Limit theorems for random Dirichlet series with summation over primes, with an application to Rademacher random multiplicative functions

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Abstract

It is shown that two conjectures put forward in the recent article Iksanov and Kostohryz (2025) are true. Namely, we prove a functional central limit theorem (FCLT) and a law of the iterated logarithm (LIL) for a random Dirichlet series $\sum_p \frac{\eta_p}{p^{1/2+s}}$ as $s \to 0+$, where η_1, $\eta_2, \ldots$ are independent identically distributed random variables with zero mean and finite variance, and $\sum_p$ denotes the summation over the prime numbers. As a consequence, an FCLT and an LIL are obtained for $\log \sum_{n\geq 1} \frac{f(n)}{n^{1/2+s}}$ as $s \to 0+$, where f is a Rademacher random multiplicative function.

Key words: functional limit theorem; law of the iterated logarithm; Rademacher random multiplicative function; random Dirichlet series

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1 Introduction and main results

Let $\eta_1, \eta_2, \ldots$ be independent copies of a random variable η with zero mean and finite variance, which live on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We present in this article limit theorems for a random Dirichlet series $X(s) := \sum_p \frac{\eta_p}{p^{1/2+s}}$ as $s \to 0+$. Here and hereafter, $\sum_p$ denotes summation over the set $\mathcal{P}$ of prime numbers. By Kolmogorov's three series theorem, for each $s > 0$, the series defining $X(s)$ converges almost surely (a.s.) and absolutely diverges a.s.

Let $\alpha \geq -1/2$. Functional central limit theorems (FCLTs) and laws of the iterated logarithm (LILs) for a random Dirichlet series $\sum_{k\geq 2} \frac{(\log k)^\alpha}{k^{1/2+s}} \eta_k$ have attracted some attention in the recent past, see [1] for an LIL in the case $\alpha = 0$ and $\mathbb{P}\{\eta = \pm 1\} = 1/2$, [4] and [12] for an FCLT and an LIL in the cases $\alpha > -1/2$ and $\alpha = -1/2$, respectively. Theorem 4.1 in [16] is an LIL-like result under the assumption that the distribution of η is symmetric γ-stable for $\gamma \in (0, 2]$.

Here are our main results. We start with an FCLT on $C[0,\infty)$ the space of real-valued continuous functions defined on $[0,\infty)$ equipped with the topology of locally uniform convergence. As usual, $\Longrightarrow$ denotes weak convergence in a function space.

Theorem 1. *Assume that* $\mathbb{E}[\eta] = 0$ *and* $\sigma^2 = \mathbb{E}[\eta^2] \in (0,\infty)$. *Then*

$$\Big(\frac{1}{(\log 1/s)^{1/2}} \sum_p \frac{\eta_p}{p^{1/2+s^t}}\Big)_{t\geq 0} \implies (\sigma B(t))_{t\geq 0}, \quad s \to 0+$$

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on $C[0,\infty)$, where $(B(t))_{t\geq 0}$ is a standard Brownian motion.

We proceed with an LIL. For a family (x_s) of real numbers denote by $C((x_s))$ the set of its limit points as $s \to 0+$.

Theorem 2. *Assume that $\mathbb{E}[\eta]=0$ and $\sigma^2=\mathbb{E}[\eta^2]\in(0,\infty)$. Then*

$$C\Big(\Big(\Big(\frac{1}{2\sigma^2 \log 1/s\ \log\log\log 1/s}\Big)^{1/2}\sum_p \frac{\eta_p}{p^{1/2+s}} : s\in(0,\mathrm{e}^{-\mathrm{e}})\Big)\Big)=[-1,1] \quad \text{a.s.} \tag{1}$$

In particular,

$$\limsup_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log\log\log 1/s}\Big)^{1/2}\sum_p \frac{\eta_p}{p^{1/2+s}}=\sqrt{2}\sigma \quad \text{a.s.} \tag{2}$$

and

$$\liminf_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log\log\log 1/s}\Big)^{1/2}\sum_p \frac{\eta_p}{p^{1/2+s}}=-\sqrt{2}\sigma \quad \text{a.s.} \tag{3}$$

Although a triple logarithm appears in formulae (2) and (3), Theorem 2 is indeed an LIL. According to relation (5) given below, $\mathbb{E}[(X(s))^2]\sim\sigma^2\log 1/s$ as $s\to 0+$, whence

$$\log\log\mathbb{E}[(X(s))^2]\ \sim\ \log\log\log 1/s, \quad s\to 0+.$$

Here, $y(s)\sim z(s)$ as $s\to 0+$ means that $\lim_{s\to 0+}(y(s)/z(s))=1$.

Theorem 2 provides a significant improvement over Lemma 2.1 in [2] and Theorem 1.3 in [6].

Theorems 1 and 2 were given as Conjectures 1 and 2 in the recent article [12]. It was shown in the cited paper that the aforestated FCLT and LIL hold true with $\sum_{k\geq 2}\frac{\eta_k}{(\log k)^{1/2}k^{1/2+s}}$ replacing $\sum_p \frac{\eta_p}{p^{1/2+s}}$. Actually, Conjectures 1 and 2 in [12] were due to a comment of one of the anonymous referees of that paper. The authors of [12] did not know, and neither do the present authors, how to *derive* the limit theorems for $\sum_p \frac{\eta_p}{p^{1/2+s}}$ from the corresponding results for $\sum_{k\geq 2}\frac{\eta_k}{(\log k)^{1/2}k^{1/2+s}}$. Let $p_1<p_2<\ldots$ be the prime numbers arranged in the order of increase. The prime number theorem ensures that $p_k\sim k\log k$ as $k\to\infty$. In view of this, one expects intuitively that

$$\sum_k \frac{\eta_{p_k}}{p_k^{1/2+s}}\asymp\sum_k\frac{\eta_{p_k}}{(k\log k)^{1/2+s}}\asymp\sum_k\frac{\eta_k}{(\log k)^{1/2}k^{1/2+s}}, \quad s\to 0+,$$

where $\asymp$ denotes some form of asymptotic closeness, without precise meaning. Not being able to make these asymptotic relations precise, we prove Theorems 1 and 2 by following the line of reasoning worked out in [12]. Naturally, technical details differ at places. To keep things simple at this point, we only note that $\#\{k\in\mathbb{N}: k\leq x\}\sim x$ as $x\to\infty$, whereas, by the prime number theorem, $\#\{p\in\mathcal{P}: p\leq x\}\sim x/\log x$.

Next, we discuss an application to Probabilistic Number Theory. To match the notation used in number-theoretic articles, we write $f(p)$ in place of η_p if $\mathbb{P}\{\eta_p=\pm 1\}=1/2$ for $p\in\mathcal{P}$. Wintner in [15] introduced a Rademacher random multiplicative function f as a model for the Möbius function. The function is defined by $f(1):=1$, $f(n):=\prod_{p|n}f(p)$ for square-free n (that is, those given by products of distinct prime numbers) and $f(n):=0$ for n, which are not square-free. Thus, for instance, $f(6)=f(2)f(3)$, $f(12)=0$ and $f(30)=f(2)f(3)f(5)$. An important open problem is to prove an LIL (or an appropriate counterpart) for $\sum_{n\leq x}f(n)$ as $x\to\infty$. As far as we know, the best one-sided results available at the moment are

$$\Big|\sum_{n\leq x}f(n)\Big|=O(x^{1/2}(\log\log x)^{3/4+\varepsilon}), \quad x\to\infty \quad \text{a.s. for all } \varepsilon>0,$$

see Theorem 1.1 in [5], and that there a.s. exist arbitrarily large values of x for which

$$\Big|\sum_{n\le x} f(n)\Big| \ge x^{1/2}(\log\log x)^{1/4+o(1)},$$

see Theorem 1 in [11]. The two aforementioned papers provide surveys of earlier works, in which weaker results have been proved. Distributional convergence of $\sum_{n\le x} f(n)$, properly normalized, as $x\to\infty$ does not seem to have been proved either, even in the sense of one-dimensional convergence. It is shown on p. 99 in [10] that the distributional limit of $(\mathbb{E}[(\sum_{n\le x} f(n))^2])^{-1/2}\sum_{n\le x} f(n)$ is not Gaussian.

Recall that a Steinhaus random multiplicative function α is defined as follows. The variables $(\alpha(p))_{p\in\mathcal{P}}$ are independent and uniformly distributed on $\{z\in\mathbb{C}: |z|=1\}$, where $\mathbb{C}$ denotes the set of complex numbers. Further, if $n=\prod_p p^{b_p}$, then $\alpha(n)=\prod_p(\alpha(p))^{b_p}$. It was very recently shown in [7], see also [9] for an important partial result, that $x^{-1/2}(\log\log x)^{1/4}\sum_{n\le x}\alpha(n)$ converges in distribution to a random multiple of a random variable with the standard complex normal distribution.

We do not contribute directly to the open problems. By an application of Theorems 1 and 2, we obtain an FCLT and an LIL for the logarithm of a random Dirichlet series with weights $f(n)$. It is known that F defined by $F(s):=\sum_{n\ge 1} f(n)n^{-1/2-s}$ for $s>0$ admits an Euler product representation

$$F(s)=\prod_p\Big(1+\frac{f(p)}{p^{1/2+s}}\Big),\quad s>0. \tag{4}$$

Since the random series $\sum_p f(p)p^{-1/2-s}$ converges a.s. and the series $\sum_p p^{-1-2s}$ converges, this representation particularly ensures that the series defining $F(s)$ converges a.s. The product on the right-hand side of the last equality is strictly positive for each $s>0$. Hence, the function $s\mapsto \log F(s)$ is real-valued.

Corollary 1.

$$\Big(\frac{1}{(\log 1/s)^{1/2}}\Big(\log\sum_{n\ge 1}\frac{f(n)}{n^{1/2+s^t}}+\frac{\log\zeta(1+2s^t)}{2}\Big)\Big)_{t\ge 0} \implies (B(t))_{t\ge 0},\quad s\to 0+$$

on $C[0,\infty)$, where $(B(t))_{t\ge 0}$ is a standard Brownian motion, and $\zeta(r):=\sum_{n\ge 1} n^{-r}$ for $r>1$.

Corollary 2.

$$C\bigg(\bigg(\Big(\frac{1}{2\log 1/s\,\log\log\log 1/s}\Big)^{1/2}\Big(\log\sum_{n\ge 1}\frac{f(n)}{n^{1/2+s}}+\frac{\log\zeta(1+2s)}{2}\Big): s\in(0,\mathrm{e}^{-\mathrm{e}})\bigg)\bigg)=[-1,1]\quad \text{a.s.}$$

Recall that, for $k\ge 2$, a positive integer number is called k-free if it has no divisor which is a perfect kth power. In particular, a 2-free number is a square-free number. Define now a random multiplicative function f by $f(1):=1$, $f(n):=\prod_{p|n} f(p)$ for k-free n and $f(n):=0$ for n, which are not k-free. For instance, if $k=3$, then $f(12)=f(3)(f(2))^2=f(3)$ and $f(24)=0$. Our last result demonstrates that, for $k\ge 3$, both FCLT and LIL for $\log\sum_{n\ge 1} f(n)n^{-1/2-s}$ as $s\to 0+$ are universal and take the forms which are slightly different from those in the case $k=2$.

Corollary 3. *Let $k\ge 3$. Then the statements of Corollaries 1 and 2 hold true with '$-$' replacing '$+$'. For instance,*

$$\Big(\frac{1}{(\log 1/s)^{1/2}}\Big(\log\sum_{n\ge 1}\frac{f(n)}{n^{1/2+s^t}}-\frac{\log\zeta(1+2s^t)}{2}\Big)\Big)_{t\ge 0} \implies (B(t))_{t\ge 0},\quad s\to 0+$$

on $C[0,\infty)$.

Remark 1. *While Corollaries 1 and 2 are results on fluctuations of* $\sum_{n\geq 1}\frac{f(n)}{n^{1/2+s}}(\sum_{k\geq 1}\frac{1}{k^{1+2s}})^{1/2}$ *as* $s\to 0+$*, Corollary 3 is concerned with fluctuations of* $\sum_{n\geq 1}\frac{f(n)}{n^{1/2+s}}(\sum_{k\geq 1}\frac{1}{k^{1+2s}})^{-1/2}$.

Sums of independent random variables indexed by prime numbers and random Euler products like F in (4) pop up frequently in the number-theoretic literature, see, for instance, Section 5 in [13] and already cited Lemma 2.1 in [2] and Theorem 1.3 in [6], and Section 3 in [8]. To be more specific, we only mention that Theorem 3.1 in [8] provides a precise asymptotic behavior with remainder of $\mathbb{P}\{\prod_{p\leq y}(1-g(p)/p)^{-1}>x\}$ and $\mathbb{P}\{\prod_{p\leq y}(1-g(p)/p)^{-1}\leq 1/x\}$ as x and y tend to ∞. Here, for $p\in\mathcal{P}$, the $g(p)$ are independent random variables with $\mathbb{P}\{g(p)=\pm 1\}=p/(2(p+1))$ and $\mathbb{P}\{g(p)=0\}=1/(p+1)$.

The remainder of the paper is organized as follows. We prove Theorems 1 and 2 in Sections 2 and 3, respectively. Our proof of Theorem 2 is necessarily more technical than that of Theorem 1. On the other hand, a chaining argument used in the proof of Theorem 1 rests heavily upon the reasoning given in the proof of Theorem 2. Summarizing, it is more convenient to prove Theorem 2 first. We explain in Section 4 how to derive Corollaries 1 and 2 from Theorems 1 and 2. Finally, a comment on the proof of Corollary 3 is given in Section 5.

2 Proof of Theorem 2

Put

$$g(s):=\mathbb{E}\Big[\Big(\sum_p\frac{\eta_p}{p^{1/2+s}}\Big)^2\Big]=\sigma^2\sum_p\frac{1}{p^{1+2s}},\quad s>0.$$

By Proposition A.3 in [6],

$$g(s)\ \sim\ \sigma^2\log(1/s),\quad s\to 0+. \tag{5}$$

We obtain separately in Propositions 1 and 2 the upper bound for lim sup and the lower bound for lim sup, respectively. The reason is that the arguments beyond these two results are essentially different. From now on, we write $\log^{(3)}$ for $\log\log\log$.

Proposition 1. *Assume that* $\mathbb{E}[\eta]=0$ *and* $\sigma^2=\mathbb{E}[\eta^2]\in(0,\infty)$*. Then*

$$\limsup_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log^{(3)}1/s}\Big)^{1/2}\sum_p\frac{\eta_p}{p^{1/2+s}}\leq\sqrt{2}\sigma\quad \text{a.s.} \tag{6}$$

and

$$\liminf_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log^{(3)}1/s}\Big)^{1/2}\sum_p\frac{\eta_p}{p^{1/2+s}}\geq-\sqrt{2}\sigma\quad \text{a.s.} \tag{7}$$

Now we explain our strategy of proving (6). We split the sum $\sum_p\frac{\eta_p}{p^{1/2+s}}$ into two fragments: initial and final. It is shown in Lemma 1 that the contribution of the initial fragment vanishes. We prove in Lemma 2 that the contribution of the final fragment vanishes, too if the variables η_p are properly truncated. Lemma 3 is concerned with the final (principal) fragment of the series, in which the variables η_p are differently truncated and also centered. Here, it is proved that the desired convergence only holds along a sequence. The most involved preparatory result is Lemma 4. It is shown in it, with the help of a chaining technique, that the convergence along a sequence obtained in Lemma 3 can be upgraded to the convergence along the real numbers. We note in passing that a complicating factor here is that the finiteness of exponential moments of η is not assumed. Under the assumption $\mathbb{E}[e^{s\eta}]<\infty$ for all $s>0$, a proof would have been much easier and shorter.

As has already been stated, our proof of Theorem 2, and particularly of Proposition 1, follows the path of the proof of Theorem 2 in [12]. The adaptation runs, for the most part, smoothly. The only exception is that an extra effort is required for proving (the most troublesome) Lemma 4.

For $s \in (0, \mathrm{e}^{-\mathrm{e}})$, put

$$L(s) = \Big(\frac{1}{2\log 1/s\, \log^{(3)} 1/s}\Big)^{1/2}.$$

Let $M : (0, \infty) \to \mathbb{N}$ denote a function satisfying $\lim_{s\to 0+} M(s) = +\infty$ and

$$\lim_{s\to 0+} \frac{M(s)}{\log 1/s} = 0. \tag{8}$$

Replacing η_p with η_p/σ we can work under the assumption that $\sigma^2 = 1$. Thus, in what follows, it is tacitly assumed that $\mathbb{E}[\eta] = 0$ and $\mathbb{E}[\eta^2] = 1$.

Lemma 1. *The following limit relation holds*

$$\lim_{s\to 0+} L(s) \sum_{p\le M(s)} \frac{\eta_p}{p^{1/2+s}} = 0 \quad \text{a.s.}$$

Proof. Let $p_1 < p_2 < \ldots$ be the rearrangement of the prime numbers in the increasing order. Put $T_0 := 0$ and $T_n := \eta_{p_1} + \ldots + \eta_{p_n}$ for $n \in \mathbb{N}$, and then $T^*(x) := \sum_{p\le x} \eta_p$ for $x \ge 1$. Observe that $T^*(x) = T_{\rho(x)}$ for $x \ge 1$, where $\rho(x) := \#\{p \in \mathcal{P} : p \le x\}$.

A functional LIL for standard random walks (see, for instance, Corollary 5.3.5 on p. 294 in [14]) entails

$$\max_{k\le n} |T_k| = O\big((n \log\log n)^{1/2}\big), \quad n \to \infty \quad \text{a.s.}$$

According to the prime number theorem, $\rho(x) \sim x/\log x$ as $x \to \infty$. A combination of these two facts yields

$$|T^*(x)| = |T_{\rho(x)}| \le \max_{1\le k\le \rho(x)} |T_k| = O\big((x \log\log x/\log x)^{1/2}\big), \quad x \to \infty \quad \text{a.s.} \tag{9}$$

Integrating by parts we infer

$$\sum_{p\le M(s)} \frac{\eta_p}{p^{1/2+s}} = \int_{(1,\, M(s)]} \frac{\mathrm{d}T^*(x)}{x^{1/2+s}} = \frac{T^*(M(s))}{(M(s))^{1/2+s}} + (1/2 + s) \int_1^{M(s)} \frac{T^*(x)}{x^{3/2+s}} \mathrm{d}x.$$

Relation (8) entails $\lim_{s\to 0+}(M(s))^s = 1$. This together with (9) enables us to conclude that, as $s \to 0+$,

$$\frac{|T^*(M(s))|}{(M(s))^{1/2+s}} \sim \frac{|T^*(M(s))|}{(M(s))^{1/2}} = O((\log M(s))^{-1/2}(\log\log M(s))^{1/2}) = o(1).$$

Since $\lim_{s\to 0+} L(s) = 0$, the latter ensures that

$$\lim_{s\to 0+} L(s) \frac{|T^*(M(s))|}{(M(s))^{1/2+s}} = 0 \quad \text{a.s.}$$

We are left with showing that

$$\lim_{s\to 0+} L(s) \int_1^{M(s)} \frac{|T^*(x)|}{x^{3/2+s}} \mathrm{d}x = 0 \quad \text{a.s.}$$

To this end, write, with the help of (9),

$$\int_1^{M(s)} \frac{|T^*(x)|}{x^{3/2+s}}\mathrm{d}x \le \big(\max_{y\le M(s)} |T^*(y)|\big)\int_1^{M(s)} \frac{\mathrm{d}x}{x^{3/2+s}} = \big(\max_{k\le \rho(M(s))} |T_k|\big)O(1)$$
$$= O((M(s)\log\log M(s)/\log M(s))^{1/2}), \quad s\to 0+ \quad \text{a.s.}$$

Finally, (8) entails

$$\lim_{s\to 0+} \frac{M(s)\log\log M(s)}{\log 1/s\ \log^{(3)} 1/s} = 0.$$

The proof of Lemma 1 is complete. □

For $p\in\mathcal{P}$, $\theta>0$ and $s\in(0,\mathrm{e}^{-\mathrm{e}})$, define the event

$$A_{p,\theta}(s) := \Big\{|\eta_p| > \frac{\theta}{\log\log 1/s}\Big(\frac{p^{1+2s}\, g(s)}{\log^{(3)} 1/s}\Big)^{1/2}\Big\}.$$

Lemma 2. *For all $\theta>0$,*

$$\lim_{s\to 0+} \sum_{p\ge M(s)+1} \frac{|\eta_p|\mathbb{1}_{A_{p,\theta}(s)}}{p^{1/2+s}} = 0 \quad \text{a.s.} \tag{10}$$

and

$$\lim_{s\to 0+} \sum_{p\ge M(s)+1} \frac{\mathbb{E}\big[|\eta_p|\mathbb{1}_{A_{p,\theta}(s)}\big]}{p^{1/2+s}} = 0. \tag{11}$$

Proof. Put $h(s) := (\log\log 1/s)(\log^{(3)} 1/s)^{1/2}$. For $s\in(0,\mathrm{e}^{-\mathrm{e}})$,

$$\sum_{p\ge M(s)+1} \frac{|\eta_p|\mathbb{1}_{A_{p,\theta}(s)}}{p^{1/2+s}} \le \sum_{p\ge M(s)+1} \frac{|\eta_p|}{p^{1/2}}\mathbb{1}_{\{p^{-1/2}|\eta_p|>\theta(g(s))^{1/2}(h(s))^{-1}\}} \quad \text{a.s.}$$

The assumption $\mathbb{E}[\eta^2]<\infty$ entails

$$\lim_{k\to\infty} k^{-1/2}|\eta_k| = 0 \quad \text{a.s.} \tag{12}$$

and thereupon $\sup_{k\ge 1}(k^{-1/2}|\eta_k|)<\infty$ a.s. Since $\lim_{s\to 0+}((g(s))^{1/2}(h(s))^{-1}) = +\infty$, we infer

$$\mathbb{1}_{\{p^{-1/2}|\eta_p|>\theta(g(s))^{1/2}(h(s))^{-1}\}} \le \mathbb{1}_{\{\sup_{k\ge 1}(k^{-1/2}|\eta_k|)>\theta(g(s))^{1/2}(h(s))^{-1}\}} = 0$$

a.s. for small s. We have proved that the sum in (10) is equal to 0 a.s. for small enough s.

Relation (11) is justified as follows:

$$\sum_{p\ge M(s)+1} \frac{\mathbb{E}\big[|\eta_p|\mathbb{1}_{A_{p,\theta}(s)}\big]}{p^{1/2+s}} \le \sum_{p\ge M(s)+1} p^{-1/2}\mathbb{E}\big[|\eta|\mathbb{1}_{\{\theta^{-1}(g(s))^{-1/2}h(s)|\eta|>p^{1/2}\}}\big]$$
$$\le \mathbb{E}\Bigg[|\eta| \sum_{p\le\lfloor\theta^{-2}(g(s))^{-1}(h(s))^2\eta^2\rfloor} p^{-1/2}\Bigg] \le \mathbb{E}\Bigg[|\eta| \sum_{k\le\lfloor\theta^{-2}(g(s))^{-1}(h(s))^2\eta^2\rfloor} k^{-1/2}\Bigg]$$
$$\le 2\theta^{-1}\mathbb{E}[\eta^2](g(s))^{-1/2}h(s) = 2\theta^{-1}(g(s))^{-1/2}h(s) \ \to\ 0, \quad s\to 0+\,.$$

The proof of Lemma 2 is complete. □

In what follows, $(A_{p,\theta}(s))^c$ denotes the complement of $A_{p,\theta}(s)$, that is, for $p \in \mathcal{P}$, $\theta > 0$ and $s \in (0, \mathrm{e}^{-\mathrm{e}})$,

$$(A_{p,\theta}(s))^c := \Big\{|\eta_p| \le \frac{\theta}{\log\log 1/s}\Big(\frac{p^{1+2s}\, g(s)}{\log^{(3)} 1/s}\Big)^{1/2}\Big\}.$$

Lemma 3. *Fix any $\gamma \in (0, (\sqrt{5}-1)/2)$, pick any $\theta = \theta(\gamma)$ satisfying*

$$(1-\gamma)(1+\gamma)^2(2-\exp(2\sqrt{2}(1+\gamma)\theta)) > 1 \tag{13}$$

and put $s_n := \exp(-\exp(n^{1-\gamma}))$ for $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} L(s_n) \sum_{p \ge M(s_n)+1} \frac{\widetilde{\eta}_{p,\theta}(s_n)}{p^{1/2+s_n}} \le 1+\gamma \quad \text{a.s.},$$

where $\widetilde{\eta}_{p,\theta}(s) := \eta_p \mathbb{1}_{(A_{p,\theta}(s))^c} - \mathbb{E}\big[\eta_p \mathbb{1}_{(A_{p,\theta}(s))^c}\big]$ for $p \in \mathcal{P}$ and $s \in (0, \mathrm{e}^{-\mathrm{e}})$.

Proof. Since $(1-\gamma)(1+\gamma)^2 > 1$ whenever $\gamma \in (0, (\sqrt{5}-1)/2)$, θ satisfying (13) does indeed exist.

Put $L^*(s) := (2g(s)\log^{(3)} 1/s)^{-1/2}$ for $s \in (0, \mathrm{e}^{-\mathrm{e}})$. Since $L^*(s) \sim L(s)$ as $s \to 0+$, we can and do prove the result, with L^* replacing L. Put

$$X(s) = L^*(s) \sum_{p \ge M(s)+1} \frac{\widetilde{\eta}_{p,\theta}(s)}{p^{1/2+s}}, \quad s \in (0, \mathrm{e}^{-\mathrm{e}}).$$

Using $\mathrm{e}^x \le 1 + x + (x^2/2)\mathrm{e}^{|x|}$ for $x \in \mathbb{R}$ and $\mathbb{E}[\widetilde{\eta}_{p,\theta}(s)] = 0$ we deduce, for $u \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{uX(s)}] &= \prod_{p \ge M(s)+1} \mathbb{E}\Big[\exp\Big(uL^*(s)\frac{\widetilde{\eta}_{p,\theta}(s)}{p^{1/2+s}}\Big)\Big] \\ &\le \prod_{p \ge M(s)+1} \Big(1 + \frac{u^2 (L^*(s))^2}{2}\frac{1}{p^{1+2s}}\mathbb{E}\Big[(\widetilde{\eta}_{p,\theta}(s))^2 \exp\Big(|u| L^*(s)\frac{|\widetilde{\eta}_{p,\theta}(s)|}{p^{1/2+s}}\Big)\Big]\Big).\end{aligned}$$

The inequality

$$\begin{aligned}|\widetilde{\eta}_{p,\theta}(s)| &\le |\eta_p|\mathbb{1}_{(A_{p,\theta}(s))^c} + \mathbb{E}\big[|\eta_p|\mathbb{1}_{(A_{p,\theta}(s))^c}\big] \\ &\le \frac{2\theta p^{1/2+s}}{\log\log 1/s}\Big(\frac{g(s)}{\log^{(3)} 1/s}\Big)^{1/2} \le 2\theta p^{1/2+s}\Big(\frac{g(s)}{\log^{(3)} 1/s}\Big)^{1/2} \quad \text{a.s.},\end{aligned} \tag{14}$$

which is valid for $p \in \mathcal{P}$ and $s \in (0, \mathrm{e}^{-\mathrm{e}})$, implies that

$$\exp\Big(|u| L^*(s)\frac{|\widetilde{\eta}_{p,\theta}(s)|}{p^{1/2+s}}\Big) \le \exp\Big(\frac{\sqrt{2}\theta|u|}{\log^{(3)} 1/s}\Big) \quad \text{a.s.}$$

Together with the inequalities $\mathbb{E}[(\widetilde{\eta}_{p,\theta}(s))^2] \le 1$ and $1 + x \le \mathrm{e}^x$ for $x \in \mathbb{R}$ this gives, for $u \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{uX(s)}] &\le \prod_{p \ge M(s)+1} \exp\Big(\frac{u^2(L^*(s))^2}{2}\frac{1}{p^{1+2s}}\exp\Big(\frac{\sqrt{2}\theta|u|}{\log^{(3)} 1/s}\Big)\Big) \\ &\le \exp\Big(\frac{u^2}{4\log^{(3)} 1/s}\exp\Big(\frac{\sqrt{2}\theta|u|}{\log^{(3)} 1/s}\Big)\Big).\end{aligned} \tag{15}$$

An application of Markov's inequality yields, for $u \geq 0$,

$$\begin{aligned}\mathbb{P}\{X(s_n) > 1+\gamma\} &\leq \mathrm{e}^{-(1+\gamma)u}\mathbb{E}[\mathrm{e}^{uX(s_n)}]\\ &\leq \exp\Big(-(1+\gamma)u + \frac{u^2}{4\log^{(3)} 1/s_n}\exp\Big(\frac{\sqrt{2}\theta u}{\log^{(3)} 1/s_n}\Big)\Big).\end{aligned}$$

Putting $u = 2(1+\gamma)\log^{(3)} 1/s_n$ we conclude that

$$\begin{aligned}\mathbb{P}\{X(s_n) > 1+\gamma\} &\leq \exp(-(1+\gamma)^2(2-\exp(2\sqrt{2}(1+\gamma)\theta))\log^{(3)} 1/s_n)\\ &= \frac{1}{n^{(1-\gamma)(1+\gamma)^2(2-\exp(2\sqrt{2}(1+\gamma)\theta))}}.\end{aligned}$$

Thus, in view of (13), $\sum_{n\geq 1}\mathbb{P}\{X(s_n) > 1+\gamma\} < \infty$, and invoking the direct part of the Borel-Cantelli lemma completes the proof of Lemma 3. □

Lemma 4. *Let $(s_n)_{n\in\mathbb{N}}$ be as defined in Lemma 3, where $\gamma \in (0, 1/2)$, and*

$$M(s) = \lfloor \log 1/s / \log\log 1/s \rfloor, \quad s \in (0, \mathrm{e}^{-\mathrm{e}}).$$

For $s \in [s_{n+1}, s_n]$, the following limit relation holds

$$\lim_{n\to\infty} L(s)\Big(\sum_{p\geq M(s)+1} \frac{\eta_p}{p^{1/2+s}} - \sum_{p\geq M(s_{n+1})+1} \frac{\eta_p}{p^{1/2+s_{n+1}}}\Big) = 0 \quad \text{a.s.}$$

Proof. Using the fact that M is a nonincreasing function for the arguments close to 0, write, for $s \in [s_{n+1}, s_n]$,

$$\begin{aligned}&\sum_{p\geq M(s)+1} \frac{\eta_p}{p^{1/2+s}} - \sum_{p\geq M(s_{n+1})+1} \frac{\eta_p}{p^{1/2+s_{n+1}}}\\ &\quad= \sum_{M(s)+1\leq p\leq M(s_{n+1})} \frac{\eta_p}{p^{1/2+s}} + \sum_{k\geq M(s_{n+1})+1}\Big(\frac{1}{p^{1/2+s}} - \frac{1}{p^{1/2+s_{n+1}}}\Big)\eta_p =: I_{n,1}(s) + I_{n,2}(s).\end{aligned}$$

Analysis of $I_{n,1}(s)$. It will be proved more than we need, namely,

$$\lim_{n\to\infty} \sup_{s\in[s_{n+1},\, s_n]} |I_{n,1}(s)| = 0 \quad \text{a.s.}$$

We obtain with the help of integration by parts

$$I_{n,1}(s) = \frac{T^*(M(s_{n+1}))}{(M(s_{n+1}))^{1/2+s}} - \frac{T^*(M(s)+1)}{(M(s)+1)^{1/2+s}} + (1/2+s)\int_{M(s)+1}^{M(s_{n+1})} \frac{T^*(x)}{x^{3/2+s}}\mathrm{d}x + o(1),$$

where, as before, $T^*(x) = \sum_{p\leq x}\eta_p$ for $x \geq 1$. The term $o(1)$ is equal to 0 if $M(s)+1 \notin \mathcal{P}$ and is equal to $\eta_{M(s)+1}(M(s)+1)^{-1/2-s}$, otherwise. In view of (8) and (12), $\lim_{s\to 0+}\eta_{M(s)+1}(M(s)+1)^{-1/2-s} = 0$ a.s. Invoking formula (9) and $\lim_{n\to\infty}(M(s_{n+1}))^{s_{n+1}} = 1$ we infer

$$\begin{aligned}\frac{|T^*(M(s_{n+1}))|}{(M(s_{n+1}))^{1/2+s}} &\leq \frac{|T^*(M(s_{n+1}))|}{(M(s_{n+1}))^{1/2+s_{n+1}}} \sim \frac{|T^*(M(s_{n+1}))|}{(M(s_{n+1}))^{1/2}}\\ &= O((\log\log M(s_{n+1})/\log M(s_{n+1}))^{1/2}) \to 0, \quad n\to\infty \quad \text{a.s.}\end{aligned}$$

By a similar argument we conclude that

$$\lim_{n\to\infty}\sup_{s\in[s_{n+1},\,s_n]}\frac{|T^*(M(s)+1)|}{(M(s)+1)^{1/2+s}}=0\quad\text{a.s.}$$

Further, for $s\in[s_{n+1},s_n]$ and large n,

$$\begin{aligned}\Big|\int_{M(s)+1}^{M(s_{n+1})}\frac{T^*(x)}{x^{3/2+s}}\mathrm{d}x\Big|&\leq\int_{M(s)+1}^{M(s_{n+1})}\frac{|T^*(x)|}{x^{3/2+s}}\mathrm{d}x\leq\big(\sup_{y\leq M(s_{n+1})}|T^*(y)|\big)\int_{M(s)+1}^{M(s_{n+1})}\frac{\mathrm{d}x}{x^{3/2+s}}\\&\leq\big(\sup_{y\leq M(s_{n+1})}|T^*(y)|\big)\frac{2}{(M(s))^{1/2+s}}\leq\big(\sup_{y\leq M(s_{n+1})}|T^*(y)|\big)\frac{2}{(M(s_n))^{1/2+s_{n+1}}}\\&=O((\log\log M(s_{n+1})/\log M(s_{n+1}))^{1/2})\ \to\ 0,\quad n\to\infty\quad\text{a.s.}\end{aligned}$$

We have used (9), $\lim_{n\to\infty}(M(s_{n+1})/M(s_n))=1$ and $\lim_{n\to\infty}(M(s_n))^{s_{n+1}}=1$ for the equality.
Analysis of $I_{n,2}(s)$. We claim that

$$\lim_{n\to\infty}\sup_{s\geq s_{n+1}}\sum_{p\geq M(s_{n+1})+1}\Big(\frac{1}{p^{1/2+s_{n+1}}}-\frac{1}{p^{1/2+s}}\Big)|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}\log n\}}=0\quad\text{a.s.}\tag{16}$$

and

$$\lim_{n\to\infty}\sup_{s\geq s_{n+1}}\sum_{p\geq M(s_{n+1})+1}\Big(\frac{1}{p^{1/2+s_{n+1}}}-\frac{1}{p^{1/2+s}}\Big)\mathbb{E}\big[|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}\log n\}}\big]=0.\tag{17}$$

As before, let $p_1<p_2<\ldots$ be the sequence of prime numbers arranged in the order of increase. By the prime number theorem, $p_k/\log p_k\sim k$ as $k\to\infty$. We note for later use that this implies that, given $\delta\in(0,1)$,

$$p_k/\log p_k\geq\delta^2k\tag{18}$$

for large k. By the direct part of the Borel-Cantelli lemma, $\lim_{k\to\infty}(p_k/\log p_k)^{-1/2}|\eta_{p_k}|=0$ a.s. and thereupon $\sup_{k\geq1}(p_k/\log p_k)^{-1/2}|\eta_{p_k}|<\infty$ a.s. Hence, relation (16) follows from

$$\begin{aligned}&\sum_{k:\ p_k\geq M(s_{n+1})+1}\Big(\frac{1}{p_k^{1/2+s_{n+1}}}-\frac{1}{p_k^{1/2+s}}\Big)|\eta_{p_k}|\mathbb{1}_{\{|\eta_{p_k}|>(p_k/\log p_k)^{1/2}\log n\}}\\&\qquad\leq\sum_{k:\ p_k\geq M(s_{n+1})+1}\frac{|\eta_{p_k}|}{p_k^{1/2}}\mathbb{1}_{\{(p_k/\log p_k)^{-1/2}|\eta_{p_k}|>\log n\}}\quad\text{a.s.}\end{aligned}$$

and the fact that the summands on the right-hand side are equal to 0 for large enough n. Relation (17) follows along the lines of the proof of (11):

$$\begin{aligned}&\sum_{k:\ p_k\geq M(s_{n+1})+1}\Big(\frac{1}{p_k^{1/2+s_{n+1}}}-\frac{1}{p_k^{1/2+s}}\Big)\mathbb{E}\big[|\eta_{p_k}|\mathbb{1}_{\{|\eta_{p_k}|>(p_k/\log p_k)^{1/2}\log n\}}\big]\\&\leq\sum_{k:\ p_k\geq M(s_{n+1})+1}\frac{1}{p_k^{1/2}}\mathbb{E}\big[|\eta|\mathbb{1}_{\{|\eta|>\delta k^{1/2}\log n\}}\big]\leq\sum_{k\geq M(s_{n+1})+1}\frac{1}{k^{1/2}}\mathbb{E}\big[|\eta|\mathbb{1}_{\{k<(\delta\log n)^{-2}\eta^2\}}\big]\\&\leq\mathbb{E}\Big[|\eta|\sum_{k\leq\lfloor(\delta\log n)^{-2}\eta^2\rfloor}\frac{1}{k^{1/2}}\Big]\leq2\delta^{-1}\mathbb{E}[\eta^2](\log n)^{-1}\ \to\ 0,\quad n\to\infty.\end{aligned}$$

We have used (18) for the first inequality.

Put $\widehat{\eta}_p(n) := \eta_p \mathbb{1}_{\{|\eta_p| \le (p/\log p)^{1/2} \log n\}} - \mathbb{E}\big[\eta_p \mathbb{1}_{\{|\eta_p| \le (p/\log p)^{1/2} \log n\}}\big]$ for $p \in \mathcal{P}$ and $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and small positive u, put

$$Y_n^*(u) := \sum_{p \ge M(s_{n+1})+1} \frac{\widehat{\eta}_p(n)}{p^{1/2+u}}.$$

In view of (16) and (17), we are left with showing that, for each $s \in [s_{n+1}, s_n]$,

$$\lim_{n\to\infty} L(s_n)(Y_n^*(s) - Y_n^*(s_{n+1})) = 0 \quad \text{a.s.}$$

We shall prove an equivalent limit relation: for each $v \in [v_{n+1}, v_n]$,

$$\lim_{n\to\infty} L(s_n)(Y_n(v) - Y_n(v_{n+1})) = 0 \quad \text{a.s.}, \tag{19}$$

where $Y_n(v) := Y_n^*(\exp(-1/v))$, $v_n := 1/(\log 1/s_n) = \exp(-n^{1-\gamma})$ for $n \in \mathbb{N}$ and $v > 0$.

For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, put

$$F_j(n) := \{t_{j,m}(n) := v_{n+1} + 2^{-j} m(v_n - v_{n+1}) : 0 \le m \le 2^j\}.$$

Note that $F_j(n) \subseteq F_{j+1}(n)$ and put $F(n) := \bigcup_{j\ge 0} F_j(n)$. The set $F(n)$ is dense in the interval $[v_{n+1}, v_n]$. For any $u \in [v_{n+1}, v_n]$, put

$$u_j := \max\{v \in F_j(n) : v \le u\} = v_{n+1} + 2^{-j}(v_n - v_{n+1}) \Big\lfloor \frac{2^j(u - v_{n+1})}{v_n - v_{n+1}} \Big\rfloor.$$

Then $\lim_{j\to\infty} u_j = u$ (we omit the dependence on n in the notation). An important observation is that either $u_{j-1} = u_j$ or $u_{j-1} = u_j - 2^{-j}(v_n - v_{n+1})$. Consequently, $u_j = t_{j,m}$ for some $0 \le m \le 2^j$, which implies that either $u_{j-1} = t_{j,m}$ or $u_{j-1} = t_{j,m-1}$. Since Y_n is a.s. continuous on $[v_{n+1}, v_n]$, we obtain

$$\begin{aligned}
|Y_n(u) - Y_n(v_{n+1})| &= \lim_{l\to\infty} |Y_n(u_l) - Y_n(v_{n+1})| \\
&= \lim_{l\to\infty} \Big| \sum_{j=1}^{l} (Y_n(u_j) - Y_n(u_{j-1})) + Y_n(u_0) - Y_n(v_{n+1}) \Big| \\
&\le \lim_{l\to\infty} \sum_{j=0}^{l} \max_{1\le m\le 2^j} |Y_n(t_{j,m}) - Y_n(t_{j,m-1})| \\
&= \sum_{j\ge 0} \max_{1\le m\le 2^j} |Y_n(t_{j,m}) - Y_n(t_{j,m-1})|.
\end{aligned}$$

Thus, our purpose is to prove that, for all $\varepsilon > 0$ and sufficiently large $n_0 \in \mathbb{N}$,

$$\sum_{n\ge n_0} \mathbb{P}\Big\{ \sum_{j\ge 0} \max_{1\le m\le 2^j} f(s_n)|Y_n(t_{j,m}) - Y_n(t_{j,m-1})| > \varepsilon \Big\} < \infty.$$

Put $a_j := (j+1)2^{-j/2}$ for $j \in \mathbb{N}_0$. In view of $\sum_{j\ge 0} a_j < \infty$, it suffices to show that, for all $\varepsilon > 0$,

$$\sum_{n\ge n_0} \sum_{j\ge 0} \mathbb{P}\Big\{ \max_{1\le m\le 2^j} f(s_n)|Y_n(t_{j,m}) - Y_n(t_{j,m-1})| > \varepsilon a_j \Big\} < \infty. \tag{20}$$

The subsequent argument is similar to that used in the proof of Lemma 3. In view of this, we only give a sketch. Write, for $u \in \mathbb{R}$ and sufficiently large n,

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(\pm u(Y_n(t_{j,m}) - Y_n(t_{j,m-1}))\big)\big] \\
&\quad = \mathbb{E}\Big[\exp\Big(\pm u \sum_{p \geq M(s_{n+1})+1} \frac{1}{p^{1/2}}\Big(\frac{1}{p^{\exp(-1/t_{j,m})}} - \frac{1}{p^{\exp(-1/t_{j,m-1})}}\Big)\widehat{\eta}_p(n)\Big)\Big] \\
&\quad \leq \prod_{p \geq M(s_{n+1})+1} \Big(1 + \frac{u^2}{2p}\Big(\frac{1}{p^{\exp(-1/t_{j,m})}} - \frac{1}{p^{\exp(-1/t_{j,m-1})}}\Big)^2 \\
&\qquad \times \mathbb{E}\Big[(\widehat{\eta}_p(n))^2 \exp\Big(\frac{|u|}{p^{1/2}}\Big(\frac{1}{p^{\exp(-1/t_{j,m})}} - \frac{1}{p^{\exp(-1/t_{j,m-1})}}\Big)|\widehat{\eta}_p(n)|\Big)\Big].
\end{aligned}$$

Now we prove that, for large n, all $j \in \mathbb{N}_0$ and integer $m \in [0, 2^j]$ and some constant $C > 0$,

$$A_j(n) := \sum_{p \geq M(s_{n+1})+1} \frac{1}{p}\Big(\frac{1}{p^{\exp(-1/t_{j,m})}} - \frac{1}{p^{\exp(-1/t_{j,m-1})}}\Big)^2 \leq \frac{C2^{-j}(v_n - v_{n+1})}{v_{n+1}^2}.$$

For $x > \mathrm{e}$, put $L_{j,m}(x) = (x \log x)^{-1}(x^{-\exp(-1/t_{j,m})} - x^{-\exp(-1/t_{j,m-1})})^2$. The function $L_{j,m}$ is differentiable and decreasing on (e, ∞), whence $L'_{j,m}(x) \leq 0$ for $x > \mathrm{e}$. Also, for $x \geq \mathrm{e}$, put $A(x) := \int_{(\mathrm{e},\, x]} \log x \, \mathrm{d}\rho(x)$. By the prime number theorem, for x_0 large enough (in particular, $x_0 > \mathrm{e}$) there exists $C_0 > 0$ such that $A(x) \leq C_0 x$ for $x \geq x_0$. Hence, there exists $C_1 > C_0$ such that $A(x) \leq C_1(x - \mathrm{e})$ for $x \geq x_0$. Also, there exists $C_2 > 0$ such that $A(x) \leq C_2(x - e)$ for $x \in [\mathrm{e},\, x_0]$. Thus, putting $C := \max(C_1, C_2)$ we infer $A(x) \leq C(x - \mathrm{e})$ for $x \geq \mathrm{e}$. Integrating by parts twice, forward and then backward, we obtain

$$\begin{aligned}
A_j(n) &\leq \int_{\mathrm{e}}^{\infty} L_{j,m}(x)\mathrm{d}A(x) = \int_{\mathrm{e}}^{\infty} (-L'_{j,m}(x))A(x)\mathrm{d}x \leq C\int_{\mathrm{e}}^{\infty} (-L'_{j,m}(x))(x - \mathrm{e})\mathrm{d}x \\
&= C\int_{\mathrm{e}}^{\infty} L_{j,m}(x)\mathrm{d}x.
\end{aligned}$$

According to formula (17) in [12],

$$\int_{\mathrm{e}}^{\infty} L_{j,m}(x)\mathrm{d}x \leq \frac{1}{t_{j,m-1}} - \frac{1}{t_{j,m}} \leq \frac{2^{-j}(v_n - v_{n+1})}{v_{n+1}^2}.$$

This proves the claimed inequality for $A_j(n)$.

Next, we estimate

$$B_j(u,p,n) := \frac{|u|}{p^{1/2}}\Big(\frac{1}{p^{\exp(-1/t_{j,m-1})}} - \frac{1}{p^{\exp(-1/t_{j,m})}}\Big)|\widehat{\eta}_p(n)|$$

for $p \geq \lfloor M(s_{n+1}) \rfloor + 1$. Assume first that $2^j \geq v_{n+1}^{-2}(v_n - v_{n+1})$ for integer j. For $a > 0$ and $0 < s < t$, the following holds

$$0 \leq \exp(-a\mathrm{e}^{-1/s}) - \exp(-a\mathrm{e}^{-1/t}) \leq a\exp(-a\mathrm{e}^{-1/s})\mathrm{e}^{-1/t}(1/s - 1/t).$$

Putting $a = \log p$, $t = t_{j,m-1}$ and $s = t_{j,m}$ we obtain

$$0 \leq \frac{1}{p^{\exp(-1/t_{j,m-1})}} - \frac{1}{p^{\exp(-1/t_{j,m})}} \leq \frac{\log p}{p^{\exp(-1/t_{j,m-1})}}\mathrm{e}^{-1/t_{j,m}}\Big(\frac{1}{t_{j,m-1}} - \frac{1}{t_{j,m}}\Big).$$

Using $x\mathrm{e}^{-ax} \le (\mathrm{e}a)^{-1}$ for $x \ge 0$ and our present assumption concerning j we further conclude that the right-hand side does not exceed

$$\frac{1}{\mathrm{e}}\mathrm{e}^{1/t_{j,\,m-1}-1/t_{j,\,m}}\Big(\frac{1}{t_{j,\,m-1}}-\frac{1}{t_{j,\,m}}\Big)$$
$$\le \frac{1}{\mathrm{e}}\exp\Big(\frac{2^{-j}(v_n-v_{n+1})}{v_{n+1}^2}\Big)\frac{2^{-j}(v_n-v_{n+1})}{v_{n+1}^2} \le \frac{2^{-j}(v_n-v_{n+1})}{v_{n+1}^2}.$$

We have shown that

$$B_j(u,p,n) \le \frac{|u|2^{1-j}\log n}{(\log M(s_{n+1}))^{1/2}}\frac{v_n-v_{n+1}}{v_{n+1}^2} =: C_j(u,n)$$

whenever $p \ge \lfloor M(s_{n+1})\rfloor + 1$ and $2^j \ge v_{n+1}^{-2}(v_n-v_{n+1})$.

Assume now that $2^j \le v_{n+1}^{-2}(v_n-v_{n+1})$ for nonnegative integer j. In view of

$$\frac{1}{p^{\exp(-1/t_{j,\,m-1})}}-\frac{1}{p^{\exp(-1/t_{j,\,m})}} \le 1$$

we infer

$$B_j(u,p,n) \le \frac{2|u|\log n}{(\log M(s_{n+1}))^{1/2}} =: C_j(u,n)$$

whenever $p \ge \lfloor M(s_{n+1})\rfloor + 1$.

Using now $1+x \le \mathrm{e}^x$ for $x \in \mathbb{R}$ we arrive at

$$\mathbb{E}\big[\exp\big(\pm u(Y_n(t_{j,\,m})-Y_n(t_{j,\,m-1}))\big)\big] \le \exp\Big(\frac{u^2}{2}\frac{C2^{-j}(v_n-v_{n+1})}{v_{n+1}^2}\mathrm{e}^{C_j(u,n)}\Big), \quad u \in \mathbb{R}.$$

By Markov's inequality and the inequality $\mathrm{e}^{u|x|} \le \mathrm{e}^{ux}+\mathrm{e}^{-ux}$ for $x \in \mathbb{R}$,

$$\mathbb{P}\big\{L(s_n)|Y_n(t_{j,\,m})-Y_n(t_{j,\,m-1})| > \varepsilon a_j\big\} \le \exp\Big(-u\frac{\varepsilon a_j}{L(s_n)}\Big)\mathbb{E}[\exp(u|Y_n(t_{j,\,m})-Y_n(t_{j,\,m-1})|)]$$
$$\le 2\exp\Big(-\frac{u\varepsilon a_j}{L(s_n)}+\frac{u^2}{2}\frac{C2^{-j}(v_n-v_{n+1})}{v_{n+1}^2}\mathrm{e}^{C_j(u,n)}\Big).$$

Put

$$u = \frac{\varepsilon 2^{j/2}}{CL(s_n)}\frac{v_{n+1}^2}{v_n-v_{n+1}} \quad \text{and} \quad k_n := \frac{1}{C(L(s_n))^2}\frac{v_{n+1}^2}{v_n-v_{n+1}}.$$

As a consequence of

$$(\log M(s_{n+1}))^{-1/2} \ \sim\ n^{\gamma/2-1/2}, \ \Big(\frac{v_{n+1}}{v_n-v_{n+1}}\Big)^{1/2} \ \sim\ (1-\gamma)^{-1/2}n^{\gamma/2}, \quad n \to \infty$$

and

$$\lim_{n\to\infty}(\log 1/s_n)v_{n+1} = 1,$$

we infer, for j satisfying $2^j \ge v_{n+1}^{-2}(v_n-v_{n+1})$,

$$C_j(u,n) = \frac{|u|2^{1-j}\log n}{(\log M(s_{n+1}))^{1/2}}\frac{v_n-v_{n+1}}{v_{n+1}^2} = \frac{\varepsilon 2^{1-j/2}\log n}{CL(s_n)(\log M(s_{n+1}))^{1/2}}$$
$$\le \frac{2\varepsilon\log n}{CL(s_n)(\log M(s_{n+1}))^{1/2}}\frac{v_{n+1}}{(v_n-v_{n+1})^{1/2}}$$
$$= \frac{\varepsilon 2^{3/2}((\log 1/s_n)v_{n+1})^{1/2}(\log^{(3)} 1/s_n)^{1/2}\log n}{C(\log M(s_{n+1}))^{1/2}}\Big(\frac{v_{n+1}}{v_n-v_{n+1}}\Big)^{1/2}$$
$$\sim\ \varepsilon C^{-1}2^{3/2}n^{\gamma-1/2}(\log n)^{3/2} \ \to\ 0, \quad n \to \infty.$$

If nonnegative integer j satisfies $2^j \le v_{n+1}^{-2}(v_n - v_{n+1})$, then

$$C_j(u,n) = \frac{2|u|\log n}{(\log M(s_{n+1}))^{1/2}} = \frac{\varepsilon 2^{1+j/2}\log n}{CL(s_n)(\log M(s_{n+1}))^{1/2}}\frac{v_{n+1}^2}{v_n - v_{n+1}}$$
$$\le \frac{2\varepsilon \log n}{CL(s_n)(\log M(s_{n+1}))^{1/2}}\frac{v_{n+1}}{(v_n - v_{n+1})^{1/2}} \to 0, \quad n\to\infty.$$

Thus, we obtain, for large n satisfying $k_n > \varepsilon^{-2}\log 2$ and $\mathrm{e}^{C_j(u,n)} \le 3/2$,

$$\sum_{j\ge 0}\mathbb{P}\Big\{\max_{1\le m\le 2^j} L(s_n)|Y_n(t_{j,m}) - Y_n(t_{j,m-1})| > \varepsilon a_j\Big\}$$
$$\le \sum_{j\ge 0} 2^j 2\exp(-\varepsilon^2(j+1)k_n)\exp\big(3\varepsilon^2 k_n/4\big) = \frac{2\exp(-\varepsilon^2 k_n/4)}{1-2\exp(-\varepsilon^2 k_n)}.$$

Since $k_n \sim 2C^{-1}n^\gamma \log n$ as $n\to\infty$, (20) follows. $\square$

After all these preparations we are ready to prove Proposition 1.

Proof of Proposition 1. Recall our convention that $\sigma^2 = 1$. To prove (6), we choose any $\gamma > 0$ sufficiently close to 0, put $s_n = \exp(-\exp(n^{1-\gamma}))$ for $n \in \mathbb{N}$ and select $\theta = \theta(\gamma)$ such that (13) holds true. Let $M(s) = \lfloor \log 1/s/\log\log 1/s\rfloor$ for $s \in (0, \mathrm{e}^{-\mathrm{e}})$. Using Lemmas 2 and 3 in combination with $\mathbb{E}\big[\eta_p \mathbb{1}_{(A_{p,\theta}(s))^c}\big] = -\mathbb{E}\big[\eta_p \mathbb{1}_{A_{p,\theta}(s)}\big]$ we conclude that

$$\limsup_{n\to\infty} L(s_n)\sum_{p\ge M(s_n)+1}\frac{\eta_p}{p^{1/2+s_n}} \le 1+\gamma \quad \text{a.s.} \tag{21}$$

Relation (21) together with Lemma 4 ensures that

$$\limsup_{s\to 0+} L(s)\sum_{p\ge M(s)+1}\frac{\eta_p}{p^{1/2+s}} \le 1+\gamma \quad \text{a.s.}$$

With these at hand, an application of Lemma 1 yields

$$\limsup_{s\to 0+} L(s)\sum_{p}\frac{\eta_p}{p^{1/2+s}} \le 1+\gamma \quad \text{a.s.}$$

Sending $\gamma\to 0+$ we arrive at (6).

Relation (7) follows from (6), with $-\eta_p$ replacing η_p. $\square$

Proposition 2. *Assume that $\mathbb{E}[\eta] = 0$ and $\sigma^2 = \mathbb{E}[\eta^2] \in (0,\infty)$. Then*

$$\limsup_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log^{(3)} 1/s}\Big)^{1/2}\sum_{p}\frac{\eta_p}{p^{1/2+s}} \ge \sqrt{2}\sigma \quad \text{a.s.} \tag{22}$$

and

$$\liminf_{s\to 0+}\Big(\frac{1}{\log 1/s\ \log^{(3)} 1/s}\Big)^{1/2}\sum_{p}\frac{\eta_p}{p^{1/2+s}} \le -\sqrt{2}\sigma \quad \text{a.s.} \tag{23}$$

At this point it is reasonable to explain the argument justifying (22). The sum $\sum_p \frac{\eta_p}{p^{1/2+s}}$ will now be split into three fragments: initial, intermediate and final. It will be shown in Lemma 5 that the initial and the final fragments do not contribute to the LIL provided that the variables η_p within each fragment are properly truncated and centered. Lemma 6 treats the intermediate fragment which gives a principal contribution to the LIL. Our proof is based on the converse part of the Borel-Cantelli lemma, which requires independence. The independence requirement complicates to some extent a selection of the intermediate fragment. Unlike in the proof of (6), it suffices to prove (22) for a suitable sequence. This is a simplifying feature of (22) in comparison to (6).

In what follows we assume without further notice that $\mathbb{E}[\eta] = 0$ and $\sigma^2 = \mathrm{Var}\,[\eta] = 1$. Also, we use the sets $A_{p,\theta}(s)$ and the corresponding variables $\widetilde{\eta}_{p,\theta}(s)$ with $\theta = 1$.

Lemma 5. *Fix any $\gamma > 0$ and put $\mathfrak{s}_n := \exp(-\exp(n^{1+\gamma}))$ for $n \geq 1$. Let N_1 and N_2 be functions which take positive integer values, may depend on γ and satisfy*[1] $\lim_{s\to 0+} N_1(s) = \infty$, $\lim_{s\to 0+}(\log\log N_1(s)/\log 1/s) = 0$, $\lim_{s\to 0+}(\log\log N_2(s)/\log 1/s) = 1$ *and* $\lim_{s\to 0+} s\log N_2(s) = \infty$. *Then*

$$\lim_{n\to\infty} L(\mathfrak{s}_n) \sum_{p\leq N_1(\mathfrak{s}_n)} \frac{\widetilde{\eta}_{p,1}(\mathfrak{s}_n)}{p^{1/2+\mathfrak{s}_n}} = 0 \quad \text{a.s.} \tag{24}$$

and

$$\lim_{n\to\infty} L(\mathfrak{s}_n) \sum_{p\geq N_2(\mathfrak{s}_n)+1} \frac{\widetilde{\eta}_{p,1}(\mathfrak{s}_n)}{p^{1/2+\mathfrak{s}_n}} = 0 \quad \text{a.s.} \tag{25}$$

Proof. As in the proof of Lemma 3, we obtain the result, with L^* replacing L. For $s > 0$ close to 0, put

$$Z_1(s) := L^*(s) \sum_{p\leq N_1(s)} \frac{\widetilde{\eta}_{p,1}(s)}{p^{1/2+s}}.$$

The reasoning used to derive both (24) and (25) is similar to the one applied in the proof of Lemma 3. Therefore, we give a proof of (24) and indicate the only minor change needed for a proof of (25).

Regarding (24), in view of the direct part of the Borel–Cantelli lemma, it is sufficient to demonstrate that, for all $\varepsilon > 0$,

$$\sum_{n\geq 1} \mathbb{P}\{Z_1(\mathfrak{s}_n) > \varepsilon\} < \infty. \tag{26}$$

To this end, we obtain a counterpart of (15), for $u \in \mathbb{R}$,

$$\mathbb{E}[\mathrm{e}^{uZ_1(s)}] \leq \exp\Big(\frac{u^2(L^*(s))^2}{2} \sum_{p\leq N_1(s)} \frac{1}{p^{1+2s}} \exp\Big(\frac{\sqrt{2}|u|}{\log^{(3)} 1/s}\Big)\Big).$$

Recalling the notation $\rho(x) = \#\{p \in \mathcal{P} : p \leq x\}$ for $x \geq 1$ we obtain with the help of integration by parts

$$\sum_{p\leq N_1(s)} p^{-1-2s} = \int_{(1,\, N_1(s)]} x^{-1-2s}\mathrm{d}\rho(x) = (N_1(s))^{-1-2s}\rho(N_1(s)) + (1+2s)\int_1^{N_1(s)} x^{-2-2s}\rho(x)\mathrm{d}x.$$

[1] For instance, one can take $N_1(s) = \lfloor(\log 1/s)^{1/2}\rfloor$ or $N_1(s) = \lfloor\exp(\mathrm{e}^{\log 1/s/\log\log 1/s})\rfloor$ and $N_2(s) = \lfloor\exp((1/s)\log 1/s)\rfloor$.

The assumption $\lim_{s\to 0+}(\log\log N_1(s)/\log 1/s) = 0$ entails $\lim_{s\to 0+} s\log N_1(s) = 0$. By the prime number theorem, $\rho(x) \sim x/\log x$ as $x\to\infty$. As a consequence,

$$\lim_{s\to 0+}(N_1(s))^{-1-2s}\rho(N_1(s)) = 0.$$

By another application of the prime number theorem, as $s\to 0+$,

$$\int_1^{N_1(s)} x^{-2-2s}\rho(x)\mathrm{d}x \ \sim\ \int_2^{N_1(s)} x^{-1-2s}(\log x)^{-1}\mathrm{d}x = \int_{(2\log 2)s}^{2s\log N_1(s)} y^{-1}\mathrm{e}^{-y}\mathrm{d}y \ \sim\ \log\log N_1(s).$$

Thus, we have proved that

$$\sum_{p\le N_1(s)} p^{-1-2s} \ \sim\ \log\log N_1(s), \quad s\to 0+. \tag{27}$$

By assumption, $\lim_{s\to 0+}(\log\log N_1(s)/g(s)) = 0$. Hence, there exists an $r>0$ close to 0 such that

$$\sum_{p\le N_1(s)} p^{-1-2s} \le rg(s)$$

for small positive s, and also $(1-\delta)(1+\gamma) > 1$, where $\delta := r(4\varepsilon^2)^{-1}\exp(\sqrt{2}\varepsilon^{-1})$ with ε appearing in (26). Then, for $u\in\mathbb{R}$ and small $s>0$,

$$\mathbb{E}[\mathrm{e}^{uZ_1(s)}] \le \exp\Big(\frac{ru^2 g(s)(L^*(s))^2}{2}\exp\Big(\frac{\sqrt{2}|u|}{\log^{(3)} 1/s}\Big)\Big) = \exp\Big(\frac{ru^2}{4\log^{(3)} 1/s}\exp\Big(\frac{\sqrt{2}|u|}{\log^{(3)} 1/s}\Big)\Big).$$

Put $u = (1/\varepsilon)\log^{(3)} 1/\mathfrak{s}_n$. By Markov's inequality, we infer, for large n,

$$\mathbb{P}\{Z_1(\mathfrak{s}_n) > \varepsilon\} \le \mathrm{e}^{-u\varepsilon}\mathbb{E}[\mathrm{e}^{uZ_1(\mathfrak{s}_n)}] \le \exp\big(-\big(1 - r(4\varepsilon^2)^{-1}\mathrm{e}^{\sqrt{2}\varepsilon^{-1}}\big)\log^{(3)} 1/\mathfrak{s}_n\big) = \frac{1}{n^{(1-\delta)(1+\gamma)}}.$$

This proves (26) and (24).

We omit a proof of (25), which is analogous to that of (24), and only point out a counterpart of (27):

$$\sum_{p\ge N_2(s)+1} p^{-1-2s} = o(1), \quad s\to 0+. \tag{28}$$

This is justified as follows:

$$\sum_{p\ge N_2(s)+1} p^{-1-2s} = \int_{N_2(s)+1}^{\infty} x^{-1-2s}\mathrm{d}\rho(x) = -\frac{\rho(N_2(s)+1)}{(N_2(s)+1)^{1+2s}} + (1+2s)\int_{N_2(s)+1}^{\infty} x^{-2-2s}\rho(x)\mathrm{d}x.$$

The first summand is $o(1)$ as $s\to 0+$. By the prime number theorem, as $s\to 0+$,

$$\int_{N_2(s)+1}^{\infty} x^{-2-2s}\rho(x)\mathrm{d}x \ \sim\ \int_{N_2(s)+1}^{\infty} x^{-1-2s}(\log x)^{-1}\mathrm{d}x = \int_{2s\log(N_2(s)+1)}^{\infty} \mathrm{e}^{-y}y^{-1}\mathrm{d}y = o(1).$$

We have used the assumption $\lim_{s\to 0+} s\log(N_2(s)+1) = \infty$ for the last equality. □

Lemma 6. *Fix sufficiently small $\delta>0$, pick $\gamma>0$ satisfying $(1+\gamma)(1-\delta^2/8) < 1$ and let, as before, $\mathfrak{s}_n = \exp(-\exp(n^{1+\gamma}))$ for $n\in\mathbb{N}$. Then*

$$\limsup_{n\to\infty} L(\mathfrak{s}_n)\sum_p \frac{\widetilde{\eta}_{p,1}(\mathfrak{s}_n)}{p^{1/2+\mathfrak{s}_n}} \ge 1-\delta \quad \text{a.s.}$$

Proof. For any function r such that $\lim_{s\to 0+} r(s) = +\infty$, we can choose a function N_1 in Lemma 5 satisfying

$$\lim_{s\to 0+} \frac{\log\log N_1(s)}{\log 1/s} r(s) = +\infty.$$

Thus, relation $\lim_{n\to\infty}(\mathfrak{s}_{n+1}/\mathfrak{s}_n) = \infty$ ensures that there exists an N_1 satisfying

$$\frac{\log\log N_1(\mathfrak{s}_{n+1})}{\log 1/\mathfrak{s}_n} = \frac{\log\log N_1(\mathfrak{s}_{n+1})}{\log 1/\mathfrak{s}_{n+1}} \frac{\log 1/\mathfrak{s}_{n+1}}{\log 1/\mathfrak{s}_n} \quad \to \quad +\infty, \quad n \to \infty.$$

For instance, one can take $N_1(s) = \lfloor \exp(\mathrm{e}^{\log 1/s/\log\log 1/s}) \rfloor$.

Let N_2 be any function satisfying the assumptions of Lemma 5. Since

$$\lim_{n\to\infty} \frac{\log\log N_2(\mathfrak{s}_n)}{\log 1/\mathfrak{s}_n} = 1,$$

we conclude that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\log\log N_1(\mathfrak{s}_{n+1})}{\log 1/\mathfrak{s}_n} \geq \frac{\log\log N_2(\mathfrak{s}_n)}{\log 1/\mathfrak{s}_n} \quad \text{for all } n \geq n_0,$$

whence

$$N_1(\mathfrak{s}_{n+1}) \geq N_2(\mathfrak{s}_n) \quad \text{for all } n \geq n_0. \tag{29}$$

Put, for small $s > 0$,

$$Z_2(s) := L(s) \sum_{N_1(s) < p \leq N_2(s)} \frac{\widetilde{\eta}_{p,1}(s)}{p^{1/2+s}}.$$

Lemma 5 ensures it is enough to prove that

$$\limsup_{n\to\infty} Z_2(\mathfrak{s}_n) \geq 1 - \delta \quad \text{a.s.} \tag{30}$$

Our plan is to show that there exists $\overline{s} > 0$ such that, for all $s \in (0, \overline{s})$,

$$\mathbb{P}\{Z_2(s) > 1 - \delta\} \geq 3^{-1} \mathrm{e}^{-(1-\delta^2/8)\log^{(3)} 1/s}. \tag{31}$$

Pick $n_1 \geq n_0$ such that $\mathfrak{s}_n < \overline{s}$ for $n \geq n_1$. Inequality (31) together with $(1+\gamma)(1-\delta^2/8) < 1$ secures

$$\sum_{n\geq n_1} \mathbb{P}\{Z_2(\mathfrak{s}_n) > 1 - \delta\} \geq 3^{-1} \sum_{n \geq n_1} \frac{1}{n^{(1+\gamma)(1-\delta^2/8)}} = \infty.$$

In view of (29), the random variables $Z_2(\mathfrak{s}_{n_1})$, $Z_2(\mathfrak{s}_{n_1+1}), \ldots$ are independent. Thus, by the converse part of the Borel-Cantelli lemma, divergence of the latter series entails (30).

When proving (31) we use the event

$$U_s := \big\{1 - \delta < Z_2(s) \leq 1\big\} = \big\{(1-\delta)V(s) < W(s)/(g(s))^{1/2} \leq V(s)\big\},$$

where $V(s) = (2\log^{(3)} 1/s)^{1/2}$ and

$$W(s) := \frac{Z_2(s)}{L(s)} = \sum_{N_1(s) < p \leq N_2(s)} \frac{\widetilde{\eta}_{p,1}(s)}{p^{1/2+s}}.$$

For $u\in\mathbb{R}$ and small $s>0$, let $\mathbb{Q}_{s,u}$ be a probability measure on $(\Omega,\mathfrak{F})$ defined by

$$\mathbb{Q}_{s,u}(A)=\frac{\mathbb{E}\big[\mathrm{e}^{uW(s)/(g(s))^{1/2}}\mathbb{1}_A\big]}{\mathbb{E}\big[\mathrm{e}^{uW(s)/(g(s))^{1/2}}\big]},\quad A\in\mathfrak{F}. \tag{32}$$

Then

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{u(W(s)/(g(s))^{1/2}-V(s))}\big]\mathbb{Q}_{s,u}(U_s)&=\mathrm{e}^{-uV(s)}\mathbb{E}\big[\mathrm{e}^{uW(s)/(g(s))^{1/2}}\mathbb{1}_{U_s}\big]\\&\le\mathbb{P}(U_s)\le\mathbb{P}\{Z_2(s)>1-\delta\}.\end{aligned} \tag{33}$$

We claim that

$$\mathbb{Q}_{s,u}(U_s)\ge 1/3 \tag{34}$$

and

$$\mathbb{E}\big[\mathrm{e}^{u(W(s)/(g(s))^{1/2}-V(s))}\big]\ge\mathrm{e}^{-(1-\delta^2/8)\log^{(3)}1/s} \tag{35}$$

provided that

$$u=u(s)=\sqrt{2}(1-\delta/2)(\log^{(3)}1/s)^{1/2}. \tag{36}$$

The argument given in the proof of Lemma 6 in [12], which justifies an inequality analogous to (34), applies without any changes. Hence, it is omitted. Now we show that (35) holds. Inequality (30) is an immediate consequence of (34) and (35).

Proof of (35). We first prove that, with $u=O((\log^{(3)}1/s)^{1/2})$,

$$\mathbb{E}\big[\mathrm{e}^{uW(s)/(g(s))^{1/2}}\big]\ \sim\ \mathrm{e}^{u^2/2+u^2h(s)},\quad s\to0+ \tag{37}$$

for some function h satisfying $\lim_{s\to0+}h(s)=0$. Put

$$\xi_p(s)=\frac{\widetilde{\eta}_{p,1}(s)}{(g(s))^{1/2}p^{1/2+s}},\quad p\in\mathcal{P},\quad s\in(0,\mathrm{e}^{-\mathrm{e}}).$$

As a consequence, for $u\in\mathbb{R}$,

$$uW(s)/(g(s))^{1/2}=\sum_{N_1(s)<p\le N_2(s)}u\xi_p(s).$$

According to the second inequality in (14),

$$|u\xi_p(s)|=O(1/\log\log 1/s)=o(1),\quad s\to0+\quad\text{a.s.}$$

for each $p\in\mathcal{P}$. Invoking

$$\mathrm{e}^x=1+x+x^2/2+o(x^2)\quad\text{and}\quad\log(1+x)=x+O(x^2),\quad x\to0,$$

we obtain

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{uW(s)/(g(s))^{1/2}}\big]&=\prod_{N_1(s)<p\le N_2(s)}\mathbb{E}[\exp(u\xi_p(s))]\\&=\prod_{N_1(s)<p\le N_2(s)}\mathbb{E}\big[1+u\xi_p(s)+u^2\xi_p^2(s)(1/2+o(1))\big]\\&=\exp\sum_{N_1(s)<p\le N_2(s)}\log\big(1+u^2\mathbb{E}\big[\xi_p^2(s)(1/2+o(1))\big]\big)\\&=\exp\Big(u^2\big(1/2+o(1)\big)\sum_{N_1(s)<p\le N_2(s)}\mathbb{E}[\xi_p^2(s)]+u^4O\Big(\sum_{N_1(s)<p\le N_2(s)}(\mathbb{E}[\xi_p^2(s)])^2\Big)\Big).\end{aligned} \tag{38}$$

Here, we have used the fact that the $o(1)$ term can be chosen nonrandom. In view of (27) and (28),

$$\sum_{N_1(s)<p\le N_2(s)} p^{-1-2s} \;\sim\; g(s), \quad s\to 0+.$$

The limit relation

$$\mathbb{E}[\widetilde{\eta}^2_{p,1}(s)] = \mathbb{E}[\eta_p^2 \mathbb{1}_{(A_{p,1}(s))^c}] - \big(\mathbb{E}[\eta_p \mathbb{1}_{(A_{p,1}(s))^c}]\big)^2 \;\to\; 1, \quad s\to 0+$$

holds uniformly in $p\in[N_1(s)+1, N_2(s)]$, $p\in\mathcal{P}$. A combination of these two facts yields

$$\sum_{N_1(s)<p\le N_2(s)} \mathbb{E}[\xi_p^2(s)] = \frac{1}{g(s)} \sum_{N_1(s)<p\le N_2(s)} \frac{\mathbb{E}[\widetilde{\eta}^2_{p,1}(s)]}{p^{1+2s}} \;\to\; 1, \quad s\to 0+. \tag{39}$$

Finally,

$$\begin{aligned} u^2 \sum_{N_1(s)<p\le N_2(s)} (\mathbb{E}[\xi_p^2(s)])^2 &= \frac{u^2}{(g(s))^2} \sum_{N_1(s)<p\le N_2(s)} \frac{(\mathbb{E}[\widetilde{\eta}^2_{p,1}(s)])^2}{p^{2+4s}} \\ &\le \frac{u^2}{(g(s))^2} \sum_{p>N_1(s)+1} \frac{1}{p^2} = o(1), \quad s\to 0+. \end{aligned} \tag{40}$$

Now (37) follows from (38), (39) and (40).

Finally, formula (37), with u as in (36), entails

$$\mathbb{E}[\mathrm{e}^{u(W(s)/(g(s))^{1/2}-V(s))}] = \mathrm{e}^{-(1-\delta^2/4)\log^{(3)}1/s+o(\log^{(3)}1/s)} \ge \mathrm{e}^{-(1-\delta^2/8)\log^{(3)}1/s}$$

for small $s>0$.

The proof of Lemma 6 is complete. □

Proof of Proposition 2. To prove (22), pick sufficiently small $\delta>0$ and $\gamma>0$, and put $\mathfrak{s}_n = \exp(-\exp(n^{1+\gamma}))$ for $n\ge 1$. Using $\mathbb{E}[\eta]=0$, write

$$\begin{aligned} L(\mathfrak{s}_n)\sum_p \frac{\eta_p}{p^{1/2+\mathfrak{s}_n}} &= L(\mathfrak{s}_n) \sum_{p\le\lfloor(\log 1/\mathfrak{s}_n)^{1/2}\rfloor} \frac{\eta_p}{p^{1/2+\mathfrak{s}_n}} \\ &\quad - L(\mathfrak{s}_n) \sum_{p\le\lfloor(\log 1/\mathfrak{s}_n)^{1/2}\rfloor} \frac{1}{p^{1/2+\mathfrak{s}_n}}(\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c} - \mathbb{E}[\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c}]) \\ &\quad + L(\mathfrak{s}_n) \sum_{p\ge\lfloor(\log 1/\mathfrak{s}_n)^{1/2}\rfloor+1} \frac{1}{p^{1/2+\mathfrak{s}_n}}(\eta_p\mathbb{1}_{A_{p,1}(\mathfrak{s}_n)} - \mathbb{E}[\eta_p\mathbb{1}_{A_{p,1}(\mathfrak{s}_n)}]) \\ &\qquad + L(\mathfrak{s}_n)\sum_p \frac{1}{p^{1/2+\mathfrak{s}_n}}(\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c} - \mathbb{E}[\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c}]). \end{aligned}$$

The first term on the right-hand side converges to 0 a.s. as $n\to\infty$ by Lemma 1, with $M(s) = \lfloor(\log 1/s)^{1/2}\rfloor$. The second term does so according to formula (24) of Lemma 5, with $N_1(s) = \lfloor(\log 1/s)^{1/2}\rfloor$. Finally, the third term vanishes by Lemma 2, again with $M(s) = \lfloor(\log 1/s)^{1/2}\rfloor$. By Lemma[2] 6,

$$\varliminf_{n\to\infty} f(\mathfrak{s}_n)\sum_p \frac{1}{p^{1/2+\mathfrak{s}_n}}(\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c} - \mathbb{E}[\eta_p\mathbb{1}_{(A_{p,1}(\mathfrak{s}_n))^c}]) \ge 1-\delta \quad \text{a.s.}$$

[2] Just in case, we attract the reader's attention to a somewhat delicate point. Although we apply Lemma 5, with $M(s) = \lfloor(\log 1/s)^{1/2}\rfloor$, we are not supposed to take the same M while proving Lemma 6. Actually, M used in the proof of Lemma 6 grows much faster than $s\mapsto\lfloor(\log 1/s)^{1/2}\rfloor$.

and thereupon

$$\limsup_{s\to 0+} L(s)\sum_p \frac{\eta_p}{p^{1/2+s}} \geq \limsup_{n\to\infty} L(\mathfrak{s}_n)\sum_p \frac{\eta_p}{p^{1/2+\mathfrak{s}_n}} \geq 1-\delta \quad \text{a.s.}$$

Letting $\delta \to 0+$ completes the proof of (22). Relation (23) follows from (22), with $-\eta_p$ replacing η_p. $\square$

Proof of Theorem 2. Relations (2) and (3) are secured by Propositions 1 and 2, respectively.

It remains to prove (1). As a preparation, put $H := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and

$$X(z) := \sum_p \frac{\eta_p}{p^{1/2+z}}, \quad z \in H.$$

The so defined X is a random analytic function. This implies that the random functions $s \to X(s) = \sum_p p^{-1/2-s}\eta_p$ and $s \mapsto (2\sigma^2 \log 1/s \log^{(3)} 1/s)^{-1/2} X(s)$ are a.s. continuous on $(0,\infty)$ and $(0, \mathrm{e}^{-\mathrm{e}})$, respectively. Now (1) follows from (2) and (3) with the help of the intermediate value theorem for continuous functions. $\square$

3 Proof of Theorem 1

We shall prove the result in an equivalent form

$$\Big(\frac{1}{s^{1/2}}\sum_p \frac{\eta_p}{p^{1/2+\exp(-ts)}}\Big)_{t\geq 0} \implies (\sigma B(t))_{t\geq 0}, \quad s\to+\infty$$

on $C[0,\infty)$. As before, we can and do assume that $\sigma^2 = 1$.

We use a standard approach, which consists of two steps: (a) proving weak convergence of finite-dimensional distributions; (b) checking tightness.

(a) If $t = 0$, then, for $s > 0$, $X(\mathrm{e}^{-ts}) = X(1) = \sum_p p^{-3/2}\eta_p$, and $s^{-1/2}X(1)$ converges in probability to $B(0) = 0$ as $s \to +\infty$.

Thus, it suffices to show that, for $t_1, t_2 \in (0,\infty)$ (we do not need to consider $t = 0$),

$$\mathbb{E}\big[X(\mathrm{e}^{-t_1 s})X(\mathrm{e}^{-t_2 s})\big] \;\sim\; \min(t_1,t_2)s, \quad s\to+\infty \tag{41}$$

and check the Lindeberg-Feller condition: for all $\varepsilon > 0$ and each fixed $t > 0$,

$$\lim_{s\to+\infty} \frac{1}{s}\sum_p \mathbb{E}\Big[\Big(\frac{\eta_p}{p^{1/2+\exp(-ts)}}\Big)^2 \mathbb{1}_{\{|\eta_p|>\varepsilon p^{1/2+\exp(-ts)}s^{1/2}\}}\Big] = 0. \tag{42}$$

Proof of (41). Using (5) we obtain

$$\begin{aligned}\mathbb{E}\Big[\sum_p \frac{\eta_p}{p^{1/2+\exp(-t_1 s)}}\sum_{p_*}\frac{\eta_{p_*}}{p_*^{1/2+\exp(-t_2 s)}}\Big] &= \sum_p \frac{1}{p^{1+\exp(-t_1 s)+\exp(-t_2 s)}} \\ &\sim\; -\log(\mathrm{e}^{-t_1 s}+\mathrm{e}^{-t_2 s}) \;\sim\; \min(t_1,t_2)s, \quad s\to+\infty.\end{aligned}$$

Proof of (42). For each $p \in \mathcal{P}$, each $s > 0$ and each $t > 0$, $p^{-1/2-\exp(-ts)} \leq 1$. Hence, the expression under the limit on the left-hand side of (42) does not exceed

$$\frac{\mathbb{E}[\eta^2 \mathbb{1}_{\{|\eta|>\varepsilon s^{1/2}\}}]}{s}\sum_p \frac{1}{p^{1+2\exp(-ts)}}.$$

As shown in the proof of (41), $\sum_p p^{-1-2\exp(-ts)} \sim ts$ as $s \to +\infty$. Further, $\mathbb{E}[\eta^2] < \infty$ entails $\lim_{s\to+\infty} \mathbb{E}[\eta^2 \mathbb{1}_{\{|\eta|>\varepsilon s^{1/2}\}}] = 0$. With these at hand, (42) follows.
(b) We have to prove tightness on $C[0,T]$ (the set of continuous functions defined on $[0,T]$) for each $T > 0$. Since $(B(t))_{t\in[0,T]}$ has the same distribution as $T^{1/2}(B(t))_{t\in[0,1]}$, it is enough to investigate the case $T = 1$ only.

As in the proof of Lemma 1, write, for $t \in [0,1]$ and $s \geq 1$,

$$\Big|\sum_{p\leq\lfloor s\rfloor} \frac{\eta_p}{p^{1/2+\exp(-ts)}}\Big| = \Big|\frac{T^*(\lfloor s\rfloor)}{(\lfloor s\rfloor)^{1/2+\exp(-ts)}} + (1/2+\exp(-ts))\int_1^{\lfloor s^{1/2}\rfloor} \frac{T^*(x)}{x^{3/2+\exp(-ts)}}\mathrm{d}x\Big|$$
$$\leq \frac{|T^*(\lfloor s\rfloor)|}{(\lfloor s\rfloor)^{1/2}} + 3/2 \max_{y\leq\lfloor s\rfloor} |T^*(y)| \int_1^\infty \frac{\mathrm{d}x}{x^{3/2}} \quad \text{a.s.}$$

Donsker's functional limit theorem (see, for instance, Theorem 14.1 on p. 146 in [3]) entails $n^{-1/2}\max_{k\leq n}|T_k| \overset{\mathrm{d}}{\to} |\mathcal{N}(0,1)|$ as $n \to \infty$, where $\overset{\mathrm{d}}{\to}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ denotes a random variable with the normal distribution of mean 0 and variance 1. This together with the prime number theorem ensures that

$$s^{-1/2}(\log s)^{1/2}\max_{y\leq\lfloor s\rfloor}|T^*(y)| = s^{-1/2}(\log s)^{1/2}\max_{k\leq\rho(\lfloor s\rfloor)}|T_k| \overset{\mathrm{d}}{\to} |\mathcal{N}(0,1)|, \quad s\to\infty.$$

As a consequence,

$$\lim_{s\to+\infty}\frac{1}{s^{1/2}}\sup_{t\in[0,1]}\Big|\sum_{p\leq\lfloor s\rfloor}\frac{\eta_p}{p^{1/2+\exp(-ts)}}\Big| = 0 \quad \text{in probability.} \tag{43}$$

Put $a(s) := (\log\lfloor s\rfloor)^{1/2}$ for $s \geq 1$. Arguing as in the proofs of (16) and (17) we infer

$$\lim_{s\to+\infty}\sup_{t\in[0,\,1]}\sum_{p\geq\lfloor s\rfloor+1}\frac{1}{p^{1/2+\exp(-ts)}}|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}a(s)\}} = 0 \quad \text{a.s.} \tag{44}$$

and

$$\lim_{s\to\infty}\sup_{t\in[0,\,1]}\sum_{p\geq\lfloor s\rfloor+1}\frac{1}{p^{1/2+\exp(-ts)}}\mathbb{E}\big[|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}a(s)\}}\big] = 0. \tag{45}$$

For instance, for $s \geq 1$,

$$\sup_{t\in[0,1]}\sum_{p\geq\lfloor s\rfloor+1}\frac{1}{p^{1/2+\exp(-ts)}}|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}a(s)\}} \leq \sum_{p\geq\lfloor s\rfloor+1}\frac{1}{p^{1/2}}|\eta_p|\mathbb{1}_{\{|\eta_p|>(p/\log p)^{1/2}a(s)\}} \quad \text{a.s.,}$$

and the right-hand side is equal to 0 for large enough s a.s.

Put

$$X^*(t,s) := \sum_{p\geq\lfloor s\rfloor+1}\frac{\eta_p^*(s)}{p^{1/2+\exp(-ts)}}, \quad t\geq 0,\ s>0,$$

where $\eta_p^*(s) := \eta_p\mathbb{1}_{\{|\eta_p|\leq(p/\log p)^{1/2}a(s)\}} - \mathbb{E}\big[\eta_p\mathbb{1}_{\{|\eta_p|\leq(p/\log p)^{1/2}a(s)\}}\big]$ for $p \in \mathcal{P}$ and $s > 0$. Relations (43), (44) and (45) guarantee that we are left with proving tightness of the distributions of $(X^*(t,s))_{t\in[0,1]}$ for large $s > 0$. By formula (7.8) on p. 82 in [3], it is enough to show that, for all $\varepsilon > 0$,

$$\lim_{i\to\infty}\limsup_{s\to\infty}\mathbb{P}\{\sup_{u,v\in[0,1],\,|u-v|\leq 2^{-i}}|X^*(u,s)-X^*(v,s)| > \varepsilon s^{1/2}\} = 0. \tag{46}$$

The proof of (46) is analogous to the last part of the proof of Lemma 4. We use dyadic partitions of $[0,1]$ by points $t^*_{j,m} := 2^{-j}m$ for $j \in \mathbb{N}_0$ and $m = 0, 1, \ldots, 2^j$. Similarly to the argument preceding formula (20) we infer

$$\sup_{u,v\in[0,1],\,|u-v|\leq 2^{-i}} |X^*(u,s) - X^*(v,s)| \leq \sum_{j\geq i} \max_{1\leq m\leq 2^j} |X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)|.$$

Thus, it suffices to prove that, for all $\varepsilon > 0$,

$$\lim_{i\to\infty} \limsup_{s\to+\infty} \mathbb{P}\Big\{ \sum_{j\geq i} \max_{1\leq m\leq 2^j} |X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)| > \varepsilon s^{1/2} \Big\} = 0.$$

Put $a^*_j := 2^{-j/2} j^2$ for $j \in \mathbb{N}_0$. The last limit relation follows if we can show that, for all $\varepsilon > 0$,

$$\lim_{i\to\infty} \limsup_{s\to+\infty} \sum_{j\geq i} \mathbb{P}\Big\{ \max_{1\leq m\leq 2^j} |X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)| > \varepsilon a^*_j s^{1/2} \Big\} = 0.$$

Denote by $A^*_j(s)$ and $B^*_j(u,p,s)$ the counterparts of $A_j(n)$ and $B_j(u,p,n)$, namely, for small $s > 0$, $j \in \mathbb{N}_0$ and $m \in [0, 2^j]$,

$$A^*_j(s) = \sum_{p\geq \lfloor s\rfloor+1} \frac{1}{p} \Big(\frac{1}{p^{\exp(-t^*_{j,m}s)}} - \frac{1}{p^{\exp(-t^*_{j,m-1}s)}} \Big)^2$$

and

$$B^*_j(u,p,s) = \frac{|u|}{p^{1/2}} \Big(\frac{1}{p^{\exp(-t^*_{j,m}s)}} - \frac{1}{p^{\exp(-t^*_{j,m-1}s)}} \Big) |\eta^*_p(s)|, \quad p \geq \lfloor s \rfloor + 1.$$

Then $A^*_j(s) \leq C2^{-j}s$ and, if $2^j \geq s$, $B^*_j(u,p,s) \leq |u|2^{1-j}s =: C^*_j(u,s)$, if $2^j \leq s$, $B^*_j(u,p,s) \leq 2|u| =: C^*_j(u,s)$. With these at hand we obtain

$$\mathbb{E}\big[\exp\big(\pm u(X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)) \big) \big] \leq \exp\Big(\frac{C2^{-j}su^2}{2} \mathrm{e}^{C^*_j(u,s)} \Big), \quad u \in \mathbb{R}$$

and thereupon

$$\begin{aligned}
\mathbb{P}\big\{ |X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)| > \varepsilon a^*_j s^{1/2} \big\} \\
\leq \exp(-u\varepsilon a^*_j s^{1/2}) \mathbb{E}[\exp(u|X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)|)] \\
\leq 2\exp\Big(-u\varepsilon 2^{-j/2} j^2 s^{1/2} + \frac{C2^{-j}su^2}{2} \mathrm{e}^{C^*_j(u,s)} \Big).
\end{aligned}$$

Putting $u = \varepsilon 2^{j/2} s^{-1/2}$ we conclude that $C^*_j(u,s) \leq 2\varepsilon$ and further

$$\begin{aligned}
\sum_{j\geq i} \mathbb{P}\Big\{ \max_{1\leq m\leq 2^j} |X^*(t^*_{j,m}, s) - X^*(t^*_{j,m-1}, s)| > \varepsilon a^*_j s^{1/2} \Big\} \\
\leq 2\exp(C\varepsilon^2 \mathrm{e}^{2\varepsilon}/2) \sum_{j\geq i} 2^j \mathrm{e}^{-\varepsilon^2 j^2} \ \to \ 0, \quad i \to \infty.
\end{aligned}$$

The proof of Theorem 1 is complete.

4 Proofs of Corollaries 1 and 2

Put $R(s) := \sum_p \sum_{k\geq 3} \frac{(-1)^{k+1}(f(p))^k}{kp^{k(1/2+s)}}$ for $s>0$. In the proof of Proposition 3.2 in [6] (see also Lemma 2.4 in [2]) it is shown that

$$\log \sum_{n\geq 1} \frac{f(n)}{n^{1/2+s}} + \frac{\log \zeta(1+2s)}{2} = \sum_p \frac{f(p)}{p^{1/2+s}} + R(s) + O(1) = \sum_p \frac{f(p)}{p^{1/2+s}} + O(1), \quad s\to 0+ \quad \text{a.s.}$$

Corollary 2 follows from this relation and Theorem 2.

Corollary 1 follows from Theorem 1 and a relation that we are now going to prove: for all $T>0$,

$$\lim_{s\to 0+} \frac{\sup_{t\in[0,T]} |R(s^t)|}{\log 1/s} = 0 \quad \text{in probability.}$$

Indeed, for $s\in(0,1)$ and $t\in[0,T]$,

$$|R(s^t)| \leq \sum_p \sum_{k\geq 3} \frac{1}{kp^{k(1/2+s^t)}} \leq \sum_p \frac{1}{p^{3(1/2+s^t)}} \frac{p^{1/2+s^t}}{p^{1/2+s^t}-1} \leq \frac{2^{1/2}}{2^{1/2}-1} \sum_p \frac{1}{p^{3(1/2+s^T)}}.$$

The right-hand side converges to $\frac{2^{1/2}}{2^{1/2}-1}\sum_p \frac{1}{p^{3/2}} < \infty$ as $s\to 0+$. This justifies the claimed limit relation.

5 Proof of Corollary 3

Starting with an Euler product representation

$$\sum_{n\geq 1} \frac{f(n)}{n^{1/2+s}} = \prod_p \Big(1 + \frac{f(p)}{p^{1/2+s}} + \frac{(f(p))^2}{p^{1+2s}} + \ldots + \frac{(f(p))^{k-1}}{p^{(1/2+s)(k-1)}}\Big)$$

we obtain

$$\begin{aligned}\log \sum_{n\geq 1} \frac{f(n)}{n^{1/2+s}} &= \sum_p \Big(\frac{f(p)}{p^{1/2+s}} + \frac{1}{p^{1+2s}}\Big) - \frac{1}{2}\sum_p \Big(\frac{f(p)}{p^{1/2+s}} + \frac{1}{p^{1+2s}}\Big)^2 + O(1) \\ &= \sum_p \frac{f(p)}{p^{1/2+s}} + \frac{1}{2}\sum_p \frac{1}{p^{1+2s}} + O(1) = \sum_p \frac{f(p)}{p^{1/2+s}} + \frac{1}{2}\log\zeta(1+2s) + O(1), \quad s\to 0+ \quad \text{a.s.}\end{aligned}$$

Put

$$R^*(s) := \sum_p \sum_{\ell=3}^{k-1} \frac{(f(p))^\ell}{p^{(1/2+s)\ell}} + \frac{1}{2}\sum_p \frac{1}{p^{1+2s}} + \sum_p \sum_{j\geq 1} \frac{(-1)^{j+1}}{j}\Big(\sum_{\ell=1}^{k-1} \frac{(f(p))^\ell}{p^{(1/2+s)\ell}}\Big)^j, \quad s>0,$$

so that

$$\log \sum_{n\geq 1} \frac{f(n)}{n^{1/2+s}} - \frac{\log\zeta(1+2s)}{2} = \sum_p \frac{f(p)}{p^{1/2+s}} + R^*(s) + O(1), \quad s\to 0+\,.$$

The fact that, for all $T>0$,

$$\lim_{s\to 0+} \frac{\sup_{t\in[0,T]} |R^*(s^t)|}{\log 1/s} = 0 \quad \text{in probability}$$

follows along the lines of the previous proof.
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