First passage times for decoupled random walks

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Abstract

Motivated by a connection to the infinite Ginibre point process, decoupled random walks were introduced in a recent article Alsmeyer, Iksanov and Kabluchko (2025). The decoupled random walk is a sequence of independent random variables, in which the nth variable has the same distribution as the position at time n of a standard random walk with nonnegative increments. We prove distributional convergence in the Skorokhod space equipped with the J_1-topology of the running maxima and the first passage times of decoupled random walks. We show that there exist five different regimes, in which distinct limit theorems arise. Rather different functional limit theorems for the number of visits of decoupled standard random walk to the interval $[0,t]$ as $t \to \infty$ were earlier obtained in the aforementioned paper Alsmeyer, Iksanov and Kabluchko (2025). While the limit processes for the first passage times are inverse extremal-like processes, the limit processes for the number of visits are stationary Gaussian.

Key words: decoupled random walk; extremal process; first passage time; functional limit theorem; tail behavior.
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1 Introduction and main results

Let $\xi_1, \xi_2, \ldots$ be independent copies of a nonnegative random variable ξ with a nondegenerate distribution. Put $S_n = \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$. The random sequence $(S_n)_{n\geq 1}$ is called *random walk* with nonnegative jumps. Let $\hat{S}_1, \hat{S}_2, \ldots$ be independent random variables such that, for each $n \in \mathbb{N}$, $\hat{S}_n$ has the same distribution as S_n. Following [3], we call the sequence $(\hat{S}_n)_{n\geq 1}$ *decoupled random walk*.

For $t \geq 0$, put

$$\hat{N}(t) := \sum_{n\geq 1} \mathbb{1}_{\{\hat{S}_n \leq t\}} \quad \text{and} \quad \hat{\tau}(t) := \inf\{n \geq 1 : \hat{S}_n > t\}.$$

Observe that, for each $t \geq 0$, both $\hat{N}(t)$ and $\hat{\tau}(t)$ are almost surely (a.s.) finite. The equality $\mathbb{P}\{\hat{N}(t) < \infty\} = 1$ is a consequence of $\sum_{n\geq 1} \mathbb{P}\{\hat{S}_n \leq t\} = \sum_{n\geq 1} \mathbb{P}\{S_n \leq t\} < \infty$. Furthermore, the inequality $\hat{\tau}(t) - 1 \leq \hat{N}(t)$ a.s. implies that $\hat{\tau}(t)$ is a.s. finite, too. The function $t \mapsto \sum_{n\geq 1} \mathbb{P}\{S_n \leq t\}$ on $[0,\infty)$ is called *renewal function*. It is a standard result of Renewal Theory

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that the renewal function is finite for all nonnegative arguments under the sole assumption $\mathbb{P}\{\xi = 0\} < 1$. We call the random processes $\hat{N} := (\hat{N}(t))_{t\geq 0}$ and $(\hat{\tau}(t))_{t\geq 0}$ *decoupled renewal process* and *decoupled first passage time process*, respectively. We note in passing that for the classical walk $(S_n)_{n\geq 1}$ with nonnegative increments one has $\inf\{n \geq 1 : S_n > t\} - 1 = \sum_{n\geq 1} \mathbb{1}_{\{S_n \leq t\}}$.

Some decoupled random walks arise naturally in an investigation of certain determinantal point processes. This connection has served as a motivation behind introducing the decoupled random walks in [3]. Now we discuss this connection in more details. Let $\mathbb{C}$ denote the set of complex numbers. For $\rho > 0$, define the kernel C_ρ by

$$C_\rho(z, w) = \frac{\rho}{2\pi} \mathrm{E}_{2/\rho,\, 2/\rho}(z\bar{w}) \mathrm{e}^{-|z|^\rho/2 - |w|^\rho/2}, \quad z, w \in \mathbb{C}.$$

Here, $\bar{w}$ denotes the complex conjugate of w and, for $a, b > 0$, $\mathrm{E}_{a,\,b}$ denotes the Mittag-Leffler function with parameters a and b given by

$$\mathrm{E}_{a,\,b}(z) := \sum_{k\geq 0} \frac{z^k}{\Gamma(ak + b)}, \quad z \in \mathbb{C},$$

and Γ is the Euler gamma-function. Denote by Θ_ρ a simple point process on $\mathbb{C}$ such that, for any $k \in \mathbb{N}$ and any mutually disjoint Borel subsets $B_1, \ldots, B_k$ of $\mathbb{C}$,

$$\mathbb{E}\Big[\prod_{j=1}^{k} \Theta(B_j)\Big] = \int_{B_1 \times \ldots \times B_k} \det(C_\rho(z_i, z_j))_{1\leq i,j\leq k}\, \mathrm{d}z_1 \ldots \mathrm{d}z_k,$$

where det denotes the determinant. The point process Θ_ρ is a *determinantal point process* with kernel C_ρ with respect to Lebesgue measure on $\mathbb{C}$. The process Θ_2 (which corresponds to $\rho = 2$) is known in the literature as the *infinite Ginibre point process*. We refer to the monograph [4], which contains a wealth of information on determinantal point processes. A discussion of the Ginibre point processes can be found in Sections 4.3.7 and 4.7 of that book and in Part I of the very recent monograph [9]. It is stated on pp. 3-4 in [1] that the set of absolute values of atoms of Θ_ρ has the same distribution as $((\hat{S}_n)^{1/\rho})_{n\geq 1}$, where $\hat{S}_1$ has the gamma distribution with parameters $2/\rho$ and 1, that is,

$$\mathbb{P}\{\hat{S}_1 \in \mathrm{d}x\} = \frac{1}{\Gamma(2/\rho)} x^{2/\rho - 1} \mathrm{e}^{-x}\, \mathbb{1}_{(0,\infty)}(x) \mathrm{d}x. \tag{1}$$

For each $t \geq 0$, let $\Theta_\rho(D_t)$ denote the number of atoms of Θ_ρ inside the disk $D_t := \{z \in \mathbb{C} : |z| < t\}$. Then

$$(\Theta_\rho(D_t))_{t\geq 0} \;\text{ has the same distribution as }\; (\hat{N}(t^\rho))_{t\geq 0} = \Big(\sum_{n\geq 1} \mathbb{1}_{\{\hat{S}_n \leq t^\rho\}}\Big)_{t\geq 0}, \tag{2}$$

with $\hat{S}_1$ as in (1). Also, we note, without going into details, that, according to Theorem 3.1 in [2], the set of absolute values of atoms of a generalized infinite Ginibre point process parameterized by $m \in \mathbb{N}$ has the same distribution as $((\hat{S}_n^{(1)} \hat{S}_n^{(2)} \cdot \ldots \cdot \hat{S}_n^{(m)})^{1/2})_{n\geq 1}$, where $(\hat{S}_n^{(1)})_{n\geq 1}, \ldots, (\hat{S}_n^{(m)})_{n\geq 1}$ are independent copies of $(\hat{S}_n)_{n\geq 1}$, with $\hat{S}_1$ having the exponential distribution of unit mean.

In [3], a functional limit theorem for $\hat{N}(t)$, properly scaled, normalized and centered, was proved under the assumption that the distribution of ξ belongs to the domain of attraction of a stable distribution with finite mean. If the distribution of ξ is exponential, the process $\hat{N}$

may be called *decoupled Poisson process*. A functional central limit theorem for the decoupled Poisson process was earlier obtained in Proposition 1.4 of [11]. A law of the *single* logarithm for the decoupled Poisson process can be found in Theorem 3.1 of [8].

In this article we give a fairly complete picture of the asymptotic properties of the distributions of both $\max_{1\le k\le n}\hat{S}_k$ and $\hat{\tau}(t)$, properly scaled, normalized and centered. There are five different regimes, which are regulated by the right tail of ξ. The first and second are treated in Theorems 1.1 and 1.6; the third in Theorems 1.2 and 1.7, the fourth in Theorems 1.3 and 1.8, and the fifth (boundary regime) in Theorems 1.5 and 1.9.

We do not see any natural applications of our present results to the point processes Θ_ρ. The reason is that, as far as we understand, the absolute values of atoms of Θ_ρ cannot be enumerated in a useful way.

For an interval I, finite or infinite, open or closed, denote by $D(I)$ the Skorokhod space, that is, the set of càdlàg functions defined on I. We shall use the standard J_1-topology on $D(I)$. Comprehensive information on the J_1-topology can be found in [6, 10]. We write $\Longrightarrow$ to denote weak convergence in a functional space.

Theorem 1.1. *Assume that*

$$\mathbb{P}\{\xi > v\} \sim v^{-\alpha}\ell(v), \quad v\to\infty \tag{3}$$

for some $\alpha\in(0,2]$ and some ℓ slowly varying at ∞. Let $a(v)$ be any positive function satisfying $\lim_{v\to\infty} v^2\mathbb{P}\{\xi > a(v)\} = 1$. Denote by (t_k, j_k) the atoms of a Poisson random measure $\mathcal{P}_\alpha^{(1)}$ on $[0,\infty)\times(0,\infty]$ with mean measure $\theta\times\nu_\alpha$, where θ and ν_α are measures on $[0,\infty)$ and $(0,\infty]$, respectively, defined by

$$\theta([0,\,x]) = x^2/2 \quad \textit{and} \quad \nu_\alpha((x,\infty]) = x^{-\alpha}.$$

If $\alpha\in(0,2)$ or $\alpha = 2$ and $\lim_{v\to\infty}\ell(v) = \infty$, then

$$\Big(\frac{\max_{1\le k\le\lfloor tv\rfloor}\hat{S}_k}{a(v)}\Big)_{t\ge0} \implies \Big(\sup_{k:\,t_k\le t} j_k\Big)_{t\ge0} =: (X_1(t))_{t\ge0}, \quad v\to\infty \tag{4}$$

in the J_1-topology on $D([0,\infty))$, whereas, if $\alpha = 2$ and $\liminf_{v\to\infty}\ell(v) < \infty$, then relation (4) holds with $\hat{S}_k - \mu k$ replacing $\hat{S}_k$, where $\mu := \mathbb{E}[\xi] < \infty$. The one-dimensional distributions of the limit process are Frêchet distributions given by

$$\mathbb{P}\{X_1(t)\le y\} = \exp(-t^2y^{-\alpha}/2), \quad y>0$$

and $\mathbb{P}\{X_1(t)\le y\} = 0$ for $y\le 0$.

A realization of $(X_1(t))_{t\ge0}$ is shown in the left panel of Figure 1.

Theorem 1.2. *Assume that (3) holds either for some $\alpha\in(2,3)$ and some ℓ slowly varying at ∞ or for $\alpha = 3$ and slowly varying ℓ satisfying $\lim_{v\to\infty}(\ell(v)/\log v) = \infty$. Let $a(v)$ be any positive function satisfying $\lim_{v\to\infty} va(v)\mathbb{P}\{\xi > a(v)\} = 1$. Denote by (t_k, j_k) the atoms of a Poisson random measure $\mathcal{P}_\alpha^{(2)}$ on $\mathbb{R}\times(0,\infty]$ with mean measure $\mathrm{Leb}\times\nu_\alpha$, where Leb denotes Lebesgue measure on $\mathbb{R}$ and ν_α is a measure $(0,\infty]$ defined by $\nu_\alpha((x,\infty]) = x^{-\alpha}$. Then*

$$\Big(\frac{\max_{1\le k\le\lfloor v+ta(v)\rfloor}\hat{S}_k - \mu v}{a(v)}\Big)_{t\in\mathbb{R}} \implies \Big(\sup_{k:\,t_k\le t}(\mu t_k + j_k)\Big)_{t\in\mathbb{R}} =: (X_2(t))_{t\in\mathbb{R}}, \quad v\to\infty \tag{5}$$

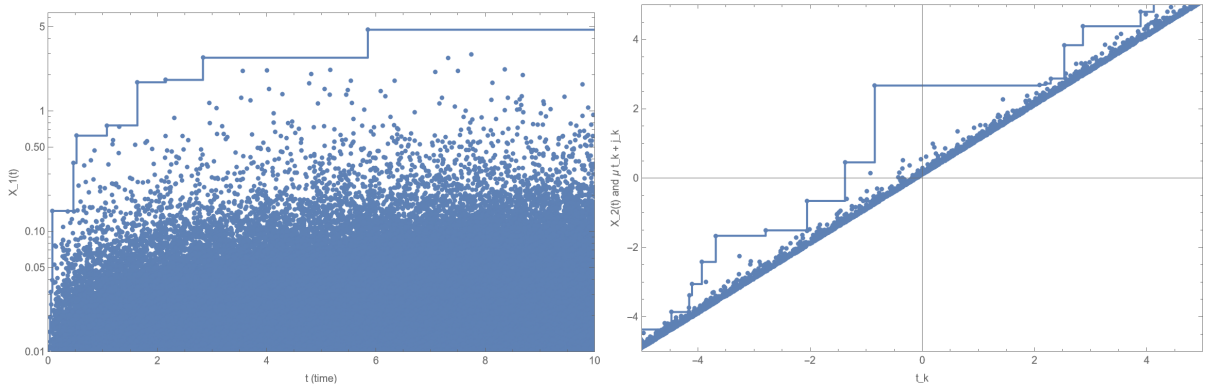

Figure 1: Left: A sample path of the process $(X_1(t))_{t\geq 0}$ from Theorem 1.1, together with the points (t_k, j_k) of the underlying Poisson random measure $\mathcal{P}_\alpha^{(1)}$. The vertical axis is displayed on a logarithmic scale. Right: A sample path of the process $(X_2(t))_{t\in\mathbb{R}}$ from Theorem 1.2, together with the points $(t_k, \mu t_k + j_k)$.

in the J_1-topology on $D(\mathbb{R})$, where $\mu = \mathbb{E}[\xi] < \infty$. The one-dimensional distributions of the limit process are (truncated) Frêchet distributions given by

$$\mathbb{P}\{X_2(t) \leq y\} = \exp\Big(-\frac{1}{\mu(\alpha-1)(y-\mu t)^{\alpha-1}}\Big), \quad t \in \mathbb{R},\ y > \mu t$$

and $\mathbb{P}\{X_2(t) \leq y\} = 0$ for $t \in \mathbb{R}$ and $y \leq \mu t$.

A realization of $(X_2(t))_{t\in\mathbb{R}}$ is shown in the right panel of Figure 1.

Let Φ denote the standard normal distribution function, that is,

$$\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y, \quad x \in \mathbb{R}.$$

The function Φ is strictly increasing and continuous. Hence, its inverse Φ^{-1} is well-defined.

Theorem 1.3. *Assume that either $\mathbb{E}[\xi^3] < \infty$ or (3) holds with $\alpha = 3$ and some ℓ slowly varying at ∞ which satisfies $\lim_{v\to\infty}(\ell(v)/\log v) = 0$. Put $a(v) := \sigma(v/\log v)^{1/2}$ and*

$$m(v) := \sigma(\mu v + v^{1/2}\Phi^{-1}(1 - 1/a(v)))$$

for $v > 1$, where $\mu = \mathbb{E}[\xi] < \infty$ and $\sigma^2 := \mathrm{Var}\,[\xi] \in (0, \infty)$. Denote by (t_k, j_k) the atoms of a Poisson random measure $\mathcal{P}^{(3)}$ on $\mathbb{R}\times\mathbb{R}$ with mean measure $\rho\times\nu$, where ρ and ν are measures on $\mathbb{R}$ defined by $\rho(\mathrm{d}x) := \mathrm{e}^{\mu x}\mathrm{d}x$ and $\nu(\mathrm{d}y) := \mathrm{e}^{-y}\mathrm{d}y$ for $x, y \in \mathbb{R}$. Then

$$\Big(\frac{\max_{1\leq k\leq \lfloor v+ta(v)\rfloor}\hat{S}_k - m(v)}{a(v)}\Big)_{t\in\mathbb{R}} \implies \Big(\sup_{k:\, t_k\leq t} j_k\Big)_{t\in\mathbb{R}} =: (X_3(t))_{t\in\mathbb{R}}, \quad v\to\infty \tag{6}$$

in the J_1-topology on $D(\mathbb{R})$. The one-dimensional distributions of the limit process are (shifted) Gumbel distributions given by

$$\mathbb{P}\{X_3(t) \leq y\} = \exp\Big(-\mu^{-1}\mathrm{e}^{\mu t - y}\Big), \quad t, y \in \mathbb{R}.$$

A sample path of $(X_3(t))_{t\in\mathbb{R}}$ is shown in the left panel of Figure 2.

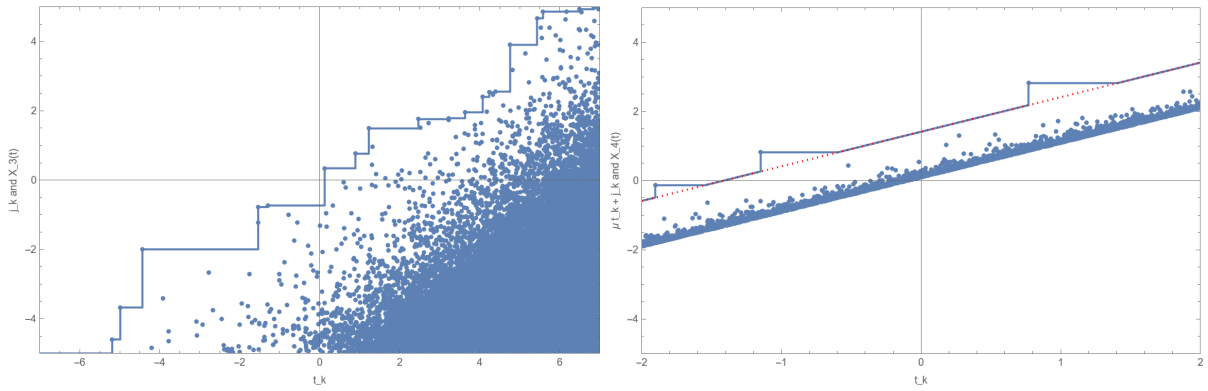

Figure 2: Left: A sample path of the process $(X_3(t))_{t\in\mathbb{R}}$ from Theorem 1.3, together with the atoms (t_k, j_k) of the underlying Poisson random measure $\mathcal{P}^{(3)}$. Right: A sample path of the process $(X_4(t))_{t\in\mathbb{R}}$ from Theorem 1.5, together with the points $(t_k, \mu t_k + j_k)$.

Remark 1.4. It can be checked that

$$\Phi^{-1}(1-h) = (2\log 1/h)^{1/2} - \frac{\log\log 1/h + \log 4\pi}{(8\log 1/h)^{1/2}} + o\Big(\frac{1}{(\log 1/h)^{1/2}}\Big), \quad h \to 0+.$$

This entails

$$\Phi^{-1}(1-1/a(v)) \;=\; (\log v)^{1/2} - \frac{\log\log v}{(\log v)^{1/2}} - \frac{\log 2\pi}{(8\log v)^{1/2}} + o\Big(\frac{1}{(\log v)^{1/2}}\Big), \quad v\to\infty.$$

Thus, the centering $m(v)$ in Theorem 1.3 could have been replaced with

$$\sigma\Big(\mu v + v^{1/2}\Big((\log v)^{1/2} - \frac{\log\log v}{(\log v)^{1/2}} - \frac{\log 2\pi}{(8\log v)^{1/2}}\Big)\Big).$$

Finally, we give a result dealing with a boundary case with respect to the cases treated in Theorems 1.2 and 1.3.

Theorem 1.5. *Assume that* $\mathbb{P}\{\xi > v\} \sim Av^{-3}\log v$ *as* $v\to\infty$ *for some* $A>0$. *Put* $a(v) = ((A/2)v\log v)^{1/2}$ *and denote by* (t_k, j_k) *the atoms of a Poisson random measure* $\mathcal{P}_3^{(2)}$ *(that is,* $\mathcal{P}_\alpha^{(2)}$ *with* $\alpha = 3$*) restricted to* $\mathbb{R}\times((2/A)^{1/2},\infty)$. *Then, as* $v\to\infty$,

$$\Big(\frac{\max_{1\le k\le \lfloor v+ta(v)\rfloor}\hat{S}_k - \mu v}{a(v)}\Big)_{t\in\mathbb{R}} \implies \big(\max\big\{\mu t + (2/A)^{1/2}, \sup_{k:\, t_k\le t}(\mu t_k + j_k)\big\}\big)_{t\in\mathbb{R}} =: (X_4(t))_{t\in\mathbb{R}} \tag{7}$$

in the J_1*-topology on* $D(\mathbb{R})$, *where* $\mu = \mathbb{E}[\xi] < \infty$. *The one-dimensional distributions of the limit process are (truncated) Frêchet distributions with atoms at* $\mu t + (2/A)^{1/2}$ *given by*

$$\mathbb{P}\{X_4(t)\le y\} = \exp\Big(-\frac{1}{2\mu(y-\mu t)^2}\Big), \quad t\in\mathbb{R},\ y\ge \mu t + (2/A)^{1/2}$$

and $\mathbb{P}\{X_4(t)\le y\} = 0$ *for* $t\in\mathbb{R}$ *and* $y < \mu t + (2/A)^{1/2}$.

A sample path of $(X_4(t))_{t\in\mathbb{R}}$ is shown in the right panel of Figure 2.

Now we present our results on the weak convergence of the first passage time processes. For a monotone nondecreasing function $f:\mathbb{R}\to\mathbb{R}$ denote by $f^{\leftarrow}$ its generalized inverse, that is, $f^{\leftarrow}(y) := \inf\{x\in\mathbb{R}:\ f(x) > y\}$ for $y\in\mathbb{R}$.

Theorem 1.6. *Under the assumptions of Theorem 1.1, if $\alpha \in (0,2)$ or $\alpha = 2$ and $\lim_{v\to\infty} \ell(v) = \infty$, then*

$$\big((\mathbb{P}\{\xi > v\})^{1/2}\hat{\tau}(tv)\big)_{t\geq 0} \implies (X_1^{\leftarrow}(t))_{t\geq 0} := \big(\inf\{z \geq 0 : \sup_{k:\, t_k \leq z} j_k > t\}\big)_{t\geq 0}$$
$$= (\inf\{t_k : \ j_k > t\})_{t\geq 0}, \quad v \to \infty \tag{8}$$

in the J_1-topology on $D([0,\infty))$, whereas, if $\alpha = 2$ and $\liminf_{v\to\infty} \ell(v) < \infty$, then relation (8) holds with $\inf\{n \geq 1 : \hat{S}_n - \mu n > t\}$ replacing $\hat{\tau}(t)$. The one-dimensional distributions of the limit process are given by

$$\mathbb{P}\{X_1^{\leftarrow}(t) \leq y\} = 1 - \exp(-t^{-\alpha}y^2/2), \quad y > 0 \tag{9}$$

and $\mathbb{P}\{X_1^{\leftarrow}(t) \leq y\} = 0$ for $y \leq 0$. Thus, the distribution of $(X_1^{\leftarrow}(t))^2$ is exponential.

Theorem 1.7. *Under the assumptions of Theorem 1.2, as $v \to \infty$,*

$$\Big(\frac{\hat{\tau}(v + ta(\mu^{-1}v)) - \mu^{-1}v}{\mu^{-1/(\alpha-1)}a(v)}\Big)_{t\in\mathbb{R}} \implies (X_2^{\leftarrow}(t))_{t\in\mathbb{R}} := \big(\inf\{z \in \mathbb{R} : \sup_{k:\, t_k \leq z} (\mu t_k + j_k) > t\}\big)_{t\in\mathbb{R}}$$
$$= (\inf\{t_k : \ \mu t_k + j_k > t\})_{t\in\mathbb{R}} \tag{10}$$

in the J_1-topology on $D(\mathbb{R})$. The one-dimensional distributions of the limit process are given by

$$\mathbb{P}\{X_2^{\leftarrow}(t) \leq y\} = 1 - \exp\Big(\frac{1}{\mu(\alpha-1)(t-\mu y)^{\alpha-1}}\Big), \quad t \in \mathbb{R},\ y < \mu^{-1}t \tag{11}$$

and $\mathbb{P}\{X_2^{\leftarrow}(t) \leq y\} = 1$ for $t \in \mathbb{R}$ and $y \geq \mu^{-1}t$.

Theorem 1.8. *Under the assumptions of Theorem 1.3, as $v \to \infty$,*

$$\Big(\frac{\hat{\tau}(m(v) + ta(v)) - v}{a(v)}\Big)_{t\in\mathbb{R}} \implies (X_3^{\leftarrow}(t))_{t\in\mathbb{R}} := \big(\inf\{z \in \mathbb{R} : \sup_{k:\, t_k \leq z} j_k > t\}\big)_{t\in\mathbb{R}}$$
$$= \big(\inf\{t_k : \ j_k > t\}\big)_{t\in\mathbb{R}}$$

in the J_1-topology on $D(\mathbb{R})$. The one-dimensional distributions of the limit process are given by

$$\mathbb{P}\{X_3^{\leftarrow}(t) \leq y\} = 1 - \exp\Big(-\mu^{-1}\mathrm{e}^{\mu y - t}\Big), \quad t, y \in \mathbb{R}.$$

Theorem 1.9. *Under the assumptions of Theorem 1.5, as $v \to \infty$,*

$$\Big(\frac{\hat{\tau}(v + ta(\mu^{-1}v)) - \mu^{-1}v}{\mu^{-1/2}a(v)}\Big)_{t\in\mathbb{R}} \implies (X_4^{\leftarrow}(t))_{t\in\mathbb{R}}$$
$$:= \big(\min\{\mu^{-1}(t - (2/A)^{1/2}), \inf\{t_k : \ \mu t_k + j_k > t\}\}\big)_{t\in\mathbb{R}} \tag{12}$$

in the J_1-topology on $D(\mathbb{R})$. The one-dimensional distributions of the limit process are given by

$$\mathbb{P}\{X_4^{\leftarrow}(t) \leq y\} = 1 - \exp\Big(\frac{1}{2\mu(t-\mu y)^2}\Big), \quad t \in \mathbb{R},\ y < \mu^{-1}(t - (2/A)^{1/2}) \tag{13}$$

and $\mathbb{P}\{X_4^{\leftarrow}(t) \leq y\} = 1$ for $t \in \mathbb{R}$ and $y \geq \mu^{-1}(t - (2/A)^{1/2})$.

2 The first passage time into (t,∞) vs the number of visits to $(-\infty,t]$

Let $(S_n)_{n\ge1}$ be a standard random walk, with not necessarily nonnegative jumps. For $t\in\mathbb{R}$, put

$$N(t):=\sum_{n\ge1}\mathbb{1}_{\{S_n\le t\}}\quad\text{and}\quad\tau(t):=\inf\{n\ge1:S_n>t\}.$$

If $\mu=\mathbb{E}[S_1]\in(0,\infty)$, then, for each $t\in\mathbb{R}$, these random variables are a.s. finite. It is known, see, for instance, Proposition A.1 in [13], that if $\mu\in(0,\infty)$ and $\sigma^2=\mathrm{Var}\,[S_1]\in(0,\infty)$, then

$$\Big(\frac{M(vt)-\mu^{-1}vt}{(\sigma^2\mu^{-3}t)^{1/2}}\Big)_{t\ge0}\implies(B(t))_{t\ge0},\quad v\to\infty$$

in the J_1-topology on $D([0,\infty))$. Here, M is either N or τ, and B is a standard Brownian motion. In words, the distributional asymptotic behaviors of the number of visits to $(-\infty,t]$ and the first passage time into (t,∞) are identical for standard random walks with a positive mean and a finite variance. This is not totally surprising, since the major contribution to $N(t)$ is made by the positions S_k with $k\le\tau(t)-1$, whereas the contribution of the subsequent positions is $o(t^{1/2})$. The argument is particularly simple in the case of one-dimensional distributions: just write

$$N(t)-(\tau(t)-1)=\sum_{n\ge\tau(t)+1}\mathbb{1}_{\{S_n\le t\}}\le\sum_{n\ge1}\mathbb{1}_{\{S_{n+\tau(t)}-S_{\tau(t)}\le0\}}\quad\text{a.s.}$$

and note that the latter variable has the same distribution as $N(0)$.

As far as decoupled standard random walks are considered, the situation is drastically different. This change is caused by the fact that the sequence $(\hat S_n)_{n\ge1}$ is not monotone nondecreasing. This leads to the following effect: the number of visits $\hat N$ and the first passage time $\hat\tau$ exhibit absolutely different distributional behaviors. For instance, according to Theorem 2.1 in [3], under the assumption that the distribution of ξ belongs to the domain of attraction of a stable distribution with finite mean, the limit processes for $\hat N$ are stationary Gaussian. On the other hand, it follows from the results of the previous section that the limit processes for $\hat\tau$ are inverse extremal-like processes. Specifically, assume that $\mathbb{E}[\xi^3]<\infty$. By formula (9) in [3],

$$\Big(\frac{\hat N(\sigma^2(t+v)^2)-\mu^{-1}\sigma^2(t+v)^2}{(\mu^{-3/2}\sigma^2v)^{1/2}}\Big)_{t\ge0}\implies(W(t))_{t\ge0},\quad v\to\infty$$

in the J_1-topology on $D([0,\infty))$ provided that the function $x\mapsto\sum_{n\ge1}\mathbb{P}\{S_n\le x\}$ is Lipschitz continuous. Here, W is a centered stationary Gaussian process with explicitly known covariance. A very different distributional limit theorem for $\hat\tau$ can be found in Theorem 1.7. Again, the striking difference outlined above is not surprising. The asymptotic behaviors of $\hat N$ and $\hat\tau$ are driven by the variables $\hat S_k$ with $k=\lfloor\mu^{-1}v+tv^{1/2}\rfloor$ as t varies over $[0,\infty)$ and $k=\lfloor v+t(v/\log v)^{1/2}\rfloor$ as t varies over $\mathbb{R}$, respectively.

3 Proofs of the main results

3.1 Large deviation probabilities in the case $\alpha\le2$

Lemma 3.1. *Assume that (3) holds with some $\alpha\le2$. Then, for fixed $t,y>0$,*

$$\lim_{v\to\infty}v\mathbb{P}\{S_{\lfloor tv\rfloor}>a(v)y\}=ty^{-\alpha}\tag{14}$$

if $\mathbb{E}[\xi] = \infty$ *and*

$$\lim_{v\to\infty} v\mathbb{P}\{S_{\lfloor tv\rfloor} - \mu\lfloor tv\rfloor > a(v)y\} = ty^{-\alpha} \tag{15}$$

if $\mu = \mathbb{E}[\xi] < \infty$.

Proof. It is a standard fact (see, for instance, Lemma 6.1.3 in [12]) that a is regularly varying at ∞ of index $2/\alpha$. In particular, a grows faster than the normalizing sequence in the limit theorem for S_n. Then the desired relations in (14) and (15) in the case $\alpha < 2$ follow from Theorem 2.1 in [5].

Now we treat the case $\alpha = 2$. We start by deriving lower bounds for $\mathbb{P}\{S_n^0 > x\}$, where $S_n^0 := S_n - \mu n$ for $n \in \mathbb{N}$. Put also $\xi_k^0 := \xi_k - \mu$ for $k \in \mathbb{N}$. For each $\varepsilon > 0$ and $x > 0$,

$$\begin{aligned}
\mathbb{P}\{S_n^0 > x\} &\geq n\mathbb{P}\{\xi_1^0 > (1+\varepsilon)x\}\mathbb{P}\big\{S_{n-1}^0 \geq -\varepsilon x, \max_{j\leq n-1}\xi_j^0 \leq (1+\varepsilon)x\big\} \\
&\geq \mathbb{P}\{\xi_1^0 > (1+\varepsilon)x\}\big(1 - \mathbb{P}\{S_{n-1}^0 < -\varepsilon x\} - \mathbb{P}\big\{\max_{j\leq n-1}\xi_j^0 > (1+\varepsilon)x\big\}\big) \\
&\geq \mathbb{P}\{\xi_1^0 > (1+\varepsilon)x\}\left(1 - n\mathbb{P}\{\xi_1^0 > (1+\varepsilon)x\} - \mathbb{P}\{S_{n-1}^0 < -\varepsilon x\}\right).
\end{aligned}$$

Since $\xi_k^0 \geq -\mu$, an application of the Bernstein inequality yields

$$\mathbb{P}\{S_{n-1}^0 < -\varepsilon x\} \to 0$$

provided that $xn^{-1/2} \to \infty$. This leads to the relation

$$\liminf_{n\to\infty} \frac{\mathbb{P}\{S_n^0 > x\}}{n\mathbb{P}\{\xi_1^0 > x\}} \geq (1+\varepsilon)^{-\alpha}$$

for x satisfying $xn^{-1/2} \to \infty$ and $n\mathbb{P}\{\xi_1 > x\} \to 0$. Letting here $\varepsilon \to 0+$, we obtain

$$\liminf_{n\to\infty} \frac{\mathbb{P}\{S_n^0 > x\}}{n\mathbb{P}\{\xi_1^0 > x\}} \geq 1 \tag{16}$$

for x satisfying $xn^{-1/2} \to \infty$ and $n\mathbb{P}\{\xi_1 > x\} \to 0$.

To obtain the corresponding upper bound we apply one of the Fuk-Nagaev inequalities. By Theorem 1.2 in [15], with $t = 2$ and $y = x/r$ for some $r > 1$,

$$\mathbb{P}\{S_n^0 > x\} \leq n\mathbb{P}\{\xi_1^0 > x/r\} + \mathrm{e}^r\left(\frac{rn\sigma^2(x/r)}{x^2}\right)^r,$$

where

$$\sigma^2(y) := \mathbb{E}\left[(\xi_1^0)^2\,\mathbb{1}_{\{|\xi_1^0|\leq y\}}\right], \quad y \geq 0.$$

The assumption (3) ensures that the function σ^2 is slowly varying at infinity. Consequently, there exists a constant C_r such that

$$\mathbb{P}\{S_n^0 > x\} \leq n\mathbb{P}\{\xi_1^0 > x/r\} + C_r\left(\frac{n\sigma^2(x)}{x^2}\right)^r.$$

This bound implies that if $x \geq n^{1/2+\delta}$ with some $\delta > 0$, then

$$\limsup_{n\to\infty} \frac{\mathbb{P}\{S_n^0 > x\}}{n\mathbb{P}\{\xi_1^0 > x\}} \leq r^\alpha.$$

Letting now $r \to 1+$, we conclude that

$$\limsup_{n\to\infty} \frac{\mathbb{P}\{S_n^0 > x\}}{n\mathbb{P}\{\xi_1^0 > x\}} \le 1 \tag{17}$$

provided that $x \ge n^{1/2+\delta}$. Combining (16) and (17) and recalling that a is regularly varying of index $2/\alpha$, we infer that (15) is valid for $\alpha = 2$. Thus, the proof of Lemma 3.1 is complete. □

Now we argue that the centering $\mu\lfloor tv \rfloor$ in (15) may be omitted in some cases. Specifically, if $\alpha \in (1,2)$ or $\alpha = 1$ and $\mu < \infty$, then regular variation of a of index $2/\alpha$ entails $\lim_{v\to\infty} v^{-1}a(v) = \infty$. The latter limit relation also holds true if $\alpha = 2$ and $\lim_{v\to\infty} \ell(v) = \infty$ as a consequence of $(\ell(a(v)))^{1/2} \sim v^{-1}a(v)$ as $v \to \infty$. Summarizing, if $\alpha \in (0,2)$ or $\alpha = 2$ and $\lim_{v\to\infty} \ell(v) = \infty$, then (14) holds, whereas if $\alpha = 2$ and $\liminf_{v\to\infty} \ell(v) < \infty$, then (15) holds.

3.2 Proof of Theorem 1.1

For intervals I and J, denote by $M_p(I \times J)$ the set of Radon point measures on $I \times J$ endowed with the vague topology. Also, let $\varepsilon_{(t,x)}$ denote the Dirac measure with atom at (t,x). As a preparation, we need a result on vague convergence. Recall that the definition of the Poisson random measure $\mathcal{P}_\alpha^{(1)}$ was given in Theorem 1.1.

Proposition 3.2. *Under the assumptions of Theorem 1.1, if $\alpha \in (0,2)$ or $\alpha = 2$ and $\lim_{v\to\infty} \ell(v) = \infty$, then*

$$\sum_{k\ge1} \varepsilon_{(k/v, \hat{S}_k/a(v))} \implies \mathcal{P}_\alpha^{(1)}, \quad v \to \infty$$

on $M_p([0,\infty) \times (0,\infty])$, whereas if $\alpha = 2$ and $\liminf_{v\to\infty} \ell(v) < \infty$, then

$$\sum_{k\ge1} \varepsilon_{(k/v, (\hat{S}_k - \mu k)/a(v))} \mathbb{1}_{\{\hat{S}_k > \mu k\}} \implies \mathcal{P}_\alpha^{(1)}, \quad v \to \infty$$

on $M_p([0,\infty) \times (0,\infty])$, where $\mu = \mathbb{E}[\xi] < \infty$. 1.1.

Proof. The case where (14) prevails. According to Proposition 3.19 on p. 153 in [17], it suffices to prove that

$$\lim_{v\to\infty} \mathbb{E}\Big[\exp\Big(-\sum_{k\ge1} f\big(k/v, \hat{S}_k/a(v)\big)\Big)\Big] = \exp\Big(-\int_{[0,\infty)\times(0,\infty)} (1 - \mathrm{e}^{-f(t,y)})\theta(\mathrm{d}t)\nu_\alpha(\mathrm{d}y)\Big) \tag{18}$$

for any nonnegative continuous function f on $[0,\infty) \times (0,\infty]$ with compact support, with the convention that the sets $[0,a] \times [b,\infty]$ are compact on $[0,\infty) \times (0,\infty]$ for $a, b > 0$.

Fix any f as above. Then there exist $\lambda, \gamma > 0$ such that $f(t,y) = 0$ whenever either $t > \lambda$ or $y \in [0,\gamma)$. By the independence of $\hat{S}_1$, $\hat{S}_2, \ldots$,

$$\begin{aligned}\log \mathbb{E}\Big[\exp\Big(-\sum_{k\ge1} f(k/v, \hat{S}_k/a(v))\Big)\Big] &= -\sum_{k\ge1} \log \mathbb{E}\big[\exp(-f(k/v, S_k/a(v)))\big] \\ &= -\sum_{k\ge1} \log\Big(1 - \int_{(\gamma,\infty)} (1 - \mathrm{e}^{-f(k/v,y)})\mathbb{P}\{S_k/a(v) \in \mathrm{d}y\}\Big).\end{aligned}$$

Put $S_0 := 0$ and define $f(0,0)$ to be 0. Using this when passing to the integral we obtain

$$\sum_{k\geq 1}\int_{(\gamma,\infty)}(1-\mathrm{e}^{-f(k/v,y)})\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}$$
$$=\int_0^\lambda\int_{(\gamma,\infty)}(\mathrm{e}^{-f(t,y)}-\mathrm{e}^{-f(\lfloor tv\rfloor/v,y)})v\mathbb{P}\{S_{\lfloor tv\rfloor}/a(v)\in \mathrm{d}y\}\mathrm{d}t$$
$$+\int_0^\lambda\int_{(\gamma,\infty)}(1-\mathrm{e}^{-f(t,y)})v\mathbb{P}\{S_{\lfloor tv\rfloor}/a(v)\in \mathrm{d}y\}\mathrm{d}t =: I(v)+J(v).$$

For $\delta>0$, put $\omega_{\mathrm{e}^{-f}}(\delta) := \sup_{||\mathbf{x}-\mathbf{y}||\leq\delta}|\mathrm{e}^{-f(\mathbf{x})}-\mathrm{e}^{-f(\mathbf{y})}|$, where $\mathbf{x},\mathbf{y}\in[0,\infty)\times(0,\infty)$. Then

$$|I(v)|\leq \lambda\omega_{\mathrm{e}^{-f}}(\sup_{t\in[0,\lambda]}(t-\lfloor tv\rfloor/v))v\mathbb{P}\{S_{\lfloor\lambda v\rfloor}>a(v)\gamma\}\ \to\ 0,\quad v\to\infty.$$

This follows from (14), $\lim_{v\to\infty}\sup_{t\in[0,\lambda]}(t-\lfloor vt\rfloor/v)=0$ and uniform continuity of e^{-f}. Relation (14) ensures that, with $t>0$ fixed,

$$\lim_{v\to\infty}\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(t,y)}\big)v\mathbb{P}\{S_{\lfloor tv\rfloor}/a(v)\in \mathrm{d}y\}=t\int_{(0,\infty)}\big(1-\mathrm{e}^{-f(t,y)}\big)\nu_\alpha(\mathrm{d}y).$$

For each $t\in[0,\lambda]$, there exists a constant $C>0$ such that, for all $v>0$,

$$\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(t,y)}\big)v\mathbb{P}\{S_{\lfloor tv\rfloor}/a(v)\in \mathrm{d}y\}\leq v\mathbb{P}\{S_{\lfloor\lambda v\rfloor}>a(v)\gamma\}\leq \mathrm{C},$$

where the last inequality is justified by (14). With this at hand, an application of the Lebesgue dominated convergence theorem yields

$$\lim_{v\to\infty}\sum_{k\geq1}\int_{(\gamma,\infty)}(1-\mathrm{e}^{-f(k/v,y)})\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}=\lim_{v\to\infty}J(v)$$
$$=\int_{[0,\infty)\times(0,\infty)}(1-\mathrm{e}^{-f(t,y)})\theta(\mathrm{d}t)\nu_\alpha(\mathrm{d}y).\quad(19)$$

We claim that

$$\lim_{v\to\infty}\sup_{k\geq1}\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(k/v,y)}\big)\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}=0.\quad(20)$$

Then the inequality

$$x\leq-\log(1-x)\leq x+x^2,\quad x\in[0,1/2]$$

applies with $x=\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(k/v,y)}\big)\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}$ for large v. As a consequence of (19) and (20) we infer

$$\lim_{v\to\infty}\sum_{k\geq1}\Big(\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(k/v,y)}\big)\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}\Big)^2=0,$$

thereby completing the proof of (18).

Proof of (20). The integral is equal to 0 for $k\geq\lfloor\lambda v\rfloor+1$. For positive integer $k\leq\lfloor\lambda v\rfloor$,

$$\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f(k/v,y)}\big)\mathbb{P}\{S_k/a(v)\in \mathrm{d}y\}\leq\mathbb{P}\{S_{\lfloor\lambda v\rfloor}>a(v)\gamma\}\ \to\ 0,\quad v\to\infty$$

in view of (14).

THE CASE WHERE (15) PREVAILS. Invoking once again Proposition 3.19 on p. 153 in [17] we conclude that it is sufficient to prove that

$$\lim_{v\to\infty}\mathbb{E}\Big[\exp\Big(-\sum_{k\ge1}f\big(k/v,(\hat S_k-\mu k)/a(v)\big)\,\mathbb{1}_{\{\hat S_k>\mu k\}}\Big)\Big]$$
$$=\exp\Big(-\int_{[0,\infty)\times(0,\infty)}(1-\mathrm{e}^{-f(t,y)})\theta(\mathrm{d}t)\nu_\alpha(\mathrm{d}y)\Big)$$

for any nonnegative continuous function f on $[0,\infty)\times\mathbb{R}$ with compact support in $[0,\infty)\times(0,\infty]$ (in particular, $f(x,y)=0$ for $y\le 0$). The proof of this mimics that of (18), the only difference being that now we appeal to (15) rather than (14).

The proof of Proposition 3.2 is complete. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that $\mathcal{P}_\alpha^{(1)}$ denotes a Poisson random measure with atoms (t_k,j_k). Plainly, $\mathcal{P}_\alpha^{(1)}([0,\infty)\times(-\infty,0])=\mathcal{P}_\alpha^{(1)}(\{0\}\times(0,\infty])=0$ a.s., $\mathcal{P}_\alpha^{(1)}$ does not have clustered jumps a.s. and that, for all $t,s\ge0$, $t<s$ and all $\delta>0$, $\mathcal{P}_\alpha^{(1)}([0,t]\times[\delta,\infty])<\infty$ a.s. and $\mathcal{P}_\alpha^{(1)}((t,s)\times(0,\infty))\ge1$ a.s. Thus, $\mathcal{P}_\alpha^{(1)}$ satisfies with probability one all the assumptions imposed on ρ_0 in Proposition 4.1. The weak convergence stated in (4) is then an immediate consequence of Proposition 3.2 in combination with the Skorokhod representation theorem (which ensures that there are versions of the point measures from Proposition 3.2 converging a.s.) and Proposition 4.1 with $f(x)=0$ for $x\ge0$.

To complete the proof, we point out the marginal distributions of the limit process X_1: for $y\ge0$ and $t>0$,

$$\mathbb{P}\big\{\sup_{k:\ t_k\le t}j_k\le y\big\}=\mathbb{P}\big\{\mathcal{P}_\alpha^{(1)}\big((s,x):s\le t,x>y\big)=0\big\}$$
$$=\exp\big(-\mathbb{E}\big[\mathcal{P}_\alpha^{(1)}\big((s,x):s\le t,x>y\big)\big]\big)=\exp(-t^2y^{-\alpha}/2).$$

□

3.3 Proof of Theorem 1.2

We start with a proposition.

Proposition 3.3. *Under the assumptions of Theorem 1.2,*

$$\sum_{k\ge1}\varepsilon_{((k-v)/a(v),(\hat S_k-\mu k)/a(v))}\,\mathbb{1}_{\{\hat S_k>\mu k\}}\ \Longrightarrow\ \mathcal{P}_\alpha^{(2)},\quad v\to\infty$$

on $M_p(\mathbb{R}\times(0,\infty])$, where $\mu=\mathbb{E}[\xi]<\infty$ and $\mathcal{P}_\alpha^{(2)}$ is as defined in Theorem 1.2.

Proof. We start by observing that $a(v)=v^{1/(\alpha-1)}L(v)$ with L slowly varying at ∞. Now we show that

$$\lim_{v\to\infty}\frac{(L(v))^2}{\log v}=\infty \tag{21}$$

provided that $\alpha=3$ and $\lim_{v\to\infty}(\ell(v)/\log v)=\infty$. Indeed, by the definition of a, as $v\to\infty$,

$$1\ \leftarrow\ va(v)\mathbb{P}\{\xi>a(v)\}\ \sim\ v^{3/2}L(v)\frac{\ell(v^{1/2}L(v))}{v^{3/2}(L(v))^3}\ \sim\ \frac{\ell(v^{1/2}L(v))}{(L(v))^2}.$$

Thus, given $\varepsilon > 0$ the inequality $\ell(v^{1/2}L(v)) \le (1+\varepsilon)(L(v))^2$ holds for large v. Also, the assumption concerning ℓ ensures that given $B > 0$ the inequality $\ell(v) \ge B \log v$ holds for large v. Hence, $(L(v))^2 \ge (1+\varepsilon)^{-1} B \log(v^{1/2}L(v)) \ge (1+\varepsilon)^{-1}(B/2)\log v$ for large v, which proves (21). Therefore, under the conditions of Theorem 1.2,

$$\lim_{v\to\infty} \frac{a(v)}{(v\log v)^{1/2}} = \infty. \tag{22}$$

Applying Theorem 1.9 in [15] to the walk $(S_n - \mu n)_{n\ge 1}$, we infer

$$\mathbb{P}\{S_n - \mu n > x\} \;\sim\; n\mathbb{P}(\xi > x)$$

provided that $x > (\alpha - 2 + \delta)^{1/2}(n\log n)^{1/2}$ for some $\delta > 0$. Using this with $n = \lfloor v + ta(v)\rfloor$ and recalling the definition of a, we obtain

$$\lim_{v\to\infty} a(v)\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor} - \mu\lfloor v + ta(v)\rfloor > a(v)y\} = y^{-\alpha}. \tag{23}$$

As in the proof of Proposition 3.2, it suffices to show that

$$\lim_{v\to\infty} \mathbb{E}\Big[\exp\Big(-\sum_{k\ge 1} f\big((k-v)/a(v), (\hat S_k - \mu k)/a(v)\big)\,\mathbb{1}_{\{\hat S_k > \mu k\}}\Big)\Big] = \exp\Big(-\int_{\mathbb{R}\times(0,\infty)} (1-\mathrm{e}^{-f(t,y)})\mathrm{d}t\,\nu_\alpha(\mathrm{d}y)\Big)$$

for any nonnegative continuous function f on $\mathbb{R}\times\mathbb{R}$ with compact support in $\mathbb{R}\times(0,\infty]$ (that is, $f(x,y) = 0$ for $y \le 0$). For fixed f, there exist $\lambda, \gamma > 0$ such that $f(t,y) = 0$ whenever either $|t| > \lambda$ or $y < \gamma$. Using this and the independence of random variables $\hat S_1$, $\hat S_2, \dots$ we obtain

$$\begin{aligned}
&\log \mathbb{E}\Big[\exp\Big(-\sum_{k\ge 1} f((k-v)/a(v), (\hat S_k - \mu k)/a(v))\,\mathbb{1}_{\{\hat S_k > \mu k\}}\Big)\Big] \\
&\quad = -\sum_{k\ge 1} \log\big(\mathbb{E}\big[\exp(-f((k-v)/a(v), (S_k - \mu k)/a(v)))\,\mathbb{1}_{\{S_k > \mu k\}}\big] + \mathbb{P}\{S_k \le \mu k\}\big) \\
&\quad\quad = -\sum_{k\ge 1} \log\Big(1 - \int_{\mathbb{R}} (1-\mathrm{e}^{-f((k-v)/a(v),y)})\mathbb{P}\{(S_k - \mu k)/a(v) \in \mathrm{d}y\}\Big) \\
&\quad\quad\quad = -\sum_{k\ge 1} \log\Big(1 - \int_{(\gamma,\infty)} (1-\mathrm{e}^{-f((k-v)/a(v),y)})\mathbb{P}\{(S_k - \mu k)/a(v) \in \mathrm{d}y\}\Big).
\end{aligned}$$

Furthermore, for large v,

$$\begin{aligned}
&\sum_{k\ge 1} \int_{(\gamma,\infty)} (1-\mathrm{e}^{-f((k-v)/a(v),y)})\mathbb{P}\{(S_k - \mu k)/a(v) \in \mathrm{d}y\} \\
&= \int_{-\lambda}^{\lambda} \int_{(\gamma,\infty)} (\mathrm{e}^{-f(t,y)} - \mathrm{e}^{-f((\lfloor v+ta(v)\rfloor - v)/a(v),y)}) a(v)\mathbb{P}\{(S_{\lfloor v+ta(v)\rfloor} - \mu\lfloor v+ta(v)\rfloor)/a(v) \in \mathrm{d}y\}\mathrm{d}t \\
&\quad + \int_{-\lambda}^{\lambda} \int_{(\gamma,\infty)} (1-\mathrm{e}^{-f(t,y)}) a(v)\mathbb{P}\{(S_{\lfloor v+ta(v)\rfloor} - \mu\lfloor v+ta(v)\rfloor)/a(v) \in \mathrm{d}y\}\mathrm{d}t =: I(v) + J(v).
\end{aligned}$$

Relations (14) and $\lim_{v\to\infty}\sup_{t\in[-\lambda,\lambda]}(t-(\lfloor v+ta(v)\rfloor-v)/a(v))=0$ together with uniform continuity of e^{-f} entail

$$\begin{aligned}&|I(v)|\\&\le\omega_{\mathrm{e}^{-f}}\Big(\sup_{t\in[-\lambda,\lambda]}(t-(\lfloor v+ta(v)\rfloor-v)/a(v))\Big)\int_{-\lambda}^{\lambda}a(v)\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor}-\mu\lfloor v+ta(v)\rfloor>a(v)\gamma\}\mathrm{d}t\\&\qquad\qquad\to\ 0,\quad v\to\infty.\end{aligned}$$

We omit a proof of the limit relation

$$\lim_{v\to\infty}J(v)=\int_{\mathbb{R}\times(0,\infty)}(1-\mathrm{e}^{-f(t,y)})\mathrm{d}t\,\nu_\alpha(\mathrm{d}y).$$

It can be justified along the lines of the proof of the corresponding fragment of Proposition 3.2.

Referring back to the proof of Proposition 3.2 we conclude that a counterpart of (20) has to be obtained, whereas the remainder of the proof mimics that of Proposition 3.2. Thus, we are left with showing that

$$\lim_{v\to\infty}\sup_{k\ge1}\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f((k-v)/a(v),y)}\big)\mathbb{P}\{(S_k-\mu k)/a(v)\in\mathrm{d}y\}=0.$$

This is a consequence of

$$\begin{aligned}&\sup_{v-\lambda a(v)\le k\le v+\lambda a(v)}\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f((k-v)/a(v),y)}\big)\mathbb{P}\{(S_k-\mu k)/a(v)\in\mathrm{d}y\}\\&\qquad\le\mathbb{P}\Big\{\frac{\sup_{1\le k\le v+\lambda a(v)}\big(S_k-\mu k\big)}{v^{1/2}}>\frac{\gamma a(v)}{v^{1/2}}\Big\}\ \to\ 0,\quad v\to\infty.\end{aligned}$$

Here, the last relation is secured by $\lim_{v\to\infty}v^{-1/2}a(v)=\infty$ and the weak convergence as $v\to\infty$ of $\frac{\sup_{1\le k\le v+\lambda a(v)}(S_k-\mu k)}{(\mathrm{Var}\,[\xi]v)^{1/2}}$ to the absolute value of a random variable with the standard normal distribution.

The proof of Proposition 3.3 is complete. □

Proof of Theorem 1.2. We start with a representation

$$\frac{\max_{1\le k\le v+ta(v)}\hat S_k-\mu v}{a(v)}=\max_{k:\ (k-v)/a(v)\le t}\Big(\frac{\mu(k-v)}{a(v)}+\frac{\hat S_k-\mu k}{a(v)}\Big),$$

which holds for each $t\in\mathbb{R}$ and sufficiently large v. In view of this, we intend to apply Proposition 4.1 with $f(x)=\mu x$ for $x\in\mathbb{R}$ and $\rho_n=\sum_k\varepsilon_{\{(k-v_n)/a(v_n),(\hat S_k-\mu k)/a(v_n)\}}$, where $(v_n)_{n\ge1}$ is any sequence of positive numbers diverging to ∞. It can be checked that $\mathcal{P}_\alpha^{(2)}$ satisfies with probability one all the assumptions imposed on ρ_0 in Proposition 4.1. The weak convergence stated in (5) then follows from Proposition 3.3 in combination with the Skorokhod representation theorem, and Proposition 4.1.

Finally, we discuss the marginal distributions of the limit process X_2: for $t\in\mathbb{R}$ and $y>\mu t$,

$$\begin{aligned}\mathbb{P}\Big\{\sup_{k:\ t_k\le t}(\mu t_k+j_k)\le y\Big\}&=\mathbb{P}\big\{\mathcal{P}_\alpha^{(2)}\big((s,x):s\le t,\mu s+x>y\big)=0\big\}\\&=\exp\big(-\mathbb{E}\big[\mathcal{P}_\alpha^{(2)}\big((s,x):s\le t,\mu s+x>y\big)\big]\big).\end{aligned}$$

Further,

$$\mathbb{E}\big[\mathcal{P}_\alpha^{(2)}\big((s,x): s\le t, \mu s + x > y\big)\big] = \int_{-\infty}^t \int_{[0,\infty)} \mathbb{1}_{\{\mu s + x > y\}}\, \nu_\alpha(\mathrm{d}x)\mathrm{d}s = \int_{-\infty}^t (y-\mu s)^{-\alpha}\mathrm{d}s$$
$$= (\mu(\alpha-1))^{-1}(y-\mu t)^{1-\alpha}.$$

□

3.4 Proof of Theorem 1.3

Proposition 3.4. *Under the assumptions of Theorem 1.3,*

$$\sum_{k\ge 1} \varepsilon_{((k-v)/a(v),(\hat{S}_k - m(v))/a(v))} \implies \mathcal{P}^{(3)}, \quad v\to\infty$$

on $M_p(\mathbb{R}\times(-\infty,\infty])$.

Proof. Without loss of generality, we can and do assume that $\sigma^2 = 1$.

We claim that, for fixed $t, y \in \mathbb{R}$,

$$\lim_{v\to\infty} a(v)\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor} > m(v) + ya(v)\} = \mathrm{e}^{\mu t - y}. \tag{24}$$

Assume first that $\mathbb{P}\{\xi > v\} \sim v^{-3}\ell(v)$ as $v\to\infty$ and $\lim_{v\to\infty}(\ell(v)/\log v) = 0$. According to Theorem 1.9 in [15],

$$\mathbb{P}\{S_n - \mu n > x\} = \big(1-\Phi(xn^{-1/2})\big)(1+o(1)) + n\mathbb{P}\{\xi > x\}(1+o(1)), \quad n\to\infty \tag{25}$$

uniformly in $x \ge n^{1/2}$. Using this with

$$n = \lfloor v + ta(v)\rfloor \quad \text{and} \quad x = \frac{m(v) - \mu\lfloor v+ta(v)\rfloor + ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}, \tag{26}$$

we intend to show that the normal distribution tail dominates. To this end, we first prove that, for fixed $t, y\in\mathbb{R}$,

$$\lim_{v\to\infty} a(v)\Big(1-\Phi\Big(\frac{m(v)-\mu\lfloor v+ta(v)\rfloor + ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big)\Big) = \mathrm{e}^{\mu t - y}. \tag{27}$$

Indeed, as a consequence of $1-\Phi(x) \sim (2\pi)^{-1/2}x^{-1}\mathrm{e}^{-x^2/2}$ as $x\to\infty$ and

$$\frac{m(v)-\mu v}{v^{1/2}} = \Phi^{-1}(1-1/a(v)) \ \sim\ (\log v)^{1/2}, \quad v\to\infty \tag{28}$$

(see Remark 1.4) we obtain

$$1-\Phi\Big(\frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big) \ \sim\ 1-\Phi\Big(\frac{m(v)-\mu v}{v^{1/2}} + \frac{(y-\mu t)a(v)}{v^{1/2}}\Big)$$
$$= 1-\Phi\Big(\Phi^{-1}(1-1/a(v)) + \frac{(y-\mu t)a(v)}{v^{1/2}}\Big)$$
$$\sim\ \frac{1}{(2\pi)^{1/2}\Phi^{-1}(1-1/a(v))}\mathrm{e}^{-(\Phi^{-1}(1-1/a(v)))^2/2}\mathrm{e}^{\Phi^{-1}(1-1/a(v))(\mu t-y)(\log v)^{-1/2}}$$
$$\sim\ 1-\Phi(\Phi^{-1}(1-1/a(v)))\mathrm{e}^{\mu t-y} = \frac{\mathrm{e}^{\mu t-y}}{a(v)}, \quad v\to\infty.$$

On the other hand, using (28) we conclude that

$$a(v)v\mathbb{P}\{\xi > m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)\} \sim a(v)v\mathbb{P}\{\xi > m(v)-\mu v\} \sim \frac{v^{3/2}}{(\log v)^{1/2}}\frac{\ell((v\log v)^{1/2})}{(v\log v)^{3/2}}$$
$$= \frac{\ell((v\log v)^{1/2})}{(\log v)^2} \to 0, \quad v\to\infty.$$

Here, the last limit relation is secured by $\lim_{v\to\infty}\frac{\ell(v)}{\log v}=0$. We have proved that the contribution of the normal tail dominates, and (24) follows.

Assume now that $\mathbb{E}[\xi^3]<\infty$. According to Theorem 14 on p. 125 in [16], the assumption $\mathbb{E}[\xi^3]<\infty$ ensures the existence of a universal constant $C>0$ such that, for all $x\in\mathbb{R}$ and all $n\in\mathbb{N}$,

$$\Big|\mathbb{P}\{S_n-\mu n\le xn^{1/2}\}-\Phi(x)\Big| \le \frac{C\mathbb{E}[\xi^3]}{n^{1/2}(1+|x|^3)}.$$

With the same choice of n and x as in (26) we infer

$$a(v)\Big|\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor} > m(v)+ya(v)\} - \Big(1-\Phi\Big(\frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big)\Big)\Big|$$
$$= a(v)\Big|\mathbb{P}\Big\{S_{\lfloor v+ta(v)\rfloor}-\mu\lfloor v+ta(v)\rfloor \le \frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}(\lfloor v+ta(v)\rfloor)^{1/2}\Big\}$$
$$-\Phi\Big(\frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big)\Big|$$
$$\le \frac{Ca(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big(1+\frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big)^{-1} \sim \frac{C}{(\log v)^2} \to 0, \quad v\to\infty.$$

This shows that, with $t,y\in\mathbb{R}$ fixed, as $v\to\infty$,

$$a(v)\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor} > m(v)+ya(v)\} \sim a(v)\Big(1-\Phi\Big(\frac{m(v)-\mu\lfloor v+ta(v)\rfloor+ya(v)}{(\lfloor v+ta(v)\rfloor)^{1/2}}\Big) \to \mathrm{e}^{\mu t-y}.$$

Here, the last limit relation follows from (27).

To prove the limit relation stated in Proposition 3.4 it is enough to check that

$$\lim_{v\to\infty}\mathbb{E}\Big[\exp\Big(-\sum_{k\ge1}f\big((k-v)/a(v),(\hat{S}_k-m(v))/a(v)\big)\Big)\Big]$$
$$= \exp\Big(-\int_{\mathbb{R}\times\mathbb{R}}(1-\mathrm{e}^{-f(t,y)})\mathrm{e}^{\mu t-y}\mathrm{d}t\,\mathrm{d}y\Big)$$

for any nonnegative continuous function f on $\mathbb{R}\times\mathbb{R}$ with compact support in $\mathbb{R}\times(-\infty,\infty]$. In particular, for fixed f, there exist $\lambda>0$ and $\gamma\in\mathbb{R}$ such that $f(t,y)=0$ whenever either $|t|>\lambda$ or $y<\gamma$.

The proof of the limit relation is analogous to the proof of Proposition 3.2. A slightly different argument is only needed for

$$\lim_{v\to\infty}\sup_{k\ge1}\int_{(\gamma,\infty)}\big(1-\mathrm{e}^{-f((k-v)/a(v),y)}\big)\mathbb{P}\{(S_k-m(v))/a(v)\in\mathrm{d}y\}=0.$$

This is a consequence of

$$\max_{v-\lambda a(v)\le k\le v+\lambda a(v)} \int_{(\gamma,\infty)} \big(1-\mathrm{e}^{-f((k-v)/a(v),y)}\big)\mathbb{P}\{(S_k-m(v))/a(v)\in \mathrm{d}y\}$$
$$\le \max_{1\le k\le v+\lambda a(v)} \mathbb{P}\big\{\big(S_k-m(v)\big)/a(v)>\gamma\big\} \le \mathbb{P}\{S_{\lfloor v+\lambda a(v)\rfloor}>m(v)+\gamma a(v)\} \sim \mathrm{e}^{\mu\lambda-\gamma}/a(v) \to 0$$

as $v\to\infty$. Here, the penultimate limit relation is secured by the previous part of the proof. □

Proof of Theorem 1.3. Write

$$\frac{\max_{1\le k\le\lfloor v+ta(v)\rfloor}\hat{S}_k-m(v)}{a(v)} = \max_{k\in\mathbb{N}:(k-v)/a(v)\le t}\frac{\hat{S}_k-m(v)}{a(v)}.$$

By the argument given on p. 214 in [17] the functional $T: M_p(\mathbb{R}\times(-\infty,\infty])\to D(\mathbb{R})$ defined by

$$\Big(T\sum_k \varepsilon_{(\tau_k,y_k)}(t)\Big)_{t\in\mathbb{R}} := (\max_{k:\tau_k\le t} y_k)_{t\in\mathbb{R}}$$

is almost surely continuous at $\mathcal{P}^{(3)}$ in the J_1-topology. Hence, relation (6) follows from this observation in combination with Proposition 3.4.

To find the marginal distributions of the limit process X_3, write for $t,y\in\mathbb{R}$,

$$\mathbb{E}\big[\mathcal{P}^{(3)}\big((s,x): s\le t, x>y\big)\big] = \int_{-\infty}^t \mathrm{e}^{\mu s}\mathrm{d}s\int_y^\infty \mathrm{e}^{-x}\mathrm{d}x = \mu^{-1}\mathrm{e}^{\mu t-y}.$$

Hence,

$$\mathbb{P}\Big\{\sup_{k:\ t_k\le t} j_k\le y\Big\} = \mathbb{P}\big\{\mathcal{P}^{(3)}\big((s,x): s\le t, x>y\big)=0\big\}$$
$$= \exp\big(-\mathbb{E}\big[\mathcal{P}^{(3)}\big((s,x): s\le t, x>y\big)\big]\big) = \exp(-\mu^{-1}\mathrm{e}^{\mu t-y}).$$

□

3.5 Proof of Theorem 1.5

Proposition 3.5. *Under the assumptions of Theorem 1.5,*

$$\sum_{k\ge1}\varepsilon_{((k-v)/a(v),(\hat{S}_k-\mu k)/a(v))} \implies \mathcal{P}_3^{(2)}, \quad v\to\infty$$

on $M_p(\mathbb{R}\times((2/A)^{1/2},\infty])$.

Proof. We prove that, for fixed $t\in\mathbb{R}$ and $y>0$,

$$\lim_{v\to\infty} a(v)\mathbb{P}\{S_{\lfloor v+ta(v)\rfloor}-\mu\lfloor v+ta(v)\rfloor > ya(v)\}$$
$$= \begin{cases} y^{-3}, & \text{if } y>(2/A)^{1/2},\\ (A/(4\pi))^{1/2}+(A/2)^{3/2}, & \text{if } y=(2/A)^{1/2},\\ +\infty, & \text{if } y<(2/A)^{1/2}. \end{cases} \tag{29}$$

We are going to use formula (25) with $n = \lfloor v + ta(v) \rfloor$ and $x = ya(v)$. The formula applies whenever the distribution right tail of ξ is regularly varying at ∞ of index $-\theta$ for $\theta > 2$, see Theorem 1.9 in [15].

It can be checked that $\lim_{v\to\infty} va(v)\mathbb{P}\{\xi > a(v)\} = 1$, whence

$$\lim_{v\to\infty} \lfloor v + ta(v) \rfloor a(v)\mathbb{P}\{\xi > ya(v)\} = y^{-3}, \quad y > 0.$$

Further,

$$a(v)\Big(1 - \Phi\Big(\frac{ya(v)}{v^{1/2}}\Big)\Big) \sim \frac{a(v)}{(A\pi \log v)^{1/2} y} \mathrm{e}^{-A(y^2/4)\log v} = \frac{1}{2^{1/2} y v^{Ay^2/4 - 1/2}}$$
$$\to \begin{cases} 0, & \text{if } y > (2/A)^{1/2}, \\ (A/(4\pi))^{1/2}, & \text{if } y = (2/A)^{1/2}, \\ +\infty, & \text{if } y < (2/A)^{1/2}. \end{cases}$$

With these at hand, an application of (25) yields (29).

We omit the remaining part of the proof, for it mimics the proof of Proposition 3.3. □

Proof of Theorem 1.5. Put $A_v(t) := \{k \in \mathbb{N} : k - v \le ta(v),\ \hat{S}_k - \mu k \le (2/A)^{1/2} a(v)\}$ and $B_v(t) := \{k \in \mathbb{N} : k - v \le ta(v),\ \hat{S}_k - \mu k > (2/A)^{1/2} a(v)\}$. We use a representation

$$\frac{\max_{1\le k\le \lfloor v+ta(v)\rfloor} \hat{S}_k - \mu v}{a(v)}$$
$$= \max\Big(\max_{k\in A_v(t)} \Big(\frac{\mu(k-v)}{a(v)} + \frac{\hat{S}_k - \mu k}{a(v)} \Big), \max_{k\in B_v(t)} \Big(\frac{\mu(k-v)}{a(v)} + \frac{\hat{S}_k - \mu k}{a(v)} \Big) \Big)$$
$$=: \max(Y_{1,v}(t), Y_{2,v}(t)).$$

Proposition 3.5 in combination with the Skorokhod representation theorem and Proposition 4.1 ensure that

$$(Y_{2,v}(t))_{t\in\mathbb{R}} \implies (\sup_{k:\, t_k \le t} (\mu t_k + j_k))_{t\in\mathbb{R}}, \quad v \to \infty$$

in the J_1-topology on $D(\mathbb{R})$. Thus, the claim in Theorem 1.5 follows if we can show that

$$(Y_{1,v}(t))_{t\in\mathbb{R}} \implies (\mu t + (2/A)^{1/2})_{t\in\mathbb{R}}, \quad v \to \infty \tag{30}$$

in the J_1-topology on $D(\mathbb{R})$.

Since, for each $v > 1$, $Y_{1,v}$ is a.s. nondecreasing, and the limit process is continuous, it suffices to prove weak convergence of the one-dimensional distributions. Fix any $t \in \mathbb{R}$. Since $Y_{1,v}(t) \le \mu t + (2/A)^{1/2}$ a.s., we are left with checking that $\lim_{v\to\infty} \mathbb{P}\{Y_{1,v}(t) \le y\} = 0$ for

$y < \mu t + (2/A)^{1/2}$. To this end, write

$$\begin{aligned}
\mathbb{P}\{Y_{1,v}(t) \le y\} &= \prod_{k=1}^{\lfloor v+ta(v)\rfloor} \mathbb{P}\{S_k \le \mu v + ya(v),\, S_k \le \mu k + (2/A)^{1/2} a(v)\} \\
&= \prod_{v+\mu^{-1}(y-(2/A)^{1/2})a(v)\le k\le v+ta(v)} \mathbb{P}\{S_k \le \mu v + ya(v)\} \\
&\quad \times \prod_{1\le k< v+\mu^{-1}(y-(2/A)^{1/2})a(v)} \mathbb{P}\{S_k \le \mu k + (2/A)^{1/2} a(v)\} \\
&\le \prod_{v+\mu^{-1}(y-(2/A)^{1/2})a(v)\le k\le v+ta(v)} \mathbb{P}\{S_k \le \mu v + ya(v)\} \\
&\le \exp\Big(- \sum_{v+\mu^{-1}(y-(2/A)^{1/2})a(v)\le k\le v+ta(v)} \mathbb{P}\{S_k > \mu v + ya(v)\}\Big).
\end{aligned}$$

It remains to prove that

$$\begin{aligned}
&\sum_{v+\mu^{-1}(y-(2/A)^{1/2})a(v)\le k\le v+ta(v)} \mathbb{P}\{S_k > \mu v + ya(v)\} \\
&= a(v) \int_{(\lfloor v+\mu^{-1}(y-(2/A)^{1/2})a(v)\rfloor - v)/a(v)}^{(\lfloor v+ta(v)\rfloor - v+1)/a(v)} \mathbb{P}\{S_{\lfloor v+xa(v)\rfloor} - \mu\lfloor v+xa(v)\rfloor > \mu v - \mu\lfloor v+xa(v)\rfloor + ya(v)\}\mathrm{d}x \\
&\qquad\qquad \to\ \infty, \quad v\to\infty.
\end{aligned}$$

According to (29),

$$\lim_{v\to\infty} a(v)\mathbb{P}\{S_{\lfloor v+xa(v)\rfloor} - \mu\lfloor v+xa(v)\rfloor > (y-\mu x)a(v)\} = \infty$$

whenever $y - \mu x < (2/A)^{1/2}$. By Fatou's lemma,

$$\begin{aligned}
&\liminf_{v\to\infty} a(v) \\
&\times \int_{(\lfloor v+\mu^{-1}(y-(2/A)^{1/2})a(v)\rfloor - v)/a(v)}^{(\lfloor v+ta(v)\rfloor - v+1)/a(v)} \mathbb{P}\{S_{\lfloor v+xa(v)\rfloor} - \mu\lfloor v+xa(v)\rfloor > \mu v - \mu\lfloor v+xa(v)\rfloor + ya(v)\}\mathrm{d}x \\
&\qquad \ge \int_{\mu^{-1}(y-(2/A)^{1/2})}^{t} \liminf_{v\to\infty} a(v)\mathbb{P}\{S_{\lfloor v+xa(v)\rfloor} - \mu\lfloor v+xa(v)\rfloor > (y-\mu x)a(v)\}\mathrm{d}x = \infty.
\end{aligned}$$

This completes the proof of (30).

Passing to the discussion of the marginal distributions of X_4 we first observe that $X_4(t) \ge \mu t + (2/A)^{1/2}$ a.s., so that $\mathbb{P}\{X_4(t) \le y\} = 0$ for $t \in \mathbb{R}$ and $y < \mu t + (2/A)^{1/2}$. Since we assume that the distribution functions are right-continuous, it suffices to consider the case $t \in \mathbb{R}$ and $y > \mu t + (2/A)^{1/2}$. For this range, we may copy a formula from the proof of Theorem 1.2: $\mathbb{E}\big[\mathcal{P}_3^{(2)}\big((s,x) : s \le t, \mu s + x > y\big)\big] = (2\mu)^{-1}(y-\mu t)^{-2}$. The claimed formula for $\mathbb{P}\{X_4(t) \le y\}$ in this range is an immediate consequence. □

3.6 Proof of Theorem 1.6

Recall that a varies regularly at ∞ of index $2/\alpha$ for $\alpha \in (0,2]$. By Theorem 1.8.3 in [7], there exists a continuous and strictly increasing function b satisfying $b(v) \sim a(v)$ as $v \to \infty$. As a

consequence, $\lim_{v\to\infty} v^2\mathbb{P}\{\xi > b(v)\} = 1$. Thus, without loss of generality we can and do assume that a is continuous and strictly increasing.

Assume that $\alpha \in (0,2)$ or $\alpha = 2$ and $\lim_{v\to\infty} \ell(v) = \infty$. By the Skorokhod representation theorem, after passing to a suitable probability space we may replace the weak convergence in Theorem 1.6 (respectively, Proposition 3.2) by almost sure convergence of random elements in the Skorokhod space $D([0,\infty))$ equipped with the J_1-topology (respectively, in $M_p([0,\infty)\times(0,\infty])$ equipped with the vague topology). A general result of Whitt [18] states that the map which assigns to a function $f \in D([0,\infty))$ with $\limsup_{t\to\infty} f(t) = \infty$ its generalized inverse $f^{\leftarrow}$ is continuous in the M_1-topology but not in the J_1-topology (for a counterexample, see p. 419 in [18]). Recall also that the M_1-topology is weaker than the J_1-topology. Consequently, starting from Theorem 1.6 and passing to the associated first-passage-time processes yields

$$\begin{aligned}\big(v^{-1}\hat\tau(ta(v))\big)_{t\ge0} &= \big(\inf\{z\ge0:\ \hat S_{\lfloor zv\rfloor} > ta(v)\}\big)_{t\ge0}\\ &\longrightarrow \Big(\inf\big\{z\ge0:\ \sup_{k:\,t_k\le z} j_k > t\big\}\Big)_{t\ge0} = \big(\inf\{t_k:\ j_k>t\}\big)_{t\ge0}, \qquad v\to\infty \quad (31)\end{aligned}$$

on $D([0,\infty))$ endowed with the M_1-topology. To show that the same convergence holds in the J_1-topology, we invoke Lemma 2.4(b) in [14]. Recall that $X_1(t) = \sup_{k:\,t_k\le t} j_k$ and $X_1^{\leftarrow}(j) = \inf\{t_k:\ j_k > j\}$. Let j' be a discontinuity point of $X_1^{\leftarrow}$; equivalently, j' is a value attained on a constancy interval of X_1. More precisely, there exist two atoms (t',j') and (t'',j'') of the point process $\sum_{k\ge1}\delta_{(t_k,j_k)}$ such that $X_1(t) = j'$ for all $t\in[t',t'')$, $X_1(t'-) < j'$, and $X_1(t'') > j'$. By Proposition 3.2,

$$\sum_{\ell\ge1}\varepsilon_{(\ell/v,\ \hat S_\ell/a(v))} \longrightarrow \sum_{k\ge1}\varepsilon_{(t_k,j_k)}$$

vaguely as $v\to\infty$. Hence there exist atoms $(t'(v),j'(v))$ and $(t''(v),j''(v))$ of $\sum_{\ell\ge1}\varepsilon_{(\ell/v,\ \hat S_\ell/a(v))}$ converging to (t',j') and (t'',j''), respectively, as $v\to\infty$, and for all sufficiently large v the first-passage-time process $\big(\inf\{z\ge0:\ \hat S_{\lfloor zv\rfloor} > ja(v)\}\big)_{j\ge0}$ has a jump at $j = j'(v)$ with left limit $t'(v)$ and right limit $t''(v)$. Since $t'(v)\to t'$ and $t''(v)\to t''$, the conditions of Lemma 2.4(b) in [14] are satisfied, and the M_1-convergence in (31) can be lifted to J_1-convergence.

Let a^{-1} be the inverse function of a. Observe that a is asymptotic generalized inverse of $x\mapsto 1/(\mathbb{P}\{\xi>x\})^{1/2}$, which particularly implies that $a^{-1}(v)\sim 1/(\mathbb{P}\{\xi>v\})^{1/2}$ as $v\to\infty$. Substituting now $a^{-1}(v)$ in place of v on the left-hand side of (31) we arrive at (8). The argument in the case $\alpha=2$ and $\liminf_{v\to\infty}\ell(v) < \infty$ is analogous.

For $s,u\ge0$, put $F(s,u) := \mathbb{P}\{X_1(s)\le u\} = \exp(-s^2u^{-\alpha}/2)$, where the last equality follows from Theorem 1.1. Since $X_1^{\leftarrow}$ is a nonnegative process, we conclude that $\mathbb{P}\{X_1^{\leftarrow}(t)\le y\} = 0$ for $y<0$. For $y\ge0$, formula (9) is a consequence of $\mathbb{P}\{X_1^{\leftarrow}(t)\le y\} = 1-F(y,t)$.

3.7 Proofs of Theorems 1.7 and 1.9

We only give a complete proof of Theorem 1.7. Theorem 1.9 follows along similar lines.

Using Proposition 3.3 in combination with the result of Whitt [18] mentioned above, we conclude with the help of the continuous mapping theorem that

$$\begin{aligned}\Big(\frac{\hat\tau(\mu v+ta(v))-v}{a(v)}\Big)_{t\in\mathbb{R}} &= \big(\inf\{z\in\mathbb{R}:\ \max_{1\le k\le\lfloor v+za(v)\rfloor}\hat S_k > \mu v+ta(v)\}\big)_{t\in\mathbb{R}}\\ &\Longrightarrow \Big(\inf\big\{z\in\mathbb{R}:\ \sup_{k:\ t_k\le z} j_k > t\big\}\Big)_{t\in\mathbb{R}}, \quad v\to\infty \quad (32)\end{aligned}$$

in the M_1-topology on $D(R)$. Arguing as in the proof of Theorem 1.6, this M_1-convergence can be lifted to J_1-convergence. Substituting now $\mu^{-1}v$ in place of v on the left-hand side of (32) and recalling that the function a is regularly varying at ∞ of index $1/(\alpha-1)$ we obtain (10).

Fix any $t \in \mathbb{R}$. Since $X_2(t) > \mu t$ a.s., we infer $X_2^{\leftarrow}(t) < \mu^{-1}t$ a.s. Hence, $\mathbb{P}\{X_2^{\leftarrow}(t) \le y\} = 1$ for $y \ge \mu^{-1}t$. For $y < \mu^{-1}t$, formula (11) follows from (9) and $\mathbb{P}\{X_2^{\leftarrow}(t) \le y\} = \mathbb{P}\{X_2(y) > t\}$.

3.8 Proof of Theorem 1.8

The functional $Q : M_p(\mathbb{R} \times (-\infty, \infty]) \to D(\mathbb{R})$ defined by

$$\Big(Q \sum_k \varepsilon_{(\tau_k, y_k)}(t)\Big)_{t\in\mathbb{R}} := \big(\inf\{z \in \mathbb{R} : \sup_{k:\, \tau_k \le z} y_k > t\}\big)_{t\in\mathbb{R}} = (\inf\{\tau_k : y_k > t\})_{t\in\mathbb{R}}$$

is almost surely continuous at $\mathcal{P}^{(3)}$ in the J_1-topology, see the proof of Corollary 4.21 on p. 216 in [17]. Using Proposition 3.4 in combination with this observation we conclude with the help of the continuous mapping theorem that

$$\begin{aligned}\Big(\frac{\hat\tau(m(v)+ta(v))-v}{a(v)}\Big)_{t\in\mathbb{R}} &= \big(\inf\{z \in \mathbb{R} : \max_{1\le k\le \lfloor v+za(v)\rfloor} \hat S_k > m(v)+ta(v)\}\big)_{t\in\mathbb{R}} \\ &\Longrightarrow \big(\inf\{z \in \mathbb{R} : \sup_{k:\, t_k\le z} j_k > t\}\big)_{t\in\mathbb{R}} = \big(\inf\{t_k :\ j_k > t\}\big)_{t\in\mathbb{R}}, \quad v\to\infty\end{aligned}$$

in the J_1-topology on $D(R)$.

The formula for the marginal distributions of $X_3^{\leftarrow}$ follows as in the proof of Theorem 1.7.

4 Appendix

We give a deterministic continuity result which is essentially used when passing from the vague convergence of random point measures to the weak convergence of maxima on the Skorokhod space. The result is a slight extension of Theorem 1.3.17 in [12] which covers point measures on both $[0,\infty) \times (-\infty,\infty]$ and $\mathbb{R} \times (-\infty,\infty]$ rather than just on $[0,\infty) \times (-\infty,\infty]$.

Let I denote either $\mathbb{R}$ or $[0,\infty)$ and M_p^* denote the set of point measures ρ on $I \times (-\infty,\infty]$ which satisfy

$$\rho(I_T \times ((-\infty,-\delta] \cup [\delta,\infty])) < \infty \tag{33}$$

for all $\delta > 0$ and all $T > 0$, where $I_T = [0,T]$ if $I = [0,\infty)$ and $I_T = [-T,T]$ if $I = \mathbb{R}$. The set M_p^* is endowed with the vague topology. For a continuous function $f : I \to \mathbb{R}$, define the mapping $\mathcal{F}$ from M_p^* to $D(I)$ by

$$\mathcal{F}(\rho)(t) := \begin{cases} \sup_{k:\ \theta_k \le t} (f(\theta_k) + y_k), & \text{if } \theta_k \le t \text{ for some } k, \\ f(0), & \text{otherwise,} \end{cases}$$

where $\rho = \sum_k \varepsilon_{(\theta_k, y_k)}$. Assumption (33) ensures that $\mathcal{F}(\rho) \in D(I)$. If (33) does not hold, $\mathcal{F}(\rho)$ may lose right-continuity.

Proposition 4.1. *For $n \in \mathbb{N}$, let $\rho_n \in M_p$. Assume that*

- *$\rho_0(I \times (-\infty, 0]) = 0$ and, if $I = [0,\infty)$, that $\rho_0(\{0\} \times (-\infty, +\infty]) = 0$,*
- *$\rho_0((r_1, r_2) \times (0,\infty]) \ge 1$ for all $r_1, r_2 \in I$ such that $r_1 < r_2$,*

- $\rho_0 = \sum_k \varepsilon_{\left(\theta_k^{(0)}, y_k^{(0)}\right)}$ *does not have clustered jumps, that is,* $\theta_k^{(0)} \neq \theta_j^{(0)}$ *for* $k \neq j$.

If

$$\lim_{n\to\infty} \rho_n|_{I\times(0,\infty]} = \rho_0$$

on M_p^**, then*

$$\lim_{n\to\infty} \mathcal{F}(\rho_n) = \mathcal{F}(\rho_0)$$

in the J_1*-topology on* $D(I)$.

The proof is omitted, for it only requires obvious modifications of the proof of Theorem 1.3.17 in [12].

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