp(-q_{N_r,a_r}/2).$$

These estimates, (6.24), (6.25) and (6.26) prove (6.28) and (6.29).

The Berry-Esseen theorem for sums of independent, not necessarily identically distributed random variables (see, for instance, Theorem 3 on p. 111 in [24]) ensures that

$$\begin{aligned}&\sup_{u\in\mathbb{R}}\Big|\mathbb{P}_{r,t}\{X_{r,j}(t) - \mathbb{E}_{r,t}[X_{r,j}(t)] \le u\} - \Phi\big((\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}\,u\big)\Big|\\ &= O\Big(\frac{\sum_{\exp(q_{j+1,a_r})<p\le\exp(q_{j,a_r})}\mathbb{E}_{r,t}\big[|\xi_p(t) - \mathbb{E}_{r,t}[\xi_p(t)]|^3\big]}{(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{3/2}}\Big) = O\big((\log\log a_r)^{-A}\big), \quad r\to\infty\end{aligned} \tag{6.30}$$

for each fixed $A > 0$, uniformly in $(\log\log a_r)^{-2} \le |t| \le (\log\log a_r)^2 + 2$ and $0 \le j < N_r$, $j \in \mathbb{Z}$. Here, the last equality follows from the second part of (6.28) and (6.29) and, as before, Φ denotes the standard normal distribution function. Now we prove that, as $r \to \infty$,

$$\sup_{u\in\mathbb{R}}\Big|\Phi\big((\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}(u + 1 - \mathbb{E}_{r,t}[X_{r,j}(t)])\big) - \Phi(2^{1/2}\,u)\Big| = O\big((\log\log a_r)^{-A}\big). \tag{6.31}$$

Indeed, for all sufficiently large r, both $(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}$ and $2^{1/2}$ are contained in the interval $[1,2]$. Hence, for each $u \in \mathbb{R}$, by the mean value theorem for differentiable functions,

$$\begin{aligned}\Big|\Phi\big((\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}\,u\big) - \Phi\big(2^{1/2}\,u\big)\Big| &\le \big|(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2} - 2^{1/2}\big| \sup_{v\in[1,2]} |u|\Phi'(vu) \\ &\le \big|(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2} - 2^{1/2}\big| \sup_{z\in\mathbb{R}} |z|\Phi'(z) = \frac{\big|(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2} - 2^{1/2}\big|}{(2\pi\mathrm{e})^{1/2}} \\ &= O\big((\log\log a_r)^{-A}\big)\end{aligned}$$

as $r \to \infty$. Another application of the mean value theorem yields

$$\begin{aligned}\sup_{u\in\mathbb{R}} \Big|\Phi\big((\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}(u + 1 - \mathbb{E}_{r,t}[X_{r,j}(t)])\big) - \Phi\big((\mathrm{Var}_{r,t}[X_{r,j}(t)])^{-1/2}\,u\big)\Big| \\ \le \frac{|\mathbb{E}_{r,t}[X_{r,j}(t)] - 1|}{(\mathrm{Var}_{r,t}[X_{r,j}(t)])^{1/2}(2\pi)^{1/2}} = O\big((\log\log a_r)^{-A}\big), \quad r \to \infty.\end{aligned}$$

The O-estimates in both formulas are secured by (6.28). Combining the last two displayed formulas proves (6.31). For all sufficiently large r and each t satisfying $(\log\log a_r)^{-2} \le |t| \le (\log\log a_r)^2 + 2$, let H_j denote the distribution function of $X_{r,j}(t) - 1$ under $\mathbb{P}_{r,t}$ and put $\Psi(u) := \Phi\big(2^{1/2}\,u\big)$ for $u \in \mathbb{R}$. Then (6.30) with $A = 100$, together with (6.31), entails

$$\max_{0\le j<N_r} \sup_{u\in\mathbb{R}} |H_j(u) - \Psi(u)| \le C_2(\log\log a_r)^{-100} =: \varepsilon_r \tag{6.32}$$

for a universal constant $C_2 > 0$ and all sufficiently large r.

We now derive a direct comparison of barrier probabilities from (6.32). For fixed t in the stated range, put

$$Y_{r,i}(t) := X_{r,N_r-i}(t) - 1, \quad 1 \le i \le N_r, \quad Z_{r,0}(t) := 0, \quad Z_{r,k}(t) := \sum_{i=1}^{k} Y_{r,i}(t), \quad 1 \le k \le N_r.$$

Let $G_1, \dots, G_{N_r}$ be independent normal random variables of mean zero and variance $1/2$. We write $\mathbb{P}_G$ for their probability measure and put $S_0 := 0$ and $S_k := \sum_{i=1}^{k} G_i$ for $1 \le k \le N_r$.

For integers $1 \le m < N_r$ and $h, a \ge 0$, define

$$\begin{aligned}D_h &:= \{z \in \mathbb{R}^{N_r} : \ \textstyle\sum_{i=1}^{k} z_i \le h, \ 1 \le k \le N_r\} \ \text{ and} \\ D_{h,a,m} &:= D_h \cap \{z \in \mathbb{R}^{N_r} : \ \textstyle\sum_{i=1}^{k} z_i \le -a, \ m \le k \le N_r\}.\end{aligned}$$

For fixed values of all coordinates except one, the corresponding section of either D_h or $D_{h,a,m}$ is empty or a half-line of the form $(-\infty, u]$. We shall write D for either D_h or $D_{h,a,m}$. We claim that

$$|\mathbb{P}_{r,t}\{(Y_{r,i}(t))_{1\le i\le N_r} \in D\} - \mathbb{P}_G\{(G_i)_{1\le i\le N_r} \in D\}| \le N_r\varepsilon_r. \tag{6.33}$$

Indeed, extend the probability space by adjoining $G_1, \dots, G_{N_r}$, independently of $(Y_{r,i}(t))_{1\le i\le N_r}$, and denote the resulting probability measure by $\mathbb{P}^*_{r,t}$. Put $\boldsymbol{Z}^{(0)} := (Y_{r,1}(t), \dots, Y_{r,N_r}(t))$, $\boldsymbol{Z}^{(N_r)} := (G_1, \dots, G_{N_r})$ and $\boldsymbol{Z}^{(i)} := (G_1, \dots, G_i, Y_{r,i+1}(t), \dots, Y_{r,N_r}(t))$ for $1 \le i < N_r$. By the triangle inequality,

$$\begin{aligned}&|\mathbb{P}_{r,t}\{(Y_{r,i}(t))_{1\le i\le N_r} \in D\} - \mathbb{P}_G\{(G_i)_{1\le i\le N_r} \in D\}| \\ &\qquad \le \sum_{i=1}^{N_r} \Big|\mathbb{P}^*_{r,t}\{\boldsymbol{Z}^{(i-1)} \in D\} - \mathbb{P}^*_{r,t}\{\boldsymbol{Z}^{(i)} \in D\}\Big| \le N_r\varepsilon_r.\end{aligned}$$

To obtain the last inequality, it suffices to show that each modulus does not exceed ε_r. To check this for the ith summand, we condition on the common coordinates

$G_1, \dots, G_{i-1}, Y_{r,i+1}(t), \dots, Y_{r,N_r}(t)$. If the corresponding section is empty, the two conditional probabilities are zero. Otherwise, their difference is $H_{N_r-i}(u) - \Psi(u)$ for the endpoint u of that section. By independence and (6.32), the absolute value of that difference does not exceed ε_r. Taking expectations proves the claimed bound for the ith summand, hence also for the whole sum.

In particular, (6.33) ensures that, for each $h \geq 0$,

$$\Big|\mathbb{P}_{r,t}\{\max_{0\leq k\leq N_r} Z_{r,k}(t) \leq h\} - \mathbb{P}_G\{\max_{0\leq k\leq N_r} S_k \leq h\}\Big| \leq N_r\varepsilon_r. \tag{6.34}$$

Since

$$D_h \setminus D_{h,a,m} = \{z : \max_{1\leq k\leq N_r} \textstyle\sum_{i=1}^k z_i \leq h,\ \max_{m\leq k\leq N_r} \sum_{i=1}^k z_i > -a\},$$

applying (6.33) to D_h and then to $D_{h,a,m}$ gives

$$\Big|\mathbb{P}_{r,t}\{\max_{0\leq k\leq N_r} Z_{r,k}(t) \leq h,\ \max_{m\leq k\leq N_r} Z_{r,k}(t) > -a\} \\ -\mathbb{P}_G\{\max_{0\leq k\leq N_r} S_k \leq h,\ \max_{m\leq k\leq N_r} S_k > -a\}\Big| \leq 2N_r\varepsilon_r. \tag{6.35}$$

Formulas (6.33), (6.34) and (6.35) hold uniformly in $(\log\log a_r)^{-2} \leq |t| \leq (\log\log a_r)^2 + 2$ as well as in the indicated barriers and indices. Since $N_r \leq \log\log a_r$, we have

$$N_r\varepsilon_r \leq C_2(\log\log a_r)^{-99} = o((\log\log a_r)^{-19}), \quad r \to \infty.$$

6.5 From weak to strong barriers

Lemma 6.7 bounds the second moment of the Euler product under the weak barriers and the contribution from realizations satisfying the weak barriers but violating at least one strong barrier. Formula (6.37) is a variant of Multiplicative Chaos Result 2 in Section 2.4 of [16], which, as stated, does not cover the barriers and the bound considered here. In view of this, we provide a full discussion.

Lemma 6.7. *For fixed $B \geq 1$ and $b \in (0,1)$, uniformly in $(\log\log a_r)^{-2} \leq |t| \leq (\log\log a_r)^2 + 2$,*

$$\mathbb{E}\left[|F_{a_r}(t)|^2 \mathbb{1}_{\mathcal{D}_r(t)}\right] = O\left(\log a_r \, \frac{(\log\log\log a_r)^2}{(\log\log a_r)^{1/2}}\right), \quad r \to \infty \tag{6.36}$$

and

$$\mathbb{E}\left[|F_{a_r}(t)|^2 \mathbb{1}_{\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)}\right] = O\left(\log a_r \, \frac{(\log\log\log a_r)^4}{\log\log a_r}\right), \quad r \to \infty. \tag{6.37}$$

The implicit constant in (6.36) *is absolute, whereas it only depends on B and b in* (6.37).

Proof. Fix t in the stated range. By (6.23), for each event E,

$$\mathbb{E}[|F_{a_r}(t)|^2 \mathbb{1}_E] = \mathbb{E}[|F_{a_r}(t)|^2]\mathbb{P}_{r,t}(E). \tag{6.38}$$

According to (6.6), $\mathbb{E}[|F_{a_r}(t)|^2] \sim 6\pi^{-2}\mathrm{e}^{\gamma}\log a_r$ as $r \to \infty$ uniformly in $t \in \mathbb{R}$. Hence, it suffices to estimate $\mathbb{P}_{r,t}(\mathcal{D}_r(t))$ and $\mathbb{P}_{r,t}(\mathcal{D}_r(t) \setminus \mathcal{A}_r(t))$.

First we discard the event $\mathcal{L}_r(t) := \{|F_{N_r,a_r}(t)| < (\log\log a_r)^{-1}\}$. By the definition of this event, independence of the prime signs and $\mathbb{E}\big[\big|1+f(p)p^{-1/2-\mathrm{i}t}\big|^2\big] = 1+p^{-1}$,

$$\begin{aligned}\mathbb{E}\big[|F_{a_r}(t)|^2\mathbb{1}_{\mathcal{L}_r(t)}\big] &\le (\log\log a_r)^{-2}\mathbb{E}\left[\left|\frac{F_{a_r}(t)}{F_{N_r,a_r}(t)}\right|^2\right] = (\log\log a_r)^{-2}\prod_{\exp(q_{N_r,a_r})<p\le a_r}(1+p^{-1})\\ &\sim \frac{\log a_r}{(\log\log a_r)^2 q_{N_r,a_r}} = \frac{\log a_r}{(\log\log a_r)^2 \mathrm{e}^{(\log\log\log a_r)^2+O(1)}}\\ &= o\Big(\log a_r\,(\log\log a_r)^{-A}\Big), \quad r\to\infty \quad (6.39)\end{aligned}$$

for each fixed $A>0$. Here, the asymptotic equivalence follows from (6.3), the penultimate equality is justified by $\log q_{N_r,a_r} = \log\log a_r - \lfloor \log\log a_r - (\log\log\log a_r)^2\rfloor = (\log\log\log a_r)^2 + O(1)$ whereas the last equality follows from the fact that $\mathrm{e}^{(\log\log\log a_r)^2}$ grows faster than any fixed positive power of $\log\log a_r$.

The partial sums introduced in Section 6.4 satisfy

$$Z_{r,k}(t) = \sum_{i=1}^{k}(X_{r,N_r-i}(t)-1) = \log\left|\frac{F_{N_r-k,a_r}(t)}{F_{N_r,a_r}(t)}\right| - k, \quad 0\le k\le N_r.$$

Moreover, $q_{N_r-k,a_r} = \mathrm{e}^k q_{N_r,a_r}$. On $\mathcal{D}_r(t)\cap(\mathcal{L}_r(t))^c$, we therefore have, for each $0\le k\le N_r$,

$$Z_{r,k}(t) \le \log\big(\mathrm{e}^k q_{N_r,a_r}(\log\log a_r)^3\big) - k = \log q_{N_r,a_r} + 3\log\log\log a_r =: h_r.$$

Consequently,

$$\mathcal{D}_r(t)\cap(\mathcal{L}_r(t))^c \subseteq \big\{\max_{0\le k\le N_r} Z_{r,k}(t)\le h_r\big\}.$$

By (6.34), with $h=h_r$, and (6.20), with $L=\lfloor\log\log a_r\rfloor$, $n=N_r$ and $h=h_r$, we obtain

$$\begin{aligned}\mathbb{P}_{r,t}(\mathcal{D}_r(t)\cap(\mathcal{L}_r(t))^c) &\le \mathbb{P}_{r,t}\big\{\max_{0\le k\le N_r} Z_{r,k}(t)\le h_r\big\}\\ &\le \mathbb{P}_G\big\{\max_{0\le k\le N_r} S_k\le h_r\big\} + N_r\varepsilon_r \le C\Big(\frac{h_r+d_r}{N_r^{1/2}} + \lfloor\log\log a_r\rfloor^{-19}\Big) + N_r\varepsilon_r\\ &= O\Big(\frac{(\log\log\log a_r)^2}{(\log\log a_r)^{1/2}}\Big), \quad r\to\infty,\end{aligned}$$

where $d_r := (20\log\lfloor\log\log a_r\rfloor)^{1/2}$. Here, we have used $N_r\sim\log\log a_r$, $h_r\sim(\log\log\log a_r)^2$ and $N_r\varepsilon_r = O((\log\log a_r)^{-99})$. Together with (6.38) and (6.39), this proves (6.36).

Now we treat (6.37). On $\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)$, there exists $0\le j\le\lfloor(1-b)\log\log a_r\rfloor$ such that

$$|F_{j,a_r}(t)| > q_{j,a_r}(\log\log a_r)^{-B}.$$

At the same time, $|F_{N_r,a_r}(t)|\le q_{N_r,a_r}(\log\log a_r)^2$ because $N_r\le J_*(a_r)$. For all sufficiently large r, the chosen index j satisfies $j\le\lfloor(1-b)\log\log a_r\rfloor < N_r$, whence

$$Z_{r,N_r-j}(t) > \log\left(\frac{q_{j,a_r}(\log\log a_r)^{-B}}{q_{N_r,a_r}(\log\log a_r)^2}\right) - (N_r-j) = -(B+2)\log\log\log a_r.$$

Also, $N_r - j \ge b\log\log a_r - (\log\log\log a_r)^2 - 1 \ge \lfloor b\log\log a_r/2\rfloor =: m_r$ for all sufficiently large r. On $(\mathcal{D}_r(t)\setminus\mathcal{A}_r(t))\cap(\mathcal{L}_r(t))^c$, we thus have

$$\max_{0\le k\le N_r} Z_{r,k}(t)\le h_r \quad\text{and}\quad \max_{m_r\le k\le N_r} Z_{r,k}(t) > -(B+2)\log\log\log a_r.$$

The validity of the first inequality was justified in the proof of (6.36).

Put $a_r' := (B+2)\log\log\log a_r$. By (6.35), with $h = h_r$ and $a = a_r'$,

$$\begin{aligned}\mathbb{P}_{r,t}((\mathcal{D}_r(t)\setminus\mathcal{A}_r(t))\cap(\mathcal{L}_r(t))^c) &\le \mathbb{P}_{r,t}\{\max_{0\le k\le N_r} Z_{r,k}(t)\le h_r,\ \max_{m_r\le k\le N_r} Z_{r,k}(t) > -a_r'\}\\ &\le \mathbb{P}_G\{\max_{0\le k\le N_r} S_k\le h_r,\ \max_{m_r\le k\le N_r} S_k > -a_r'\} + 2N_r\varepsilon_r\\ &\le \mathbb{P}_G\{\max_{0\le k\le N_r} S_k\le h_r,\ \max_{m_r\le k\le N_r} S_k \ge -a_r'\} + 2N_r\varepsilon_r.\end{aligned}$$

Applying (6.21) with $L = \lfloor\log\log a_r\rfloor$, $n = N_r$, $m = m_r$, $h = h_r$ and $a = a_r'$, we obtain

$$\begin{aligned}&\mathbb{P}_{r,t}((\mathcal{D}_r(t)\setminus\mathcal{A}_r(t))\cap(\mathcal{L}_r(t))^c)\\ &\quad\le C'\Big(\frac{(h_r+d_r)(h_r+(B+2)\log\log\log a_r+d_r)}{(m_r(N_r-m_r))^{1/2}} + \lfloor\log\log a_r\rfloor^{-19}\Big) + 2N_r\varepsilon_r\\ &\quad= O\Big(\frac{(\log\log\log a_r)^4}{\log\log a_r}\Big),\quad r\to\infty.\end{aligned}$$

Here, we have used $m_r \sim b\log\log a_r/2$, $N_r - m_r \sim (1-b/2)\log\log a_r$, $h_r = O((\log\log\log a_r)^2)$ and $N_r\varepsilon_r = O((\log\log a_r)^{-99})$. Finally, by (6.38) and (6.39),

$$\begin{aligned}\mathbb{E}[|F_{a_r}(t)|^2\mathbb{1}_{\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)}] &\le \mathbb{E}[|F_{a_r}(t)|^2\mathbb{1}_{\mathcal{L}_r(t)}]\\ &\quad+\mathbb{E}[|F_{a_r}(t)|^2]\mathbb{P}_{r,t}((\mathcal{D}_r(t)\setminus\mathcal{A}_r(t))\cap(\mathcal{L}_r(t))^c) = O\Big(\log a_r\,\frac{(\log\log\log a_r)^4}{\log\log a_r}\Big),\end{aligned}$$

as $r\to\infty$, uniformly in the stated range of t. This proves (6.37). □

6.6 Restricting the covariance integral

In this subsection we show that, uniformly over $x, y \in \mathcal{X}_r$, the conditional covariance can be approximated by an integral over pairs of nearby arguments at which the strong barriers are satisfied. This follows the argument in Sections 3.2–3.3 of [16], see, in particular, formula (3.4) and the subsequent error estimate. Since the cutoffs, barriers and error bound in [16] differ from those used here, we provide full details.

Recalling that $\omega_r = (\log\log a_r)^4/T_r$, put

$$\Delta_r := \Big\{(v,t)\in\mathbb{R}^2:\ (\log\log a_r)^{-2}\le|v|,|t|\le(\log\log a_r)^2,\ |v-t|\le\omega_r\Big\}$$

and, for $x, y \in \mathcal{X}_r$, define

$$I_r(x,y) := \frac{1}{4\pi^2}\iint_{\Delta_r}\frac{\mathbb{1}_{\mathcal{A}_r(v)}F_{a_r}(v)x^{\mathrm{i}v}\mathbb{1}_{\mathcal{A}_r(t)}\overline{F_{a_r}(t)}y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)}K_r(v-t)\,\mathrm{d}v\,\mathrm{d}t. \tag{6.40}$$

The dependence on the fixed parameters B and b appearing in the definition of $\mathcal{A}_r(t)$ is suppressed in this notation.

Lemma 6.8. *For each fixed $B \ge 1$ and $b \in (0,1)$,*

$$\lim_{r\to\infty}\mathbb{P}\Big\{\max_{x,y\in\mathcal{X}_r}\Big|\frac{C_r(x,y)}{(xy)^{1/2}} - I_r(x,y)\Big| \le (\log\log a_r)^{-5/8}\Big\} = 1. \tag{6.41}$$

Proof. Put $g(t) := (1+|t|)^{-1}$ for $t\in\mathbb{R}$. Since

$$(1/4+t^2)^{-1/2}\le 5^{1/2}g(t),\quad t\in\mathbb{R}$$

and $|x^{\mathrm{i}v}y^{-\mathrm{i}t}| = 1$, all estimates that follow hold simultaneously for $x, y \in \mathcal{X}_r$.

We first remove the part of the integral in (6.10) in which $\max(|v|,|t|) > (\log\log a_r)^2$. Recalling that $H_r = T_r^8$, denote its absolute majorant by

$$\mathcal{R}_r^{(1)} := \iint\limits_{\substack{|v|,|t|\le H_r \\ \max(|v|,|t|)>(\log\log a_r)^2}} |F_{a_r}(v)F_{a_r}(t)|g(v)g(t)|K_r(v-t)|\,\mathrm{d}v\,\mathrm{d}t.$$

By the Cauchy-Schwarz inequality and (6.6),

$$\mathbb{E}\big[|F_{a_r}(v)F_{a_r}(t)|\big] \le \Big(\mathbb{E}[|F_{a_r}(v)|^2]\mathbb{E}[|F_{a_r}(t)|^2]\Big)^{1/2} = O(\log a_r), \quad r\to\infty \tag{6.42}$$

for all $v,t\in\mathbb{R}$. By (6.42) and (6.8),

$$\mathbb{E}[\mathcal{R}_r^{(1)}] = O(\log a_r)\iint\limits_{\substack{|v|,|t|\le H_r \\ \max(|v|,|t|)>(\log\log a_r)^2}} g(v)g(t)\min\left(1,\frac{1}{T_r|v-t|}\right)\,\mathrm{d}v\,\mathrm{d}t.$$

We further split the region of integration in $\mathcal{R}_r^{(1)}$ into subregions.

Subregion where $|v|/2\le|t|\le 2|v|$. By symmetry in $|v|$ and $|t|$, it suffices to consider the region $(\log\log a_r)^2<|v|\le H_r$ and $|v|/2\le|t|\le|v|$. On it,

$$g(v)g(t)\le\frac{2}{(1+|v|)^2}.$$

Therefore, enlarging the range of integration in t, the expected contribution from the subregion that we treat now is at most

$$O(\log a_r)\int_{(\log\log a_r)^2<|v|\le H_r}\frac{1}{(1+|v|)^2}\left[\int_{-H_r}^{H_r}\min\left(1,\frac{1}{T_r|v-t|}\right)\,\mathrm{d}t\right]\mathrm{d}v.$$

For each $|v|\le H_r$, the inner integral is bounded by

$$2\int_0^{2H_r}\min\left(1,\frac{1}{T_r u}\right)\,\mathrm{d}u = \frac{2}{T_r}(1+\log(2H_rT_r)).$$

Therefore, using $\log a_r\sim T_r/2$ we conclude that the preceding expression does not exceed

$$\begin{aligned}O(1)(1+\log(2H_rT_r))\int_{|v|>(\log\log a_r)^2}\frac{\mathrm{d}v}{(1+|v|)^2} &= O((1+\log(2H_rT_r))(\log\log a_r)^{-2})\\ &= O((\log\log a_r)^{-1}),\quad r\to\infty.\end{aligned}$$

Subregion where $|v|\ge 2|t|$ and $|v|>(\log\log a_r)^2$. Here, $|v-t|\ge|v|-|t|\ge|v|/2$. Hence the contribution of this subregion to $\mathbb{E}[\mathcal{R}_r^{(1)}]$ does not exceed

$$\begin{aligned}&O(1)\int_{(\log\log a_r)^2<|v|\le H_r}\frac{1}{|v|(1+|v|)}\left(\int_{|t|\le|v|/2}\frac{\mathrm{d}t}{1+|t|}\right)\mathrm{d}v\\ &\qquad = 4O(1)\int_{(\log\log a_r)^2}^{H_r}\frac{\log(1+v/2)}{v(1+v)}\,\mathrm{d}v\\ &\qquad \le O(1)\int_{(\log\log a_r)^2}^{\infty}\frac{\log v}{v^2}\,\mathrm{d}v = O(1)\frac{2\log\log\log a_r+1}{(\log\log a_r)^2} = O\left(\frac{\log\log\log a_r}{(\log\log a_r)^2}\right).\end{aligned}$$

Here, the inequality holds whenever $(\log\log a_r)^2\ge 2$, that is, for all sufficiently large r. The same estimate holds when $|t|\ge 2|v|$ and $|t|>(\log\log a_r)^2$. Consequently,

$$\mathbb{E}[\mathcal{R}_r^{(1)}] = O((\log\log a_r)^{-1}) \quad\text{and}\quad \mathbb{P}\{\mathcal{R}_r^{(1)} > (\log\log a_r)^{-3/4}\} = O((\log\log a_r)^{-1/4}),\quad r\to\infty$$

having utilized Markov's inequality for the second equality.

Next we remove the part for which $\max(|v|,|t|) \le (\log\log a_r)^2$ and $|v-t| > \omega_r$. Denote its absolute majorant by

$$\mathcal{R}_r^{(2)} := \iint_{\substack{|v|,|t|\le(\log\log a_r)^2 \\ |v-t|>\omega_r}} |F_{a_r}(v)F_{a_r}(t)|g(v)g(t)|K_r(v-t)| \,\mathrm{d}v\,\mathrm{d}t.$$

Lemma 6.3, together with (6.8), (6.42) and $\log a_r \sim T_r/2$, shows that $\mathbb{E}[\mathcal{R}_r^{(2)}]$ is bounded by the sum of

$$J_{r,1} := O((\log a_r)^{-1/2}) \iint_{\substack{|v|,|t|\le(\log\log a_r)^2 \\ |v-t|>\omega_r}} \frac{\min(T_r, |v-t|^{-1})^{1/2}}{|v-t|}\, g(v)g(t)\,\mathrm{d}v\,\mathrm{d}t,$$

$$J_{r,2} := O((\log a_r)^{-1/2}) \iint_{\substack{|v|,|t|\le(\log\log a_r)^2 \\ |v-t|>\omega_r}} \frac{\min(T_r, |v+t|^{-1})^{1/2}}{|v-t|}\, g(v)g(t)\,\mathrm{d}v\,\mathrm{d}t,$$

$$J_{r,3} := O((\log a_r)^{-1/2}) \iint_{\substack{|v|,|t|\le(\log\log a_r)^2 \\ |v-t|>\omega_r}} \frac{\log(2+|v|+|t|)}{|v-t|}\, g(v)g(t)\,\mathrm{d}v\,\mathrm{d}t.$$

For the first term, putting $u = v - t$, we write

$$\int_{\mathbb{R}} g(t+u)g(t)\,\mathrm{d}t \le \Big(\int_{\mathbb{R}} (g(t+u))^2\,\mathrm{d}t\Big)^{1/2} \left(\int_{\mathbb{R}} (g(t))^2\,\mathrm{d}t\right)^{1/2} = \int_{\mathbb{R}} (g(t))^2\,\mathrm{d}t = 2.$$

Since $T_r\omega_r = (\log\log a_r)^4 > 1$ for large r, we obtain

$$J_{r,1} \le 2O((\log a_r)^{-1/2}) \int_{\omega_r<|u|\le 2(\log\log a_r)^2} |u|^{-3/2}\,\mathrm{d}u = O((T_r\omega_r)^{-1/2}) = O((\log\log a_r)^{-2}).$$

For the second term, putting $u = v - t$, $s = v + t$ and using $g(v)g(t) \le 1$ gives

$$J_{r,2} \le O((\log a_r)^{-1/2}) \int_{\omega_r<|u|\le 2(\log\log a_r)^2} \frac{\mathrm{d}u}{|u|} \int_{|s|\le 2(\log\log a_r)^2} \min(T_r, |s|^{-1})^{1/2}\,\mathrm{d}s$$
$$= O\Big(\frac{(\log\log a_r)^2}{(\log a_r)^{1/2}}\Big).$$

Here, each integral is $O(\log\log a_r)$. For the first, it follows from

$$\int_{\omega_r<|u|\le 2(\log\log a_r)^2} \frac{\mathrm{d}u}{|u|} = 2\log\Big(\frac{2(\log\log a_r)^2}{\omega_r}\Big)$$
$$= 2\log T_r - 4\log\log\log a_r + 2\log 2 = O(\log\log a_r),$$

where we have used $\log T_r = \log\log a_r + O(1)$. Finally,

$$J_{r,3} \le O((\log a_r)^{-1/2}) \log\log\log a_r \int_{\omega_r<|u|\le 2(\log\log a_r)^2} \frac{\mathrm{d}u}{|u|} \int_{|t|\le(\log\log a_r)^2} \mathrm{d}t$$
$$= O\Big(\frac{(\log\log a_r)^3 \log\log\log a_r}{(\log a_r)^{1/2}}\Big).$$

Thus,

$$\mathbb{E}[\mathcal{R}_r^{(2)}] = O((\log\log a_r)^{-2}) \quad \text{and} \quad \mathbb{P}\{\mathcal{R}_r^{(2)} > (\log\log a_r)^{-1}\} = O((\log\log a_r)^{-1}), \quad r\to\infty.$$

Here, the second equality is justified by another application of Markov's inequality.

We now remove the part where $|v-t| \le \omega_r$ and $\min(|v|,|t|) < (\log\log a_r)^{-2}$. In this region, $\max(|v|,|t|) \le (\log\log a_r)^{-2} + \omega_r$. Denoting its absolute majorant by $\mathcal{R}_r^{(3)}$, invoking (6.9) and (6.42) and noting that $\omega_r = o\big((\log\log a_r)^{-2}\big)$ we arrive at

$$\mathbb{E}[\mathcal{R}_r^{(3)}] \le O(\log a_r)((\log\log a_r)^{-2} + \omega_r) \int_{-\omega_r}^{\omega_r} |K_r(u)| \, \mathrm{d}u = O\Big(\frac{\log\log\log a_r}{(\log\log a_r)^2}\Big).$$

Consequently, $\mathbb{P}\{\mathcal{R}_r^{(3)} > (\log\log a_r)^{-1}\} = O((\log\log a_r)^{-1} \log\log\log a_r)$ as $r \to \infty$. For later use, observe that, on the event $\mathcal{P}_r$ appearing in Lemma 6.4,

$$\max_{x,y\in\mathcal{X}_r} \bigg| \frac{C_r(x,y)}{(xy)^{1/2}} - \frac{1}{4\pi^2} \iint_{\Delta_r} \frac{F_{a_r}(v) x^{\mathrm{i}v} \overline{F_{a_r}(t)} y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)} K_r(v-t) \, \mathrm{d}v \, \mathrm{d}t \bigg| \le \mathcal{R}_r^{(1)} + \mathcal{R}_r^{(2)} + \mathcal{R}_r^{(3)} + O(T_r^{-1}). \quad (6.43)$$

It remains to impose the strong barriers on Δ_r. Put $E_r := \{t \in \mathbb{R} : \ (\log\log a_r)^{-2} \le |t| \le (\log\log a_r)^2\}$,

$$\mathcal{U}_r := \int_{E_r} \frac{|F_{a_r}(t)|^2 \mathbb{1}_{\mathcal{D}_r(t)}}{1/4+t^2} \, \mathrm{d}t \quad \text{and} \quad \mathcal{V}_r := \int_{E_r} \frac{|F_{a_r}(t)|^2 \mathbb{1}_{\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)}}{1/4+t^2} \, \mathrm{d}t.$$

By Fubini's theorem and Lemma 6.7,

$$\mathbb{E}[\mathcal{U}_r] = O\Big(\frac{\log a_r \, (\log\log\log a_r)^2}{(\log\log a_r)^{1/2}}\Big) \quad \text{and} \quad \mathbb{E}[\mathcal{V}_r] = O\Big(\frac{\log a_r \, (\log\log\log a_r)^4}{\log\log a_r}\Big), \quad r \to \infty.$$

Now Markov's inequality ensures that

$$\lim_{r\to\infty} \mathbb{P}\{\mathcal{U}_r \le \log a_r \, (\log\log a_r)^{-2/5}, \ \mathcal{V}_r \le \log a_r \, (\log\log a_r)^{-9/10}\} = 1. \quad (6.44)$$

On the event appearing in Lemma 6.5, the event $\mathcal{D}_r(t)$ holds for each $t \in E_r$. Thus, on the intersection of $\mathcal{P}_r$ and the event from Lemma 6.5, multiplying the integrands in (6.40) and (6.43) by $\mathbb{1}_{\mathcal{D}_r(v)}\mathbb{1}_{\mathcal{D}_r(t)}$ does not change the values of the integrals. In particular, (6.43) remains valid. For $t, u \in \mathbb{R}$, put $k_r(u) := |K_r(u)| \mathbb{1}_{[-\omega_r,\omega_r]}(u)$,

$$A_r(t) := \frac{|F_{a_r}(t)|}{(1/4+t^2)^{1/2}} \mathbb{1}_{E_r}(t) \mathbb{1}_{\mathcal{D}_r(t)} \quad \text{and} \quad B_r(t) := \frac{|F_{a_r}(t)|}{(1/4+t^2)^{1/2}} \mathbb{1}_{E_r}(t) \mathbb{1}_{\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)}.$$

With this notation laid down, on the aforementioned intersection of the events, we obtain

$$\max_{x,y\in\mathcal{X}_r} \bigg| \frac{1}{4\pi^2} \iint_{\Delta_r} \frac{F_{a_r}(v) \mathbb{1}_{\mathcal{D}_r(v)} x^{\mathrm{i}v} \overline{F_{a_r}(t)} \mathbb{1}_{\mathcal{D}_r(t)} y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)} K_r(v-t) \, \mathrm{d}v \, \mathrm{d}t - I_r(x,y) \bigg| \le \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} k_r(v-t)(B_r(v)A_r(t) + A_r(v)B_r(t)) \, \mathrm{d}v \, \mathrm{d}t \quad (6.45)$$

having utilized the inequality

$$\mathbb{1}_{\mathcal{D}_r(v)}\mathbb{1}_{\mathcal{D}_r(t)}(1 - \mathbb{1}_{\mathcal{A}_r(v)}\mathbb{1}_{\mathcal{A}_r(t)}) \le \mathbb{1}_{\mathcal{D}_r(v)\setminus\mathcal{A}_r(v)}\mathbb{1}_{\mathcal{D}_r(t)} + \mathbb{1}_{\mathcal{D}_r(v)}\mathbb{1}_{\mathcal{D}_r(t)\setminus\mathcal{A}_r(t)}.$$

Now we show that the latter double integral is bounded from above by

$$2\Big(\int_{\mathbb{R}} k_r(u) \, \mathrm{d}u\Big)\Big(\int_{\mathbb{R}} (A_r(t))^2 \, \mathrm{d}t\Big)^{1/2}\Big(\int_{\mathbb{R}} B_r(t)^2 \, \mathrm{d}t\Big)^{1/2} = 2\Big(\int_{-\omega_r}^{\omega_r} |K_r(u)| \, \mathrm{d}u\Big)(\mathcal{U}_r\mathcal{V}_r)^{1/2}.$$

Indeed, changing the variable $u = v - t$ and applying Fubini's theorem gives

$$\iint_{\mathbb{R}^2} k_r(v-t) B_r(v) A_r(t) \, \mathrm{d}v \, \mathrm{d}t = \int_{\mathbb{R}} k_r(u) \left(\int_{\mathbb{R}} B_r(t+u) A_r(t) \, \mathrm{d}t \right) \mathrm{d}u.$$

For each fixed u, the Cauchy–Schwarz inequality and translation invariance of Lebesgue measure yield

$$\int_{\mathbb{R}} B_r(t+u)A_r(t)\,\mathrm{d}t \le \Big(\int_{\mathbb{R}}(B_r(t+u))^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{\mathbb{R}}(A_r(t))^2\,\mathrm{d}t\Big)^{1/2} = \Big(\int_{\mathbb{R}}(B_r(t))^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{\mathbb{R}}(A_r(t))^2\,\mathrm{d}t\Big)^{1/2}.$$

Hence,

$$\iint_{\mathbb{R}^2} k_r(v-t)B_r(v)A_r(t)\,\mathrm{d}v\,\mathrm{d}t \le \int_{\mathbb{R}} k_r(u)\,\mathrm{d}u\Big(\int_{\mathbb{R}}(A_r(t))^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{\mathbb{R}} B_r(t)^2\,\mathrm{d}t\Big)^{1/2}.$$

The other term admits the same upper bound.

Using (6.9) and (6.44), we conclude that, with probability tending to one, the maximum in (6.45) is at most

$$10C\,T_r^{-1}\log\log T_r\log a_r\,(\log\log a_r)^{-13/20} = O(\log\log\log a_r\,(\log\log a_r)^{-13/20}) = o\big((\log\log a_r)^{-5/8}\big).$$

Finally, with probability tending to one, the maximum in (6.43) is at most

$$(\log\log a_r)^{-3/4} + 2(\log\log a_r)^{-1} + O(T_r^{-1}) = o\big((\log\log a_r)^{-5/8}\big).$$

This proves (6.41). □

6.7 How often is $|C_r(x,y)|$ large?

Next we estimate the high-order moments of the integrals $I_r(x,y)$ and use these estimates to bound, simultaneously for $x \in \mathcal{X}_r$, the number of points y for which $|C_r(x,y)|$ is large.

The unit spacing of the logarithms of the points in $\mathcal{X}_r$ gives rise to a periodic function. We first estimate its integral and then prove an auxiliary inequality for the gaps between ordered arguments. For $u \in \mathbb{R}$ and $m \in \mathbb{N}$, put $d_{2\pi}(u) := \inf_{j\in\mathbb{Z}} |u-2\pi j|$ and $D_m(u) := \min(m, \pi/d_{2\pi}(u))$ with $D_m(u) := m$ when $u \in 2\pi\mathbb{Z}$. We claim that

$$\Big|\sum_{j=0}^{m-1} \mathrm{e}^{\mathrm{i}ju}\Big| \le D_m(u), \quad m \in \mathbb{N},\ u \in \mathbb{R}. \tag{6.46}$$

Indeed, if $u \in 2\pi\mathbb{Z}$, then (6.46) is trivial. Note that $|\sin(u/2)| \ge d_{2\pi}(u)/\pi$ for $u \in \mathbb{R}$. With this at hand, if $u \notin 2\pi\mathbb{Z}$, it follows from

$$\Big|\sum_{j=0}^{m-1} \mathrm{e}^{\mathrm{i}ju}\Big| = \Big|\frac{\mathrm{e}^{\mathrm{i}mu}-1}{\mathrm{e}^{\mathrm{i}u}-1}\Big| = \Big|\frac{\sin(mu/2)}{\sin(u/2)}\Big| \le \frac{\pi}{d_{2\pi}(u)},$$

and also from $\big|\sum_{j=0}^{m-1}\mathrm{e}^{\mathrm{i}ju}\big| \le m$.

For integer $1 \le n \le 2\log\log a_r$, put

$$\mathcal{K}_{r,n}(u) := \sup_{|v-u|\le n\omega_r} D_{|\mathcal{X}_r|}(v), \quad u \in \mathbb{R},$$

where $\omega_r = (\log\log a_r)^4/T_r$. This function is 2π-periodic and does not exceed $|\mathcal{X}_r| = \lfloor T_r^{1-\lambda}\rfloor + 1$. Since $\omega_r = o((\log\log a_r)^{-1})$, we conclude that, for all sufficiently large r, $n\omega_r < 1$. Further, for such r,

$$\mathcal{K}_{r,n}(u) = \begin{cases} |\mathcal{X}_r|, & d_{2\pi}(u) \le n\omega_r, \\ \min(|\mathcal{X}_r|, \pi/(d_{2\pi}(u)-n\omega_r)), & d_{2\pi}(u) > n\omega_r. \end{cases}$$

Using symmetry about π, we obtain

$$\int_0^{2\pi} \mathcal{K}_{r,n}(u)\,\mathrm{d}u = 2\int_0^{n\omega_r} |\mathcal{X}_r|\,\mathrm{d}u + 2\int_{n\omega_r}^{\pi} \min(|\mathcal{X}_r|, \pi/(u-n\omega_r))\,\mathrm{d}u$$
$$= 2n\omega_r|\mathcal{X}_r| + 2\int_0^{\pi-n\omega_r} \min(|\mathcal{X}_r|, \pi/s)\,\mathrm{d}s \le 2(n\omega_r|\mathcal{X}_r| + \pi(1+\log|\mathcal{X}_r|))$$

having utilized

$$\int_0^{\pi} \min(m, \pi/s)\,\mathrm{d}s = \int_0^{\pi/m} m\,\mathrm{d}s + \pi\int_{\pi/m}^{\pi} s^{-1}\,\mathrm{d}s = \pi(1+\log m),$$

which holds for any $m \in \mathbb{N}$. For later use, we note that

$$0 \le \mathcal{K}_{r,n}(u) \le |\mathcal{X}_r| \quad \text{and}$$
$$\sup_{\substack{J\subset\mathbb{R}\ \text{interval}\\ |J|\le 2}} \int_J \mathcal{K}_{r,n}(u)\,\mathrm{d}u \le \int_0^{2\pi} \mathcal{K}_{r,n}(u)\,\mathrm{d}u \le 2(n\omega_r|\mathcal{X}_r| + \pi(1+\log|\mathcal{X}_r|)) \quad (6.47)$$

for all sufficiently large r, uniformly in $1 \le n \le 2\log\log a_r$.

The following inequality will control the total interaction between different gaps.

Lemma 6.9. *For each $n \in \mathbb{N}$ and all positive numbers $a_1, \dots, a_n$,*

$$\sum_{1\le j<l\le n} \frac{\min(a_j, a_l)}{a_j + \dots + a_{l-1}} \le 2n\log n. \quad (6.48)$$

Proof. Fix $1 \le j < n$ and put $b_l := \min(1, a_l/a_j)$, $A_l := a_j + \dots + a_{l-1}$ for $j < l \le n$ and $B_l := 1 + b_{j+1} + \dots + b_{l-1}$ for $j < l \le n+1$. Since $A_l \ge a_j B_l$, $B_{l+1} = B_l + b_l$ and $0 < b_l/B_l \le 1$ (just because $b_l \le 1$ and $B_l \ge 1$), we conclude that

$$\sum_{l=j+1}^{n} \frac{\min(a_j, a_l)}{A_l} \le \sum_{l=j+1}^{n} \frac{b_l}{B_l} \le 2\sum_{l=j+1}^{n} \log\frac{B_{l+1}}{B_l} = 2\log B_{n+1} \le 2\log(n-j+1)$$

having utilized $x \le 2\log(1+x)$ for $x \in [0,1]$ for the second inequality. Summing over $1 \le j \le n-1$ we obtain

$$\sum_{1\le j<l\le n} \frac{\min(a_j, a_l)}{a_j + \dots + a_{l-1}} \le 2\log(n!) \le 2n\log n,$$

thereby completing the proof. □

Next we bound joint moments of the Euler products subject to the strong barriers by a product of deterministic functions of the gaps between their arguments. For $0 < x \le (\log\log a_r)^2$, put

$$\psi_r(x) := \min\big(\log a_r,\ \max\big(1/x, (\log\log a_r)^2\big)\big)$$

and

$$\chi_r(x) := \begin{cases} (\log\log a_r)^{-2B}\psi_r(x), & 0 < x < (\log a_r)^{-b}, \\ (\log\log a_r)^4\psi_r(x), & (\log a_r)^{-b} \le x \le (\log\log a_r)^2, \end{cases}$$

where $B \ge 1$ and $b \in (0,1)$ are fixed numbers as appear in (6.18). Also, we recall the definition of events $\mathcal{A}_r(t)$ from (6.18).

Lemma 6.10. *There exists an absolute constant $C > 0$ such that, for all sufficiently large r, integers $1 \le k \le \lfloor \log\log a_r \rfloor$ and arguments*

$$(\log\log a_r)^{-2} \le t_1 < \dots < t_{2k} \le (\log\log a_r)^2,$$

we have

$$\mathbb{E}\Big[\prod_{j=1}^{2k} \big(\mathbb{1}_{\mathcal{A}_r(t_j)} |F_{a_r}(t_j)|^2\big)\Big] \le (Ck)^{Ck} (\log a_r)^{2k} \prod_{j=1}^{2k-1} \chi_r(t_{j+1} - t_j). \tag{6.49}$$

The constant C does not depend on B or b.

Proof. Put $n := 2k$ and $x_j := t_{j+1} - t_j$ for $1 \le j < n$. For these indices, define

$$h_j := \max\big(0,\ \lceil \log(x_j \log a_r) \rceil - 1\big) \quad \text{and} \quad \widetilde{h}_j := \min\big(h_j,\ \lfloor \log\log a_r - 2\log\log\log a_r \rfloor\big).$$

For all sufficiently large r, each $\widetilde{h}_j$ belongs to $\{0, \dots, J_*(a_r)\}$, where $J_*(a_r) = \lfloor \log\log a_r - 1 \rfloor$. We claim that

$$(\log\log a_r)^2 \le q_{\widetilde{h}_j, a_r} \le \mathrm{e}\psi_r(x_j). \tag{6.50}$$

Proof of (6.50). We first show that $q_{h_j,a_r} \le \mathrm{e}/x_j$. Indeed, if $h_j \ge 1$, then $h_j < \log(x_j \log a_r) \le h_j + 1$, whence $1/x_j < q_{h_j,a_r} \le \mathrm{e}/x_j$. If $h_j = 0$, then $x_j \log a_r \le \mathrm{e}$, so that $q_{0,a_r} = \log a_r \le \mathrm{e}/x_j$.

Further, using

$$\log\log a_r - 2\log\log\log a_r - 1 < H_r^* := \lfloor \log\log a_r - 2\log\log\log a_r \rfloor \le \log\log a_r - 2\log\log\log a_r,$$

we infer

$$(\log\log a_r)^2 \le q_{H_r^*, a_r} = \exp(-H_r^*) \log a_r < \mathrm{e}(\log\log a_r)^2.$$

Since the function $h \mapsto q_{h,a_r}$ is decreasing, we have $q_{\widetilde{h}_j,a_r} = \max(q_{h_j,a_r}, q_{H_r^*,a_r})$. Consequently,

$$(\log\log a_r)^2 \le q_{\widetilde{h}_j,a_r} \le \mathrm{e}\max(1/x_j, (\log\log a_r)^2).$$

This proves the left-hand inequality in (6.50). To obtain the right-hand inequality in (6.50), we combine the right-hand inequality in the latter displayed formula with $q_{\widetilde{h}_j,a_r} \le \log a_r$ which follows from $\widetilde{h}_j \ge 0$:

$$\begin{aligned} q_{\widetilde{h}_j,a_r} &\le \min\big(\log a_r,\ \mathrm{e}\max\big(1/x_j,\ (\log\log a_r)^2\big)\big) \\ &\le \mathrm{e}\min\big(\log a_r,\ \max\big(1/x_j,\ (\log\log a_r)^2\big)\big) = \mathrm{e}\psi_r(x_j). \end{aligned}$$

The proof of (6.50) is complete.

Put

$$\alpha_j := \begin{cases} (\log\log a_r)^{-B}, & x_j < (\log a_r)^{-b}, \\ (\log\log a_r)^2, & x_j \ge (\log a_r)^{-b}. \end{cases}$$

If $x_j < (\log a_r)^{-b}$, then $\widetilde{h}_j \le h_j \le \lfloor (1-b)\log\log a_r \rfloor$ for all sufficiently large r. With this at hand, irrespective of whether $x_j < (\log a_r)^{-b}$ or not, the definitions of the weak and strong barriers imply that, on $\mathcal{A}_r(t_j)$, $|F_{\widetilde{h}_j,a_r}(t_j)| \le q_{\widetilde{h}_j,a_r}\alpha_j$. Extracting these factors for $j < n$ and then discarding the remaining indicators, we obtain

$$\mathbb{E}\Big[\prod_{j=1}^{n} \big(\mathbb{1}_{\mathcal{A}_r(t_j)} |F_{a_r}(t_j)|^2\big)\Big] \le \prod_{j=1}^{n-1} \big(q_{\widetilde{h}_j,a_r}\alpha_j\big)^2 \mathbb{E}\Big[|F_{a_r}(t_n)|^2 \prod_{j=1}^{n-1} \Big|\frac{F_{a_r}(t_j)}{F_{\widetilde{h}_j,a_r}(t_j)}\Big|^2\Big]. \tag{6.51}$$

We shall prove that

$$\mathbb{E}\Big[|F_{a_r}(t_n)|^2\prod_{j=1}^{n-1}\Big|\frac{F_{a_r}(t_j)}{F_{\widetilde{h}_j,a_r}(t_j)}\Big|^2\Big]\le (C_*n)^{C_*n}\log a_r\prod_{j=1}^{n-1}\mathrm{e}^{\widetilde{h}_j} \tag{6.52}$$

for an absolute constant $C_*>0$.

The ratio with $\widetilde{h}_j=0$ is equal to one. Each other ratio contains only primes exceeding $\exp(q_{\widetilde{h}_j,a_r})\ge\exp((\log\log a_r)^2)$, see (6.50). For $p\le a_r$, put

$$\mathcal{I}_p:=\{n\}\cup\{j\in\{1,\dots,n-1\}:\ p>\exp(q_{\widetilde{h}_j,a_r})\}.$$

By independence of the prime signs,

$$\begin{aligned}\mathbb{E}\Big[|F_{a_r}(t_n)|^2\prod_{j=1}^{n-1}\Big|\frac{F_{a_r}(t_j)}{F_{\widetilde{h}_j,a_r}(t_j)}\Big|^2\Big]&=\prod_{p\le a_r}\mathbb{E}\Big[\prod_{j\in\mathcal{I}_p}(1+f(p)p^{-1/2+\mathrm{i}t_j})(1+f(p)p^{-1/2-\mathrm{i}t_j})\Big]\\&=\prod_{p\le a_r}\mathbb{E}\Big[\prod_{k=1}^{2|\mathcal{I}_p|}(1+f(p)p^{-1/2+\mathrm{i}s_{p,k}})\Big],\end{aligned}$$

where, for each prime $p\le a_r$, $(s_{p,k})_{1\le k\le|\mathcal{I}_p|}$ is an enumeration of $(t_j)_{j\in\mathcal{I}_p}$ and $s_{p,|\mathcal{I}_p|+k}:=-s_{p,k}$ for $1\le k\le|\mathcal{I}_p|$. Expanding the latter product gives

$$\prod_{k=1}^{2|\mathcal{I}_p|}(1+f(p)p^{-1/2+\mathrm{i}s_{p,k}})=\sum_{\ell=0}^{2|\mathcal{I}_p|}(f(p)^\ell)p^{-\ell/2}e_\ell(p^{\mathrm{i}s_{p,1}},\dots,p^{\mathrm{i}s_{p,2|\mathcal{I}_p|}}),$$

where e_ℓ is the elementary symmetric polynomial of degree ℓ given by

$$e_\ell(p^{\mathrm{i}s_{p,1}},\dots,p^{\mathrm{i}s_{p,2|\mathcal{I}_p|}})=\sum_{1\le k_1<\dots<k_\ell\le2|\mathcal{I}_p|}p^{\mathrm{i}(s_{p,k_1}+\dots+s_{p,k_\ell})},\quad 1\le\ell\le2|\mathcal{I}_p|$$

and $e_0\equiv1$. Put

$$L_p:=\prod_{j\in\mathcal{I}_p}(1+f(p)p^{-1/2+\mathrm{i}t_j})(1+f(p)p^{-1/2-\mathrm{i}t_j}).$$

Noting that

$$\begin{aligned}e_2(p^{\mathrm{i}s_{p,1}},\dots,p^{\mathrm{i}s_{p,2|\mathcal{I}_p|}})&=|\mathcal{I}_p|+4\sum_{\substack{j<l\\ j,l\in\mathcal{I}_p}}\cos(t_j\log p)\cos(t_l\log p)\\&=|\mathcal{I}_p|+2\sum_{\substack{j<l\\ j,l\in\mathcal{I}_p}}(\cos((t_j-t_l)\log p)+\cos((t_j+t_l)\log p))\end{aligned}$$

and $|e_\ell(p^{\mathrm{i}s_{p,1}},\dots,p^{\mathrm{i}s_{p,2|\mathcal{I}_p|}})|\le\binom{2|\mathcal{I}_p|}{\ell}$ and passing to expectations, we obtain

$$\mathbb{E}[L_p]=1+|\mathcal{I}_p|p^{-1}+2p^{-1}\sum_{\substack{j<l\\ j,l\in\mathcal{I}_p}}(\cos((t_j-t_l)\log p)+\cos((t_j+t_l)\log p))+R_p, \tag{6.53}$$

where $|R_p|\le\sum_{\ell=4}^{2|\mathcal{I}_p|}\binom{2|\mathcal{I}_p|}{\ell}p^{-\ell/2}$. Here, as usual, an empty sum is understood to be zero. Since $\binom{2|\mathcal{I}_p|}{\ell}\le(2|\mathcal{I}_p|)^\ell/\ell!$, we infer

$$|R_p|\le\sum_{\ell\ge4}\frac{(2|\mathcal{I}_p|p^{-1/2})^\ell}{\ell!}\le\Big(16\sum_{\ell\ge4}\frac{1}{\ell!}\Big)(|\mathcal{I}_p|p^{-1/2})^4 \tag{6.54}$$

whenever $2|\mathcal{I}_p|p^{-1/2} \le 1$. Similarly,

$$|\mathbb{E}[L_p] - 1| \le \sum_{\ell \ge 2} \frac{(2|\mathcal{I}_p|p^{-1/2})^\ell}{\ell!} \le \Big(4 \sum_{\ell \ge 1} \frac{1}{\ell!}\Big)(|\mathcal{I}_p|p^{-1/2})^2$$

whenever $2|\mathcal{I}_p|p^{-1/2} \le 1$. We are now working with primes satisfying $\exp((\log\log a_r)^2) < p \le a_r$. Hence,

$$|\mathcal{I}_p|p^{-1/2} \le np^{-1/2} \le 2\log\log a_r \, \exp(-(\log\log a_r)^2/2) \ \to \ 0, \quad r \to \infty,$$

and we conclude that, uniformly over the primes and arguments under consideration, $|\mathbb{E}[L_p]-1| \le 1/2$ for all sufficiently large r. Using (6.53) and (6.54), in combination with $|\log(1+v)-v| \le v^2$ for $|v| \le 1/2$, we arrive at

$$\log \mathbb{E}[L_p] = |\mathcal{I}_p|p^{-1} + 2p^{-1} \sum_{\substack{j<l \\ j,l \in \mathcal{I}_p}} (\cos((t_j - t_l)\log p) + \cos((t_j + t_l)\log p)) + O(n^4 p^{-2}), \quad (6.55)$$

where the implicit constant is absolute.

Now we are ready to prove (6.52). We first note that

$$n^4 \sum_{p > \exp((\log\log a_r)^2)} p^{-2} \le n^4 \sum_{k > \exp((\log\log a_r)^2)} k^{-2} = O(n^4 \exp(-(\log\log a_r)^2)) = o(1)$$

because $n \le 2\lfloor \log\log a_r \rfloor$. Next, we estimate the contribution to the expectation in (6.52) of the primes $p \le \exp\big((\log\log a_r)^2\big)$ and the terms $|\mathcal{I}_p|p^{-1}$ for $\exp\big((\log\log a_r)^2\big) < p \le a_r$. For $p \le \exp\big((\log\log a_r)^2\big)$, we have $\mathcal{I}_p = \{n\}$, and the contribution of one prime from the present range is $\mathbb{E}\big[\big|1 + f(p)p^{-1/2-\mathrm{i}t_n}\big|^2\big] = 1 + 1/p$. For $\exp((\log\log a_r)^2) < p \le a_r$, we have $|\mathcal{I}_p| = 1 + \sum_{j=1}^{n-1} \mathbb{1}_{\{p > \exp(q_{\widetilde{h}_j, a_r})\}}$, whence

$$\sum_{\exp((\log\log a_r)^2) < p \le a_r} |\mathcal{I}_p|p^{-1} = \sum_{\exp((\log\log a_r)^2) < p \le a_r} p^{-1} + \sum_{j=1}^{n-1} \sum_{\exp(q_{\widetilde{h}_j, a_r}) < p \le a_r} p^{-1}.$$

Therefore, the contribution under discussion is

$$\prod_{p \le \exp((\log\log a_r)^2)} (1 + p^{-1}) \exp\Big(\sum_{\exp((\log\log a_r)^2) < p \le a_r} |\mathcal{I}_p|p^{-1} \Big)$$
$$\le \exp\Big(\sum_{p \le a_r} p^{-1} \Big) \prod_{j=1}^{n-1} \exp\Big(\sum_{\exp(q_{\widetilde{h}_j, a_r}) < p \le a_r} p^{-1} \Big).$$

We have used $1 + 1/p \le \mathrm{e}^{1/p}$ for the inequality. By (4.8), for all sufficiently large r, $\exp\big(\sum_{p \le a_r} p^{-1}\big) \le C_0 \log a_r$ and, uniformly in $1 \le j < n$ and the arguments $t_1, \dots, t_n$ in the stated regions,

$$\sum_{\exp(q_{\widetilde{h}_j, a_r}) < p \le a_r} p^{-1} = \log\log a_r - \log q_{\widetilde{h}_j, a_r} + O(1) = \widetilde{h}_j + O(1).$$

Hence, $\exp\big(\sum_{\exp(q_{\widetilde{h}_j, a_r}) < p \le a_r} p^{-1}\big) \le C_1 \mathrm{e}^{\widetilde{h}_j}$. When $\widetilde{h}_j = 0$, the sum is empty and this factor is equal to 1. Summarizing, the contribution under discussion is bounded from above by $C_2^n \log a_r \prod_{j=1}^{n-1} \mathrm{e}^{\widetilde{h}_j}$, where $C_2 := \max(C_0, C_1)$.

It remains to estimate the contribution of the pairs in (6.55). Fix $j<l<n$ such that both $\widetilde{h}_j$ and $\widetilde{h}_l$ are positive. If $j,l\in\mathcal{I}_p$, then necessarily $\exp(\max(q_{\widetilde{h}_j,a_r},q_{\widetilde{h}_l,a_r}))<p\le a_r$, and an application of Lemma 6.1 and (6.2) yields

$$\sum_{\exp(\max(q_{\widetilde{h}_j,a_r},q_{\widetilde{h}_l,a_r}))<p\le a_r} p^{-1}(\cos((t_j-t_l)\log p)+\cos((t_j+t_l)\log p))$$
$$\le 6\frac{\min(\mathrm{e}^{\widetilde{h}_j},\mathrm{e}^{\widetilde{h}_l})}{(t_l-t_j)\log a_r}+4C'(\log\log a_r)^2\mathrm{e}^{-c'\log\log a_r}.$$

To obtain the last exponent, we have used (6.50) ensuring that $\max(q_{\widetilde{h}_j,a_r},q_{\widetilde{h}_l,a_r})\ge(\log\log a_r)^2$. Since $\mathrm{e}^{\widetilde{h}_j}/\log a_r\le \mathrm{e}^{h_j}/\log a_r\le x_j$, see the paragraph following (6.50), the first term is at most $6\frac{\min(x_j,x_l)}{x_j+\cdots+x_{l-1}}$. Finally, by Lemma 6.9 applied to $x_1,\dots,x_{n-1}$, $\sum_{1\le j<l<n}\frac{\min(x_j,x_l)}{x_j+\ldots+x_{l-1}}\le 2(n-1)\log(n-1)\le 2n\log n$.

For a pair (j,n) with $j<n$ and $\widetilde{h}_j>0$, we have $j,n\in\mathcal{I}_p$ if, and only if, $\exp(q_{\widetilde{h}_j,a_r})<p\le a_r$ because the index n belongs to $\mathcal{I}_p$ for each $p\le a_r$. Applying the same estimates as above, the sum of the corresponding first terms is at most

$$6\sum_{\substack{1\le j<n\\ \widetilde{h}_j>0}}\frac{\mathrm{e}^{\widetilde{h}_j}}{\log a_r\,(t_n-t_j)}\le 6(n-1).$$

Here, we have used $\mathrm{e}^{\widetilde{h}_j}/\log a_r\le x_j\le t_n-t_j$. Indices with $\widetilde{h}_j=0$ contribute no such pairs. Since $n\le 2\lfloor\log\log a_r\rfloor$, $\sum_{1\le j<l\le n}(\log\log a_r)^2\mathrm{e}^{-c'\log\log a_r}=(\log\log a_r)^2\mathrm{e}^{-c'\log\log a_r}n(n-1)/2=o(1)$ as $r\to\infty$. Exponentiating the resulting $C_3n\log n$ bound, with $C_3\ge1$, proves (6.52), with $C_*:=\max(C_2,C_3)$.

Finally, observe that

$$\log a_r\prod_{j=1}^{n-1}(q^2_{\widetilde{h}_j,a_r}\mathrm{e}^{\widetilde{h}_j}\alpha_j^2)=(\log a_r)^n\prod_{j=1}^{n-1}(q_{\widetilde{h}_j,a_r}\alpha_j^2).$$

Combining this equality with (6.50), (6.51) and (6.52) and noting that $\alpha_j^2=\chi_r(x_j)/\psi_r(x_j)$ proves (6.49), with $C:=\mathrm{e}C_*$. □

To proceed, we record three consequences of the definitions of ψ_r and χ_r. We shall show that there exist absolute constants $c_1,c_2\ge1$ such that, for all sufficiently large r,

$$\int_0^{(\log a_r)^{-b}}\chi_r(x)\,\mathrm{d}x\le c_1(\log\log a_r)^{1-2B},\qquad \int_0^{(\log\log a_r)^2}\chi_r(x)\,\mathrm{d}x\le c_2(\log\log a_r)^8,\quad(6.56)$$

and

$$\max_{(\log a_r)^{-b}\le x\le(\log\log a_r)^2}\chi_r(x)=(\log a_r)^b(\log\log a_r)^4.\quad(6.57)$$

Indeed, $(\log a_r)^{-1}<(\log a_r)^{-b}<(\log\log a_r)^{-2}$ for all sufficiently large r. Hence, $\psi_r(x)=\min(\log a_r,1/x)$ for $x\in(0,(\log a_r)^{-b})$ and

$$\int_0^{(\log a_r)^{-b}}\chi_r(x)\,\mathrm{d}x=(\log\log a_r)^{-2B}\Big(\int_0^{1/\log a_r}\log a_r\,\mathrm{d}x+\int_{1/\log a_r}^{(\log a_r)^{-b}}\frac{\mathrm{d}x}{x}\Big)$$
$$=(\log\log a_r)^{-2B}(1+(1-b)\log\log a_r).$$

On the remaining range, $\psi_r(x) = \max(1/x, (\log\log a_r)^2)$, which proves (6.57) and gives

$$\int_{(\log a_r)^{-b}}^{(\log\log a_r)^2} \chi_r(x)\,\mathrm{d}x = (\log\log a_r)^4\Big(\int_{(\log a_r)^{-b}}^{(\log\log a_r)^{-2}} \frac{\mathrm{d}x}{x} + \int_{(\log\log a_r)^{-2}}^{(\log\log a_r)^2} (\log\log a_r)^2\,\mathrm{d}x\Big)$$
$$= (\log\log a_r)^4\big(b\log\log a_r - 2\log\log\log a_r + (\log\log a_r)^4 - 1\big) \;\sim\; (\log\log a_r)^8.$$

These formulas prove (6.56), with constants independent of B and b.

Next, integrating the preceding moment bound, we obtain an estimate which controls all rows of the covariance integral $I_r(x,y)$ simultaneously.

Lemma 6.11. *There exists an absolute constant $C_0 \ge 1$, independent of B and b, such that, for all sufficiently large r and integers $1 \le k \le \lfloor \log\log a_r \rfloor$,*

$$\mathbb{E}\left[\max_{x\in\mathcal{X}_r} \sum_{y\in\mathcal{X}_r} |I_r(x,y)|^{2k}\right]$$
$$\le (C_0k)^{C_0k}(\log\log a_r)^{C_0k}\big((\log a_r)^{2b} + (\log a_r)(\log\log a_r)^{-2Bk}\big). \quad (6.58)$$

Proof. Put $n := 2k$. Throughout the proof, r is sufficiently large and $2 \le n \le 2\lfloor\log\log a_r\rfloor$. We first extend the periodic-kernel estimate to intervals of length at most $(\log\log a_r)^2$. Since, as $r \to \infty$,

$$n\omega_r|\mathcal{X}_r| \le 2(\log\log a_r)^5 T_r^{-1}(T_r^{1-\lambda}+1) \le 4(\log\log a_r)^5 T_r^{-\lambda} = o(1)$$

and

$$\log|\mathcal{X}_r| = O(\log T_r) = O(\log\log a_r),$$

formula (6.47) and periodicity yield

$$\sup_{\substack{J\subset\mathbb{R}\text{ interval}\\ |J|\le(\log\log a_r)^2}} \int_J \mathcal{K}_{r,n}(u)\,\mathrm{d}u \le \Big(1 + \frac{(\log\log a_r)^2}{2\pi}\Big)\int_0^{2\pi} \mathcal{K}_{r,n}(u)\,\mathrm{d}u \le C_1(\log\log a_r)^3. \quad (6.59)$$

Also,

$$0 \le \mathcal{K}_{r,n}(u) \le |\mathcal{X}_r| \le T_r + 1 \le 3\log a_r, \quad u\in\mathbb{R}. \quad (6.60)$$

Here, we have used $\log a_r = (T_r + T_r^{1-\lambda})/2$.

Define

$$\Delta_r^+ := \Big\{(v,t):\ (\log\log a_r)^{-2} \le v,t \le (\log\log a_r)^2,\quad |v-t| \le \omega_r\Big\}$$

and

$$J_r(x,y) := \frac{1}{4\pi^2}\iint_{\Delta_r^+} \frac{\mathbb{1}_{\mathcal{A}_r(v)}F_{a_r}(v)x^{\mathrm{i}v}\mathbb{1}_{\mathcal{A}_r(t)}\overline{F_{a_r}(t)}y^{-\mathrm{i}t}}{(1/2+\mathrm{i}v)(1/2-\mathrm{i}t)}K_r(v-t)\,\mathrm{d}v\,\mathrm{d}t. \quad (6.61)$$

Since $\omega_r < (\log\log a_r)^{-2}$, the integration domain Δ_r in the definition of $I_r(x,y)$ is the disjoint union of Δ_r^+ and $-\Delta_r^+$. The identities

$$F_{a_r}(-t) = \overline{F_{a_r}(t)},\quad \mathcal{A}_r(-t) = \mathcal{A}_r(t),\quad K_r(-t) = \overline{K_r(t)},\quad t\in\mathbb{R}$$

ensure that $I_r(x,y) = 2\,\mathrm{Re}\,J_r(x,y)$ and thereupon

$$\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|I_r(x,y)|^n \le 2^n \max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|J_r(x,y)|^n. \quad (6.62)$$

We shall use the bound

$$\frac{1}{(1/4+v^2)^{1/2}(1/4+t^2)^{1/2}} \le 4,\quad (v,t)\in\Delta_r^+. \quad (6.63)$$

Put

$$\widehat{F}_r(t) := \frac{\mathbb{1}_{\mathcal{A}_r(t)} F_{a_r}(t)}{1/2 + \mathrm{i}t}, \quad \mathscr{H}_r(v,t) := \widehat{F}_r(v)\overline{\widehat{F}_r(t)} K_r(v-t),$$

$$\sigma_j := \begin{cases} 1, & 1 \le j \le k, \\ -1, & k < j \le n, \end{cases} \quad \text{and} \quad s(\boldsymbol{t}) := \sum_{j=1}^{n} \sigma_j t_j.$$

Writing $\mathrm{d}\boldsymbol{v} := \mathrm{d}v_1 \cdots \mathrm{d}v_n$ and $\mathrm{d}\boldsymbol{t} := \mathrm{d}t_1 \cdots \mathrm{d}t_n$, an expansion of $J_r(x,y)^k \overline{J_r(x,y)}^k$ gives

$$\sum_{y \in \mathcal{X}_r} |J_r(x,y)|^n = \frac{1}{(4\pi^2)^n} \int_{(\Delta_r^+)^n} \prod_{j=1}^{k} \mathscr{H}_r(v_j,t_j) \prod_{j=k+1}^{n} \overline{\mathscr{H}_r(v_j,t_j)} \times x^{\mathrm{i}s(\boldsymbol{v})} \Big(\sum_{y \in \mathcal{X}_r} y^{-\mathrm{i}s(\boldsymbol{t})} \Big) \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}\boldsymbol{t}. \quad (6.64)$$

By (6.46),

$$\Big| \sum_{y \in \mathcal{X}_r} y^{-\mathrm{i}s(\boldsymbol{t})} \Big| = \Big| \sum_{l=0}^{|\mathcal{X}_r|-1} \mathrm{e}^{-\mathrm{i}ls(\boldsymbol{t})} \Big| \le D_{|\mathcal{X}_r|}(s(\boldsymbol{t})).$$

Consequently, with $P_r(\boldsymbol{t}) := \prod_{j=1}^{n} \mathbb{1}_{\mathcal{A}_r(t_j)} |F_{a_r}(t_j)|$,

$$\sum_{y \in \mathcal{X}_r} |J_r(x,y)|^n \le \pi^{-2n} \int_{(\Delta_r^+)^n} P_r(\boldsymbol{v}) P_r(\boldsymbol{t}) D_{|\mathcal{X}_r|}(s(\boldsymbol{t})) \prod_{j=1}^{n} |K_r(v_j - t_j)| \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}\boldsymbol{t}. \quad (6.65)$$

Using $P_r(\boldsymbol{v})P_r(\boldsymbol{t}) \le ((P_r(\boldsymbol{v}))^2 + (P_r(\boldsymbol{t}))^2)/2$, we bound the integral in (6.65) by the sum of two terms. In the term containing $(P_r(\boldsymbol{v}))^2$, we interchange $\boldsymbol{v}$ and $\boldsymbol{t}$. The domain is unchanged, and $|K_r(t_j - v_j)| = |K_r(v_j - t_j)|$. Furthermore,

$$|s(\boldsymbol{v}) - s(\boldsymbol{t})| \le \sum_{j=1}^{n} |v_j - t_j| \le n\omega_r.$$

With this at hand, the definition of $\mathcal{K}_{r,n}$ secures

$$D_{|\mathcal{X}_r|}(s(\boldsymbol{v})) \le \mathcal{K}_{r,n}(s(\boldsymbol{t})) \quad \text{and} \quad D_{|\mathcal{X}_r|}(s(\boldsymbol{t})) \le \mathcal{K}_{r,n}(s(\boldsymbol{t})).$$

As a result,

$$\sum_{y \in \mathcal{X}_r} |J_r(x,y)|^n \le \pi^{-2n} \int_{(\Delta_r^+)^n} (P_r(\boldsymbol{t}))^2 \mathcal{K}_{r,n}(s(\boldsymbol{t})) \prod_{j=1}^{n} |K_r(v_j - t_j)| \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}\boldsymbol{t}.$$

For fixed $\boldsymbol{t} \in [(\log\log a_r)^{-2}, (\log\log a_r)^2]^n$, enlarging each range of v_j to $[t_j - \omega_r, t_j + \omega_r]$ gives

$$\int \prod_{j=1}^{n} |K_r(v_j - t_j)| \, \mathrm{d}\boldsymbol{v} \le \prod_{j=1}^{n} \int_{t_j - \omega_r}^{t_j + \omega_r} |K_r(v_j - t_j)| \, \mathrm{d}v_j = \Lambda_r^n,$$

where

$$\Lambda_r := \int_{-\omega_r}^{\omega_r} |K_r(u)| \, \mathrm{d}u.$$

Thus, simultaneously for all $x \in \mathcal{X}_r$,

$$\sum_{y \in \mathcal{X}_r} |J_r(x,y)|^n \le \pi^{-2n} \Lambda_r^n \int_{\left[(\log\log a_r)^{-2}, (\log\log a_r)^2\right]^n} \prod_{j=1}^{n} (\mathbb{1}_{\mathcal{A}_r(t_j)} |F_{a_r}(t_j)|^2) \mathcal{K}_{r,n}(s(\boldsymbol{t})) \, \mathrm{d}\boldsymbol{t}. \quad (6.66)$$

The right-hand side does not depend on x. Taking expectations in (6.62) and invoking (6.66) and Fubini's theorem, we infer

$$\mathbb{E}\Big[\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|I_r(x,y)|^n\Big] \leq (2/\pi^2)^n\Lambda_r^n\int_{\left[(\log\log a_r)^{-2},(\log\log a_r)^2\right]^n}\mathbb{E}\Big[\prod_{j=1}^n\big(\mathbb{1}_{\mathcal{A}_r(t_j)}|F_{a_r}(t_j)|^2\big)\Big]\mathcal{K}_{r,n}(s(\boldsymbol{t}))\,\mathrm{d}\boldsymbol{t}. \quad (6.67)$$

For $s\in\mathbb{N}$, put $\Omega_s := \{\boldsymbol{t}\in\left[(\log\log a_r)^{-2},(\log\log a_r)^2\right]^s : t_1<\ldots<t_s\}$. For $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n$, define

$$\mathfrak{J}_{r,n}(\boldsymbol{\varepsilon}) := \int_{\Omega_n}\mathcal{K}_{r,n}\Big(\sum_{j=1}^n\varepsilon_j t_j\Big)\prod_{j=1}^{n-1}\chi_r(t_{j+1}-t_j)\,\mathrm{d}\boldsymbol{t}. \quad (6.68)$$

Now we estimate the integral in (6.67). To this end, we partition $\left[(\log\log a_r)^{-2},(\log\log a_r)^2\right]^n$, apart from the sets where two coordinates coincide, into the $n!$ regions corresponding to the possible orderings of the coordinates. On each region, we relabel the coordinates in increasing order. The product inside the expectation is unchanged, whereas the signs σ_j are permuted. Lemma 6.10 therefore gives

$$\int_{\left[(\log\log a_r)^{-2},(\log\log a_r)^2\right]^n}\mathbb{E}\Big[\prod_{j=1}^n\big(\mathbb{1}_{\mathcal{A}_r(t_j)}|F_{a_r}(t_j)|^2\big)\Big]\mathcal{K}_{r,n}(s(\boldsymbol{t}))\,\mathrm{d}\boldsymbol{t} \leq n!(Ck)^{Ck}(\log a_r)^n\max_{\boldsymbol{\varepsilon}\in\{-1,1\}^n}\mathfrak{J}_{r,n}(\boldsymbol{\varepsilon}). \quad (6.69)$$

Next, we work towards estimating (6.68). For $\boldsymbol{t}\in\Omega_n$, put $x_j := t_{j+1}-t_j$ and, for $1\leq i\leq n$, put

$$E_i := \{i-1,i\}\cap\{1,\ldots,n-1\}$$

and

$$\Omega_{n,i} := \{\boldsymbol{t}\in\Omega_n : \ x_j\geq(\log a_r)^{-b} \ \text{ for all } j\in E_i\}.$$

Thus $|E_i|=1$ if $i=1$ or $i=n$, and $|E_i|=2$ otherwise. Finally, put $\Omega_{n,0} := \Omega_n\setminus\bigcup_{i=1}^n\Omega_{n,i}$. Since the integrand in (6.68) is nonnegative,

$$\mathfrak{J}_{r,n}(\boldsymbol{\varepsilon})\leq\sum_{i=1}^n\int_{\Omega_{n,i}}\mathcal{K}_{r,n}\Big(\sum_{j=1}^n\varepsilon_j t_j\Big)\prod_{j=1}^{n-1}\chi_r(x_j)\,\mathrm{d}\boldsymbol{t}+\int_{\Omega_{n,0}}\mathcal{K}_{r,n}\left(\sum_{j=1}^n\varepsilon_j t_j\right)\prod_{j=1}^{n-1}\chi_r(x_j)\,\mathrm{d}\boldsymbol{t}. \quad (6.70)$$

Fix $1\leq i\leq n$. On $\Omega_{n,i}$, we have $\prod_{j\in E_i}\chi_r(x_j)\leq(\max_{(\log a_r)^{-b}\leq x\leq(\log\log a_r)^2}\chi_r(x))^{|E_i|}$. After using this bound, the remaining factors χ_r do not depend on t_i. For fixed values of the other coordinates, (6.59) gives

$$\int_{[(\log\log a_r)^{-2},(\log\log a_r)^2]}\mathcal{K}_{r,n}\left(\varepsilon_i t_i+\sum_{j\neq i}\varepsilon_j t_j\right)\,\mathrm{d}t_i\leq C_1(\log\log a_r)^3.$$

Here, the substitution $u=\varepsilon_i t_i+\sum_{j\neq i}\varepsilon_j t_j$ maps the integration interval onto an interval of the same length.

For $s\in\mathbb{N}$, define

$$V_{r,s} := \int_{\Omega_s}\prod_{j=1}^{s-1}\chi_r(t_{j+1}-t_j)\,\mathrm{d}t_1\ldots\mathrm{d}t_s$$

and put $V_{r,0} := 1$. The change of variables $(t_1, \dots, t_s) \mapsto (t_1, t_2 - t_1, \dots, t_s - t_{s-1})$ has unit Jacobian. Therefore,

$$V_{r,s} \le \big((\log\log a_r)^2 - (\log\log a_r)^{-2}\big)\Big(\int_0^{(\log\log a_r)^2} \chi_r(x)\,\mathrm{d}x\Big)^{s-1} \\ \le (\log\log a_r)^2\Big(\int_0^{(\log\log a_r)^2} \chi_r(x)\,\mathrm{d}x\Big)^{s-1}, \quad s \in \mathbb{N}. \tag{6.71}$$

After integrating over t_i, we discard the restrictions involving the relative positions of the coordinates to its left and to its right. The remaining domain is contained in $\Omega_{i-1} \times \Omega_{n-i}$. Consequently,

$$\int_{\Omega_{n,i}} \mathcal{K}_{r,n}\Big(\sum_{j=1}^n \varepsilon_j t_j\Big) \prod_{j=1}^{n-1} \chi_r(x_j)\ \mathrm{d}\boldsymbol{t} \\ \le C_1(\log\log a_r)^3\Big(\max_{(\log a_r)^{-b} \le x \le (\log\log a_r)^2} \chi_r(x)\Big)^{|E_i|} V_{r,i-1} V_{r,n-i} \\ \le C_1(\log\log a_r)^{3+2|E_i|}\Big(\max_{(\log a_r)^{-b} \le x \le (\log\log a_r)^2} \chi_r(x)\Big)^{|E_i|}\Big(\int_0^{(\log\log a_r)^2} \chi_r(x)\ \mathrm{d}x\Big)^{n-1-|E_i|}.$$

The last inequality follows from (6.71), because there are exactly $|E_i|$ nonempty groups of remaining coordinates. Using (6.56) and (6.57), we obtain

$$C_1(\log\log a_r)^{3+2|E_i|}\Big(\max_{(\log a_r)^{-b} \le x \le (\log\log a_r)^2} \chi_r(x)\Big)^{|E_i|}\Big(\int_0^{(\log\log a_r)^2} \chi_r(x)\ \mathrm{d}x\Big)^{n-1-|E_i|} \\ \le C_1 c_2^{n-1-|E_i|}(\log a_r)^{b|E_i|}(\log\log a_r)^{3+6|E_i|+8(n-1-|E_i|)} \le C_3^n (\log a_r)^{2b}(\log\log a_r)^{8n},$$

where $C_3 := \max(1, C_1, c_2)$. Here, we have used $1 \le |E_i| \le 2$ and

$$3 + 6|E_i| + 8(n - 1 - |E_i|) = 8n - 5 - 2|E_i| \le 8n.$$

Summing over i gives

$$\sum_{i=1}^n \int_{\Omega_{n,i}} \mathcal{K}_{r,n}\Big(\sum_{j=1}^n \varepsilon_j t_j\Big) \prod_{j=1}^{n-1} \chi_r(x_j)\ \mathrm{d}\boldsymbol{t} \le C_3^n n (\log a_r)^{2b}(\log\log a_r)^{8n}. \tag{6.72}$$

Now consider $\boldsymbol{t} \in \Omega_{n,0}$. For each i, at least one gap indexed by E_i is shorter than $(\log a_r)^{-b}$, whence

$$n \le \sum_{i=1}^n \sum_{j \in E_i} \mathbb{1}_{\{x_j < (\log a_r)^{-b}\}} = 2\sum_{j=1}^{n-1} \mathbb{1}_{\{x_j < (\log a_r)^{-b}\}}.$$

Since $n = 2k$, at least k gaps are shorter than $(\log a_r)^{-b}$. In particular,

$$\mathbb{1}_{\Omega_{n,0}} \le \sum_{\substack{J \subset \{1,\dots,n-1\} \\ |J| = k}} \prod_{j \in J} \mathbb{1}_{\{x_j < (\log a_r)^{-b}\}} \quad \text{on } \Omega_n.$$

Using (6.60) and then changing variables to $t_1, x_1, \dots, x_{n-1}$, we obtain

$$\int_{\Omega_{n,0}} \mathcal{K}_{r,n}\Big(\sum_{j=1}^n \varepsilon_j t_j\Big) \prod_{j=1}^{n-1} \chi_r(x_j)\ \mathrm{d}\boldsymbol{t} \le |\mathcal{X}_r| \sum_{\substack{J \subset \{1,\dots,n-1\} \\ |J| = k}} \int_{\Omega_n} \prod_{j=1}^{n-1} \chi_r(x_j) \prod_{j \in J} \mathbb{1}_{\{x_j < (\log a_r)^{-b}\}}\ \mathrm{d}\boldsymbol{t} \\ \le |\mathcal{X}_r|(\log\log a_r)^2 \binom{n-1}{k} \Big(\int_0^{(\log a_r)^{-b}} \chi_r(x)\mathrm{d}x\Big)^k \Big(\int_0^{(\log\log a_r)^2} \chi_r(x)\mathrm{d}x\Big)^{n-1-k}.$$

For the last inequality, we enlarged the range of each gap indexed by J to $(0,(\log a_r)^{-b})$ and that of each remaining gap to $(0,(\log\log a_r)^2)$. Since $\binom{n-1}{k}\le 2^{n-1}$ and $|\mathcal{X}_r|\le 3\log a_r$, formula (6.56) now yields

$$\int_{\Omega_{n,0}} \mathcal{K}_{r,n}\Big(\sum_{j=1}^n \varepsilon_j t_j\Big)\prod_{j=1}^{n-1}\chi_r(x_j)\,\mathrm{d}\boldsymbol{t} \le C_4^n \log a_r\,(\log\log a_r)^{2+(1-2B)k+8(n-1-k)}$$
$$\le C_4^n \log a_r\,(\log\log a_r)^{8n-2Bk}. \quad (6.73)$$

where $C_4 := 2c_1\max(c_2,3)$.

Combining (6.70), (6.72) and (6.73), we conclude that

$$\mathfrak{J}_{r,n}(\boldsymbol{\varepsilon}) \le C_5^n n(\log\log a_r)^{8n}\big((\log a_r)^{2b} + \log a_r\,(\log\log a_r)^{-2Bk}\big) \quad (6.74)$$

uniformly in $\boldsymbol{\varepsilon}$, where $C_5 := \max(C_3, C_4)$. Substituting (6.69) and (6.74) into (6.67), we obtain

$$\mathbb{E}\Big[\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|I_r(x,y)|^{2k}\Big] \le C_6^n n\,n!(Ck)^{Ck}(\Lambda_r\log a_r)^n(\log\log a_r)^{8n}$$
$$\times\big((\log a_r)^{2b} + \log a_r\,(\log\log a_r)^{-2Bk}\big),$$

where $C_6 := 2\pi^{-2}C_5$. By (6.9) and $\log a_r \le T_r \le 2\log a_r$, we have $\Lambda_r\log a_r \le C_7\log\log\log a_r$. Finally, recalling that $n = 2k$ and using $n! \le n^n$ and $\log\log\log a_r \le \log\log a_r$ for all sufficiently large r, we conclude that, for an appropriate absolute constant $C_0 \ge 1$,

$$C_6^n n\,n!(Ck)^{Ck}(\Lambda_r\log a_r)^n(\log\log a_r)^{8n} \le (C_0k)^{C_0k}(\log\log a_r)^{C_0k}.$$

This proves (6.58). □

7 Proof of (4.3)

Fix $\kappa\in(0,1)$. With C_0 as in Lemma 6.11, let $C_1 := 3C_0+10$, choose[1] $b = \kappa/8$, $B = 100 + \lceil 16(C_1+4)/\kappa\rceil$ and then put

$$k_r := \left\lfloor \frac{\log\log a_r}{B\log\log\log a_r}\right\rfloor.$$

Observe that $1\le k_r\le\lfloor\log\log a_r\rfloor$ for all sufficiently large r and thereupon

$$\log\Big[\big(2(\log\log a_r)^{3/5}\big)^{2k_r}(C_0k_r)^{C_0k_r}(\log\log a_r)^{C_0k_r}\Big] \le (2C_0+4)k_r\log\log\log a_r$$
$$\le C_1k_r\log\log\log a_r \le (C_1/B)\log\log a_r.$$

We have used the definition of k_r for the last inequality. Furthermore, $(\log\log a_r)^{-2Bk_r} = (\log a_r)^{-2+o(1)}$. With these at hand, an application of Lemma 6.11 yields

$$\mathbb{E}\Big[\big(2(\log\log a_r)^{3/5}\big)^{2k_r}\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|I_r(x,y)|^{2k_r}\Big] \le (\log a_r)^{\kappa/2} + (\log a_r)^{-1/2} \quad (7.1)$$

for all sufficiently large r. Indeed, the exponents arising from the two terms in (6.58) are

$$2b + C_1/B + o(1) < \kappa/2 \quad\text{and}\quad -1 + C_1/B + o(1) < -1/2,$$

[1]The constants C_1 and B are larger than needed for the subsequent proof. No attempt has been made to choose them optimally.

respectively.

Denote by $\mathcal{T}_r$ the event in Lemma 6.8, with the chosen B and b. Then $\lim_{r\to\infty}\mathbb{P}(\mathcal{T}_r)=1$ and, on $\mathcal{T}_r$,

$$\max_{x,y\in\mathcal{X}_r}\left|\frac{C_r(x,y)}{(xy)^{1/2}}-I_r(x,y)\right|\le(\log\log a_r)^{-5/8}.$$

Since $(\log\log a_r)^{-5/8}=o((\log\log a_r)^{-3/5})$, for all sufficiently large r, for all $x,y\in\mathcal{X}_r$,

$$\mathcal{T}_r\cap\{|C_r(x,y)|>(\log\log a_r)^{-3/5}(xy)^{1/2}\}\subseteq\{|I_r(x,y)|>(\log\log a_r)^{-3/5}/2\}.$$

Therefore, on $\mathcal{T}_r$,

$$\begin{aligned}&\max_{x\in\mathcal{X}_r}\#\left\{y\in\mathcal{X}_r\setminus\{x\}:\ |C_r(x,y)|>(\log\log a_r)^{-3/5}(xy)^{1/2}\right\}\\&\quad\le\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r\setminus\{x\}}\mathbb{1}_{\{|I_r(x,y)|>(\log\log a_r)^{-3/5}/2\}}\le\left(2(\log\log a_r)^{3/5}\right)^{2k_r}\max_{x\in\mathcal{X}_r}\sum_{y\in\mathcal{X}_r}|I_r(x,y)|^{2k_r}.\end{aligned}$$

Markov's inequality and (7.1) now give

$$\begin{aligned}&\mathbb{P}\{\max_{x\in\mathcal{X}_r}\#\{y\in\mathcal{X}_r\setminus\{x\}:\ |C_r(x,y)|>(\log\log a_r)^{-3/5}(xy)^{1/2}\}>(\log a_r)^{\kappa}\}\\&\qquad\le\mathbb{P}(\mathcal{T}_r^c)+(\log a_r)^{-\kappa/2}+(\log a_r)^{-\kappa-1/2}\ \to\ 0,\quad r\to\infty\end{aligned}$$

thereby proving (4.3).

Statement on the use of AI tools

The development of this paper involved an extended series of mathematical discussions with GPT-6 Astra (OpenAI). These discussions contributed in particular to the uniform fourth-moment estimate of Lemma 3.5 and to the conditional covariance estimate (4.3) in Proposition 4.2. GPT-6 Astra was also used during the preparation of the manuscript, for discussing the presentation, checking drafts and language editing. Claude (Anthropic) was used for proofreading and checking the final version.

The choice of the problem, the overall strategy of the proof and the formulation of the main results are due to the authors. The authors take full responsibility for the content of the paper, the correctness of its mathematical results and proofs, and any remaining errors.

References

[1] M. Aymone, S. Frómeta and R. Misturini, *Law of the iterated logarithm for a random Dirichlet series*. Electron. Commun. Probab. **25** (2020), article no. 56, 14 pp. doi:10.1214/20-ECP340.

[2] J. Basquin, *Sommes friables de fonctions multiplicatives aléatoires*. Acta Arith. **152** (2012), 243–266. doi:10.4064/aa152-3-2.

[3] A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*. Ann. Inst. Fourier (Grenoble) **20** (1970), 335–402. doi:10.5802/aif.357.

[4] D. Buraczewski and A. Iksanov, *A self-normalized law of the iterated logarithm for Rademacher random multiplicative functions*. Work in progress, 2026.

[5] R. Caich, *Almost sure upper bound for random multiplicative functions*. Acta Arith. **224** (2026), 297–325. doi:10.4064/aa250225-20-2.

[6] V. H. de la Peña, T. L. Lai and Q.-M. Shao, *Self-Normalized Processes: Limit Theory and Statistical Applications*. Probability and its Applications, Springer, Berlin, 2009. doi:10.1007/978-3-540-85636-8.

[7] B. Durkan and A. Pearce-Crump, *A sharp almost sure upper bound for partial sums of random multiplicative functions*. Preprint (2026), arXiv:2607.29429v2.

[8] R. Durrett, *Probability: Theory and Examples*, 4th edition. Cambridge University Press, Cambridge, 2010. doi:10.1017/CBO9780511779398.

[9] O. Gorodetsky and M. D. Wong, *On the limiting distribution of sums of random multiplicative functions*. Preprint (2025), arXiv:2508.12956v3.

[10] H. Gou and M. W. Xu, *Large fluctuations of extended Rademacher random multiplicative functions*. Preprint (2026), arXiv:2609.12940v2.

[11] G. Halász, *On random multiplicative functions*. In *Hubert Delange Colloquium* (Orsay, 1982), Publications Mathématiques d'Orsay 83-4, Université Paris-Sud, Orsay, 1983, pp. 74–96.

[12] P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application*. Academic Press, New York, 1980. doi:10.1016/C2013-0-10818-5.

[13] S. Hardy, *Almost sure bounds for a weighted Steinhaus random multiplicative function*. J. London Math. Soc. **110** (2024), e12979. doi:10.1112/jlms.12979.

[14] S. Hardy, *The distribution of partial sums of random multiplicative functions with a large prime factor*. Preprint (2025), arXiv:2503.06256.

[15] A. J. Harper, *Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos*. Forum Math. Pi **8** (2020), e1, 95 pp. doi:10.1017/fmp.2019.7.

[16] A. J. Harper, *Almost sure large fluctuations of random multiplicative functions*. Int. Math. Res. Not. IMRN **2023** (2023), 2095–2138. doi:10.1093/imrn/rnab299.

[17] W. B. Johnson and G. Schechtman, *Remarks on Talagrand's deviation inequality for Rademacher functions*. In *Geometric Aspects of Functional Analysis* (1989–90), Lecture Notes in Mathematics **1470**, Springer, Berlin, 1991, pp. 72–77. doi:10.1007/BFb0090214.

[18] Y.-K. Lau, G. Tenenbaum and J. Wu, *On mean values of random multiplicative functions*. Proc. Amer. Math. Soc. **141** (2013), 409–420. doi:10.1090/S0002-9939-2012-11332-2. Corrected version dated 15 October 2021: `https://tenenb.perso.math.cnrs.fr/PPP/RMF.pdf`.

[19] M. Longnecker and R. J. Serfling, *General moment and probability inequalities for the maximum partial sum*. Acta Math. Acad. Sci. Hungar. **30** (1977), 129–133. doi:10.1007/BF01895656.

[20] D. Mastrostefano, *An almost sure upper bound for random multiplicative functions on integers with a large prime factor*. Electron. J. Probab. **27** (2022), Paper No. 32, 21 pp. doi:10.1214/22-EJP751.

[21] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, Cambridge, 2007. doi:10.1017/CBO9780511618314.

[22] R. O'Donnell, *Analysis of Boolean Functions*. Cambridge University Press, Cambridge, 2014. doi:10.1017/CBO9781139814782.

[23] R. Pemantle and Y. Peres, *Critical random walk in random environment on trees*. Ann. Probab. **23** (1995), 105–140. doi:10.1214/aop/1176988379.

[24] V. V. Petrov, *Sums of Independent Random Variables*. Ergebnisse der Mathematik und ihrer Grenzgebiete **82**, Springer-Verlag, Berlin–Heidelberg–New York, 1975. doi:10.1007/978-3-642-65809-9.

[25] W. F. Stout, *A martingale analogue of Kolmogorov's law of the iterated logarithm*. Z. Wahrscheinlichkeitstheorie verw. Gebiete **15** (1970), 279–290. doi:10.1007/BF00533299.

[26] W. F. Stout, *Almost Sure Convergence*. Academic Press, New York–London, 1974.

[27] W. Verreault, *Almost sure upper bound for sums of random multiplicative functions and critical chaos*. Preprint (2026), arXiv:2608.21354v2.

[28] A. Wintner, *Random factorizations and Riemann's hypothesis*. Duke Math. J. **11** (1944), 267–275. doi:10.1215/S0012-7094-44-01122-1.