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ASYMPTOTIC FLUCTUATIONS IN SUPERCRITICAL CRUMP-MODE-JAGERS PROCESSES

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Consider a supercritical Crump–Mode–Jagers process $(\mathcal{Z}_t^\varphi)_{t\geq 0}$ counted with a random characteristic φ. Nerman's celebrated law of large numbers [*Z. Wahrsch. Verw. Gebiete* 57, 365–395, 1981] states that, under some mild assumptions, $e^{-\alpha t}\mathcal{Z}_t^\varphi$ converges almost surely as $t\to\infty$ to aW. Here, $\alpha>0$ is the Malthusian parameter, a is a constant and W is the limit of Nerman's martingale, which is positive on the survival event. In this general situation, under additional (second moment) assumptions, we prove a central limit theorem for $(\mathcal{Z}_t^\varphi)_{t\geq 0}$. More precisely, we show that there exist a constant $k\in\mathbb{N}_0$ and a function $H(t)$, a finite random linear combination of functions of the form $t^j e^{\lambda t}$ with $\alpha/2\leq \mathrm{Re}(\lambda)<\alpha$, such that $(\mathcal{Z}_t^\varphi - ae^{\alpha t}W - H(t))/\sqrt{t^k e^{\alpha t}}$ converges in distribution to a normal random variable with random variance. This result unifies and extends various central limit theorem-type results for specific branching processes.

CONTENTS

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1. Introduction. A general (Crump-Mode-Jagers) branching process starts at time 0 with a single individual, the ancestor, who is alive in the random time interval $[0,\zeta)$ for a random variable ζ, the life span, taking values in $[0,\infty]$. The ancestor produces offspring born at the points of a reproduction point process ξ on $[0,\infty)$. No particular assumption about the dependence structure between ξ and ζ is made. For each individual u that is ever born there is an independent copy (ξ_u,ζ_u) of the pair (ξ,ζ) that determines the birth times of the individual's offspring relative to u's time of birth and its life span.

The general branching process encompasses e.g. the Bienaymé-Galton-Watson process, the Yule process, the continuous-time Markov branching process, the Sevastyanov process, and the Bellman-Harris process. We refer to [25] for a more detailed account of the history of the general branching process and its predecessors.

The general branching process counted with a random characteristic at time t is the sum over all individuals ever born where the contribution of each individual to the sum is determined by some random characteristic that may take into account all aspects of the individual's life such as its age at time t, its life span, etc. This formulation makes it possible to treat at one go various quantities of interest derived from the general branching process such as the number of births up to time t, the number of individuals alive at time t, the number of individuals alive at time t which are younger than a given threshold $a>0$, etc. A formal description of the model will be given in Section 2.

General branching processes serve as models of biological populations such as humans, cells or plants [17, 25, 34, 41], as models for tumor growth [15, 34], but also for neutron chain reactions [3] or fragmentation [31] (after a change of time) to name but a few. The general branching process is also an important tool within related fields of applied probability or theoretical computer science. In fact, its applications in these fields are numerous and any attempt to give a complete survey here is hopeless. We confine ourselves to mentioning its successful application in the study of asymptotic properties of random graph growth models driven by preferential attachment dynamics [6, 9, 37, 45] and particularly random tree growth models [14, 22, 23, 32, 35, 42]. It is also used as an approximation for epidemic models [10, 47] and as a model of the initial phase of epidemics such as SARS, Ebola and SARS-CoV-2 [8, 11, 12], during which the disease spreads exponentially fast but the impact of population structure and preventive measures is still small [47].

The laws of large numbers of the supercritical general branching process counted with a random characteristic are due to Nerman [38, 39] in the single-type, non-lattice case, that is, when the reproduction point process is not concentrated on any lattice. There were earlier results for special cases, but here we refrain from sketching the history and instead refer to the introduction of [39]. The lattice version of Nerman's law of large numbers was proved by Gatzouras [16].

In view of the relevance of the general branching process in applications and the fact that the laws of large numbers date back as far as 1981, it is remarkable, and rather surprising, that there is no comprehensive central limit theorem for the general process counted with a random characteristic in the literature. However, there are results for related models indicating the intricate nature of the fluctuations that can occur. For the multi-type continuous-time Markov branching process with finite type space where individuals give birth only at the time of their death Athreya [4, 5] proved a central limit theorem and Janson [29] proved a functional central limit theorem. Asmussen and Hering [3, Section VIII.3] provide results for the asymptotic fluctuations of multi-type Markov branching processes with rather general type space. In principle, these results contain the single-type case of the general branching process since such a process can be seen as a Markov process in which the type of an individual at time t is its entire life history up to time t. However, this type space is large, and the assumptions of [3] are typically not satisfied except in special cases such as the case of the Galton-Watson process. Recently, Janson studied the asymptotic fluctuations of single-type supercritical general branching processes in the lattice case [30]. For the non-lattice case, there is a second-order result by Janson and Neininger [31] for Kolmogorov's conservative fragmentation model that may be translated into the language of general branching processes. It gives a central limit theorem for the number of individuals born up to time t, but it requires that the offspring variable $N := \xi([0,\infty))$ be bounded and the additional assumption that $\int e^{-x}\,\xi(\mathrm{d}x) = 1$ almost surely, a rather restrictive assumption in the context of general branching processes. Another related work is the paper by Charmoy, Croydon, and Hambly [13], where the authors investigate the fluctuations of the eigenvalue counting function related to certain random fractals. This problem can be addressed using limit theorems for specific Crump-Mode-Jagers processes. The random characteristics in this model are no longer assumed to be independent, which takes it beyond the scope of the present paper. It is worth noting that limit theorems for general branching processes were previously explored by Jagers and Nerman [27]. However, the conditions in this paper can be challenging to verify, even for relatively simple characteristics. Another related result is the central limit theorem for Nerman's martingale [24].

In the present paper, we close the gap in the literature and present a central limit theorem for the general branching process counted with a random characteristic. Our main result, Theorem 2.15, contains and extends all results for single-type processes summarized above. A non-exhaustive list of applications given in Section 3 contains Galton-Watson processes, Nerman's martingale and its complex-valued counterparts, epidemic models, Crump-Mode-Jagers processes with homogeneous Poisson offspring process and general lifetimes, and conservative fragmentation models.

Organization of the paper. The paper is organized as follows. In Section 2 we formally introduce the general branching process counted with a random characteristic. We further state and discuss the assumptions we are working with. In Section 2.3, we state the main result, Theorem 2.15, and its corollaries. We then apply our general results to some specific models in Section 3. Section 4 contains some preliminaries for the proofs. Nerman's martingale and further related martingales play a crucial role in our theory. All these martingales are introduced and discussed in Section 5. Section 6 is devoted to proving our main result, Theorem 2.15. Our central limit theorem is based on an asymptotic expansion of the mean of a general branching process counted with a random characteristic. Such asymptotic expansions are derived in Section 7. We close the paper with Section 8, in which possible future research directions are outlined.

2. Setup, preliminaries and main results. We continue with a formal description of the general branching process.

2.1. *The general branching process counted with a random characteristic.* We introduce the general (Crump-Mode-Jagers) branching process following Jagers [25, 26]. The process starts with a single individual, the ancestor, born at time 0. The ancestor produces offspring born at the points of a reproduction point process $\xi = \sum_{j=1}^{N} \delta_{X_j}$ on $[0,\infty)$ where $N = \xi([0,\infty))$ takes values in $\mathbb{N}_0 \cup \{\infty\}$ with $\mathbb{N}_0 := \{0,1,2,\ldots\}$ and $X_j := \inf\{t \geq 0 : \xi([0,t]) \geq j\}$. Here and throughout the paper, the infimum of the empty set is defined to be ∞. The ancestor has a random lifetime ζ, which may be dependent on ξ. Formally, ζ is a random variable assuming values in $[0,\infty]$.

Individuals are indexed by $u \in \mathcal{I} = \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$ according to their genealogy. Here, $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}^0 := \{\varnothing\}$ is the singleton set containing only the empty tuple $\varnothing$. We use the usual Ulam-Harris notation. We abbreviate a tuple $u = (u_1,\ldots,u_n) \in \mathbb{N}^n$ by $u_1 \ldots u_n$ and refer to n as the length or generation of u; we write $|u| = n$. In this context, any $u = u_1 \ldots u_n \in \mathcal{I}$ is called (potential) individual. Its ancestral line is encoded by

$$\varnothing \to u_1 \to u_1u_2 \to \ldots \to u_1 \ldots u_n = u$$

where u_1 is the u_1^{th} child of the ancestor, u_1u_2 the u_2^{th} child of u_1, etc. If $v = v_1 \ldots v_m \in \mathcal{I}$, then uv is short for $u_1 \ldots u_n v_1 \ldots v_m$. For $u \in \mathcal{I}$ and $i \in \mathbb{N}$, the individuals ui will be called children of u. Conversely, u will be called parent of ui. More generally, w will be called descendant of u (short: $u \preceq w$) iff $uv = w$ for some $v \in \mathcal{I}$. Conversely, u will be called an ancestor/progenitor of w. We write $u \prec w$ if $u \preceq w$ and $u \neq w$. Often, we shall refer to $\mathbb{N}^n$ as the (potential) n^{th} generation ($n \in \mathbb{N}_0$). With these notations, we have

$$|u| = n \quad \text{iff} \quad u \in \mathbb{N}^n \quad \text{iff} \quad u \text{ is an } n^{\text{th}} \text{ generation (potential) individual.}$$

For $u = u_1 \ldots u_n \in \mathbb{N}^n$ and $k \in \mathbb{N}_0$, let $u|_k$ denote the ancestor of u in the k^{th} generation. Formally, $u|_k$ is the restriction of the vector u to its first k components:

$$u|_k = \begin{cases} \varnothing & \text{if } k = 0, \\ u_1 \ldots u_k & \text{if } 1 \leq k \leq |u|, \\ u & \text{if } k > |u|. \end{cases} \tag{2.1}$$

For typographical reasons, we may sometimes write $v|k$ instead of $v|_k$. For $u \in \mathcal{I}$ let $u\mathcal{I}$ denote the subtree of $\mathcal{I}$ emanating from u, that is,

$$u\mathcal{I} := \{uv : v \in \mathcal{I}\} = \{w \in \mathcal{I} : w|_{|u|} = u\}.$$

For each $u \in \mathcal{I}$ there is an independent copy (ξ_u, ζ_u) of the pair (ξ, ζ) that determines the birth times of u's offspring relative to its time of birth, and the duration of its life. Quantities derived from (ξ_u, ζ_u) are indexed by u. For instance, N_u is the number of offspring of u and $X_{u,k}$ is the difference between the birth-time of the k^{th} child of u and u itself, etc. The birth-times $S(u)$ for $u \in \mathcal{I}$ are defined recursively. We set $S(\varnothing) := 0$ and, for $n \in \mathbb{N}_0$,

$$S(uj) := S(u) + X_{u,j} \quad \text{for } u \in \mathbb{N}^n \text{ and } j \in \mathbb{N}.$$

The family tree of all individuals ever born is denoted by $\mathcal{T} := \{u \in \mathcal{I} : S(u) < \infty\}$. We call

$$\mathcal{S} := \bigcap_{n\in\mathbb{N}} \{\#(\mathcal{T} \cap \mathbb{N}^n) \geq 1\}$$

the survival set and its complement $\mathcal{S}^{\mathsf{c}} = \cup_{n\in\mathbb{N}}\{\#(\mathcal{T} \cap \mathbb{N}^n) = 0\}$ the extinction set. The time of death of individual u is $S(u) + \zeta_u$. An individual u is alive at time $t \geq 0$ if it is born, but not yet dead at time t, i.e., if

$$S(u) \leq t < S(u) + \zeta_u.$$

We now construct the canonical space for the general branching process. For $u \in \mathcal{I}$, let $(\Omega_u, \mathcal{A}_u, P_u)$ be a copy of a given probability space $(\Omega_\varnothing, \mathcal{A}_\varnothing, P_\varnothing)$, the *life space* of the ancestor. An element $\omega \in \Omega_u$ is a possible life career for individual u and any property of interest of u like its mass at some age or its life span is viewed as a measurable function on the life space. In particular, ξ and ζ, the reproduction point process and the life span, are measurable functions defined on $(\Omega_\varnothing, \mathcal{A}_\varnothing)$.

From the life space, we construct the population space:

$$(\Omega, \mathcal{F}, \mathbb{P}) := \big(\bigtimes_{u\in\mathcal{I}} \Omega_u, \bigotimes_{u\in\mathcal{I}} \mathcal{A}_u, \bigotimes_{u\in\mathcal{I}} P_u\big).$$

For $u \in \mathcal{I}$, we write π_u for the projection $\pi_u : \bigtimes_{v\in\mathcal{I}} \Omega_v \to \Omega_u$ and θ_u for the shift $\theta_u((\omega_v)_{v\in\mathcal{I}}) = (\omega_{uv})_{v\in\mathcal{I}}$. To formally lift an entity χ defined on the life space, i.e. a function χ on Ω_u, to the population space, we define $\chi_u := \chi \circ \pi_u$. In particular, $\xi_u = \xi \circ \pi_u$ and $\zeta_u = \zeta \circ \pi_u$. In slight abuse of notation, if χ is defined on the life space, when working on the population space, we write χ instead of $\chi \circ \pi_\varnothing$. For instance, we sometimes write $\mathbb{P}(\zeta \le t)$ for $P_u(\zeta \le t) = \mathbb{P}(\zeta_\varnothing \le t)$. A technical remark is in order. Sometimes random variables independent of $\mathcal{F}$ appear. This means that when required, we work on a suitable extension of the space $(\Omega, \mathcal{F}, \mathbb{P})$.

We are interested in the general branching process counted with a random characteristic. A *random characteristic* φ is a random process on $(\Omega_\varnothing, \mathcal{A}_\varnothing, P_\varnothing)$ taking values in the Skorokhod space of right-continuous functions $f : \mathbb{R} \to \mathbb{R}^d$ with existing left limits at every point in $\mathbb{R}$. Such functions are called càdlàg for short. The characteristic φ may also be viewed as a stochastic process $\varphi : \Omega_\varnothing \times \mathbb{R} \to \mathbb{R}^d$, $(\omega, t) \mapsto \varphi(\omega, t)$ with right-continuous paths and existing left limits. Notice that unlike in some important references [16, 39], we allow, and actually need at some places, that $\varphi(t) \ne 0$ for some $t < 0$. It is known that such a process is product-measurable. Define $\varphi_u = \varphi \circ \pi_u$. By product measurability, $\varphi_u(t - S(u))$ is a random variable. Note that, for given $u \in \mathcal{I}$, φ_u is independent of $S(u)$. However, φ_u and $S(v)$ can be dependent, when u is an ancestor of v. The general branching process counted with characteristic φ is $\mathcal{Z}^\varphi = (\mathcal{Z}_t^\varphi)_{t\in\mathbb{R}}$ where $\mathcal{Z}_t^\varphi$ is defined by

$$\mathcal{Z}_t^\varphi := \sum_{u\in\mathcal{I}} \varphi_u(t - S(u)), \quad t \in \mathbb{R}. \tag{2.2}$$

Here, we use the convention $\varphi(-\infty) := 0$ and so the above sum involves only terms associated with individuals that are eventually born. In the special case $\varphi = \mathbb{1}_{[0,\zeta)}$,

$$\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}} = \sum_{u\in\mathcal{I}} \mathbb{1}_{[0,\zeta_u)}(t - S(u)) = \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t < S(u)+\zeta_u\}}, \tag{2.3}$$

i.e., $\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}}$ is the number of individuals alive at time t. Similarly,

$$N((t, t+a]) := \mathcal{Z}_{t+a}^{\mathbb{1}_{[0,a)}} = \sum_{u\in\mathcal{I}} \mathbb{1}_{[0,a)}(t + a - S(u)) = \sum_{u\in\mathcal{I}} \mathbb{1}_{\{t<S(u)\le t+a\}} \tag{2.4}$$

is the number of individuals born strictly after time t and up to and including time $t + a$, $a > 0$. The setup covers a wide range of possible applications. Some special cases and specific examples are covered in Section 3.

Notice that $\mathcal{Z}^\varphi$ is not well-defined *a priori*. Conditions for the finiteness of the general branching process are given in [25, Section 6.2]. For instance, the existence of a Malthusian parameter, a condition formally stated as (A1) below and assumed throughout this paper, implies that the number of individuals born up to and including time t is finite for all $t \ge 0$ almost surely, see [25, Theorem 6.2.3]. In particular, if (A1) holds, then $\mathcal{Z}_t^\varphi$ is well-defined whenever the characteristic φ vanishes on the negative half-line since in this case, the sum on the right-hand side of (2.2) has only finitely many non-vanishing summands almost surely. As, in general, we allow the characteristic φ to be real-valued and do not require that it vanishes on the negative half-line, we need a finiteness result that goes beyond [25, Theorem 6.2.3]. Jagers and Nerman [28] work under their assumption (6.1), which corresponds to our condition (A4) for $|\varphi|$ below. However, in our proofs, we shall require the well-definedness of $\mathcal{Z}_t^{\chi_\lambda}$ for a specific centered characteristic χ_λ, defined in Section 5, with $|\chi_\lambda|$ not satisfying (A4). Instead, we work under (A4) and (A5) to ensure the general branching process counted with a random characteristic to be well-defined. The corresponding result is Proposition 2.2 below.

2.2. *Assumptions.* We write $\mu(\cdot)$ for the intensity measure $\mathbb{E}[\xi(\cdot)]$ of the point process $\xi(\cdot)$, and $\mathcal{L}\mu$ for its Laplace transform, i.e.,

$$\mathcal{L}\mu(z) := \int e^{-zx}\,\mu(\mathrm{d}x) = \mathbb{E}\Big[\sum_{j=1}^{N} e^{-zX_j}\Big] \tag{2.5}$$

for all $z \in \mathbb{C}$ for which the above integral converges absolutely.

Throughout this paper we distinguish between the lattice and the non-lattice case. Here, we say that ξ is *lattice* if $\mu([0,\infty)\setminus h\mathbb{N}_0) = 0$ for some $h > 0$, and we say that ξ is *non-lattice*, otherwise. In the lattice case, without loss of generality, we assume that the lattice span is 1, $\mu([0,\infty)\setminus \mathbb{N}_0) = 0$ and $\mu([0,\infty)\setminus h\mathbb{N}_0) > 0$ for all $h > 1$. We set $\mathbb{G} := \mathbb{Z}$ in the lattice case and $\mathbb{G} := \mathbb{R}$ in the non-lattice case. We use the symbol ℓ to denote the counting measure on $\mathbb{Z}$ in the lattice case and the Lebesgue measure in the non-lattice case, respectively.

For a function $f : \mathbb{G} \mapsto \mathbb{C}$ we define the bilateral Laplace transform $\mathcal{L}f$ of f at $z \in \mathbb{C}$ by

$$\mathcal{L}f(z) := \int_{\mathbb{G}} e^{-zx} f(x)\,\ell(\mathrm{d}x)$$

whenever the integral converges absolutely.

The following assumption is essential in the law of large numbers [16, 39] and, therefore, also for the central limit theorem studied here.

(A1) There exists a *Malthusian parameter* $\alpha > 0$, i.e., an $\alpha > 0$ satisfying

$$\mathcal{L}\mu(\alpha) = \int e^{-\alpha x}\,\mu(\mathrm{d}x) = 1 \quad \text{and} \tag{2.6}$$

$$\mathbb{E}\Big[\sum_{j=1}^{N} X_j e^{-\alpha X_j}\Big] = -(\mathcal{L}\mu)'(\alpha) =: \beta \in (0,\infty). \tag{2.7}$$

Notice that (A1) implies the supercriticality of the general branching process, that is, $\mathbb{E}[N] = \mu([0,\infty)) \in (1,\infty]$, which, in turn, ensures that the underlying branching process survives with positive probability meaning that $\mathbb{P}(\mathcal{S}) > 0$. We stress that the case $\mathbb{P}(N = \infty) > 0$ is allowed. For the rest of the paper, we assume that (A1) is satisfied.

In our main results, we further assume that the Laplace transform $\mathcal{L}\mu$ is finite on an open half-space $\mathrm{Re}(z) > \vartheta$ for some $\vartheta < \frac{\alpha}{2}$:

(A2) There exists $\vartheta \in (0, \frac{\alpha}{2})$ such that

$$\mathcal{L}\mu(\vartheta) = \mathbb{E}\Big[\sum_{j=1}^{N} e^{-\vartheta X_j}\Big] < \infty. \tag{2.8}$$

For the central limit theorem, we need a second moment assumption for the point process ξ. Before we state it, we set k^* to be the maximum of all multiplicities of the roots of $\mathcal{L}\mu(z) = 1$ on the critical line $\mathrm{Re}(z) = \frac{\alpha}{2}$ or $k^* := \frac{1}{2}$ if there is no such root.

(A3) The random variable

$$\int (1 + x^{k^*-\frac{1}{2}}) e^{-\frac{\alpha}{2}x} \, \xi(\mathrm{d}x) = \sum_{j=1}^{N} \big(1 + X_j^{k^*-\frac{1}{2}}\big) e^{-\frac{\alpha}{2}X_j} \tag{2.9}$$

has finite second moment.

REMARK 2.1. Notice that Condition (A6) in [30], namely, the existence of a $\vartheta < \frac{\alpha}{2}$ such that

$$\mathbb{E}\bigg[\bigg(\sum_{j=1}^{N} e^{-\vartheta X_j}\bigg)^2\bigg] < \infty,$$

implies both our conditions (A2) and (A3). Janson's condition (A6) may be easier to check in cases where it holds.

The existence of the Malthusian parameter allows us to define a nonnegative martingale, called *Nerman's martingale*, namely,

$$W_t = W_t(\alpha) = \sum_{u \in \mathcal{C}_t} e^{-\alpha S(u)}, \quad t \geq 0 \tag{2.10}$$

where

$$\mathcal{C}_t := \{uj \in \mathcal{T} : S(u) \leq t < S(uj)\} \tag{2.11}$$

is the coming generation at time t. For the proof of the martingale property under (A1) see [39, Proposition 2.4]. We denote the almost sure limit of Nerman's martingale by W. Martingale theory implies that $\mathbb{E}[W] = 1$ iff $(W_t)_{t\geq 0}$ is uniformly integrable. Sufficient conditions for the latter can be found in [39, Corollary 3.3], [40, Theorem 2.1] and [16, Theorems 2.1 and 3.3]. In the given situation, $(W_t)_{t\geq 0}$ is uniformly integrable iff

$$\mathbb{E}[Z_1 \log_+ Z_1] < \infty \tag{$Z \log Z$}$$

holds where

$$Z_n = \sum_{|u|=n} e^{-\alpha S(u)}, \quad n \in \mathbb{N}_0. \tag{2.12}$$

The process $(Z_n)_{n\in\mathbb{N}_0}$ is also a nonnegative martingale, called Biggins' martingale, and it has the same almost sure limit W as Nerman's martingale $(W_t)_{t\geq 0}$ [16, Theorem 3.3]. Since (A3) immediately implies ($Z \log Z$), we infer that validity of (A1) and (A3) implies that both martingales, $(W_t)_{t\geq 0}$ and $(Z_n)_{n\in\mathbb{N}_0}$, converge almost surely and in L^1 to the same limit $W \geq 0$. Hence, in our theorems, ($Z \log Z$) will not be imposed explicitly, but will hold automatically whenever (A1) and (A3) are assumed to hold.

We continue with assumptions concerning the random characteristic φ. These assumptions are not made throughout the paper, but in certain results only. It will be explicitly stated, when this is the case.

Throughout the paper, if φ is a nonnegative or integrable characteristic (meaning that $\mathbb{E}[|\varphi(t)|]$ is finite for every $t \in \mathbb{R}$), then we write $\mathbb{E}[\varphi]$ for the (measurable) function that maps $t \mapsto \mathbb{E}[\varphi](t) := \mathbb{E}[\varphi(t)]$. This notation has the advantage that if X is a random variable, then we can write $\mathbb{E}[\varphi](X)$, which is again a random variable. Similarly, we write $\mathrm{Var}[\varphi]$ for the variance function $\mathbb{E}[(\varphi - \mathbb{E}[\varphi])^2]$, so $\mathrm{Var}[\varphi](t) = \mathrm{Var}[\varphi(t)]$. We start with an assumption regarding the mean of the characteristic.

(A4) $\varphi(t) \in L^1$ for every $t \in \mathbb{R}$ and $t \mapsto \mathbb{E}[\varphi](t)e^{-\alpha t}$ is directly Riemann integrable.

If (A1) is fulfilled, and if φ is a real-valued characteristic such that $|\varphi|$ satisfies (A4), then, in the non-lattice case, the law of large numbers by Nerman (see [28, Theorem 6.1]) states that

$$e^{-\alpha t}\mathcal{Z}_t^{\varphi} \to \beta^{-1}\mathcal{L}\mathbb{E}[\varphi](\alpha)W = \beta^{-1}\int e^{-\alpha x}\mathbb{E}[\varphi](x)\,\mathrm{d}x \cdot W \quad \text{as } t \to \infty \tag{2.13}$$

in probability. If, additionally, ($Z \log Z$) holds, then the convergence in (2.13) holds in L^1. To see this, first recall that ($Z \log Z$) implies $\mathbb{E}[W] = 1$ and hence $e^{-\alpha t}\mathbb{E}[\mathcal{Z}_t^{\varphi}]$ converges by the two-sided version of the key renewal theorem [2, Satz 2.5.3] to $\beta^{-1}\int e^{-\alpha x}\mathbb{E}[\varphi](x)\,\mathrm{d}x$, which is the expectation of the random variable on the right-hand side of (2.13). If φ is nonnegative, then the convergence of the first moment in combination with convergence in probability gives the convergence in L^1 by Proposition 4.12 in [33]. The case of general φ can be reduced to the case of nonnegative φ using the decomposition $\varphi = \varphi_+ - \varphi_-$ of φ into its positive part minus its negative part.

What is more, (A3) implies $\mathcal{L}\mu(\frac{\alpha}{2}) < \infty$ and hence the holomorphy of $\mathcal{L}\mu$ on the half-space $\mathrm{Re}(z) > \frac{\alpha}{2}$, which implies that all higher derivatives of $\mathcal{L}\mu$ in the point $z = \alpha$ exist. This in turn implies (5.4) in [39] (for instance with $g(t) = 1 \wedge t^{-2}$ there). Hence, Conditions 5.1 of [39], 3.2 of [16] and (3.2) and (3.4) of [36] are satisfied. This ensures that the convergence in (2.13) holds in the almost sure sense and in L^1 provided that

- φ vanishes on $(-\infty, 0)$ and satisfies Condition 5.2 of [39] in the non-lattice case;
- φ vanishes on $(-\infty, 0)$ and satisfies Condition 3.1 in [16] in the lattice case or
- φ satisfies Eq. (3.3) in [36] in the case that it does not vanish on the negative half-line.

The next two assumptions are conditions on the second moments of the characteristic φ.

(A5) $\varphi(t) \in L^2$ for every $t \in \mathbb{R}$ and $t \mapsto \mathrm{Var}[\varphi](t)e^{-\alpha t}$ is directly Riemann integrable.

(A6) For any $t \in \mathbb{R}$ there is an $\varepsilon > 0$ such that the family

$$(|\varphi(x)|^2)_{|x-t|\leq\varepsilon} \quad \text{is uniformly integrable.}$$

Notice that if φ is deterministic real-valued, then (A6) holds since φ is càdlàg, in particular, locally bounded.

In the lattice case, if $t \in \mathbb{Z}$, then $t - S(u) \in \mathbb{Z}$ for all individuals u with $S(u) < \infty$. Then $\mathcal{Z}_t^{\varphi}$ depends only on the values $\varphi_u(x)$ for $x \in \mathbb{Z}$ ($u \in \mathcal{I}$). In particular, the values of φ on $\mathbb{R} \setminus \mathbb{Z}$ are irrelevant for our purposes. Therefore, in the lattice case, we make the assumption that φ has paths that are constant on intervals of the form $[n, n+1)$, $n \in \mathbb{Z}$. With this assumption, condition (A6) is meaningful also in the lattice case, but reduces to the condition that $\varphi(x) \in L^2$ for all $x \in \mathbb{Z}$, a condition contained in (A5).

We continue with a proposition giving sufficient conditions for the general branching process counted with characteristic φ to be well-defined. Before this, we introduce the notion of an *admissible ordering* of $\mathcal{I}$. We call a sequence $u_1, u_2, \ldots \in \mathcal{I}$ an admissible ordering of $\mathcal{I}$ if

- $\mathcal{I}_n := \{u_1, \ldots, u_n\}$ is a subtree of the Ulam-Harris tree $\mathcal{I}$ of cardinality n,
- $\mathcal{I} = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$.

Admissible orderings exist. Indeed, we can construct $(u_n)_{n\in\mathbb{N}}$ recursively. First, let $u_1 = \varnothing$. If we have constructed u_i for $i = 1, \ldots, 2^k$ where $k \in \mathbb{N}_0$, then, for any $2^k < i \leq 2^{k+1}$, we set $u_i := u_{i-2^k}j$ with the smallest $j \in \mathbb{N}$ such that $u_{i-2^k}j \notin \{v_1, \ldots, u_{2^k}\}$, see Figure 1.

Recall that a series $\sum_{n\in\mathbb{N}} x_n$ in a Banach space $(X, \|\cdot\|)$ is said to converge unconditionally if, for any $\varepsilon > 0$, there is a finite $I \subseteq \mathbb{N}$ such that $\|\sum_{n\in J} x_n\| < \varepsilon$ for any finite

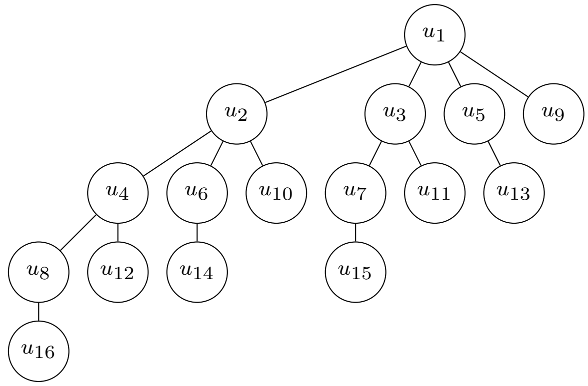

Fig 1: The subtree $\mathcal{I}_{16}$ for the particular admissible ordering of $\mathcal{I}$ given above.

$J \subseteq \mathbb{N} \setminus I$. An equivalent definition is that the series converges for any rearrangement. For this and other characterizations we refer the reader to [21].

PROPOSITION 2.2. *Suppose that (A1) holds and that φ is a random characteristic satisfying (A4) and (A5). Then, for every $t \in \mathbb{R}$,*

$$\mathcal{Z}_t^{\varphi} := \sum_{u \in \mathcal{I}} \varphi_u(t - S(u))$$

converges unconditionally in L^1 and almost surely over every admissible ordering of $\mathcal{I}$.

The proof of the proposition will be given in Section 4.2.

REMARK 2.3. Notice that $\mathcal{Z}_t^{\varphi} = \mathcal{Z}_t^{\varphi \mathbb{1}_{(-\infty,t]}}$ and thus from the proposition, we infer that $\mathcal{Z}_t^{\varphi}$ converges unconditionally in L^1 and almost surely for every $t \in \mathbb{R}$ if (A1) holds and, for every $t \in \mathbb{R}$, $\varphi \mathbb{1}_{(-\infty,t]}$ satisfies (A4) and (A5).

REMARK 2.4. Notice that by Proposition 2.2, the process $\mathcal{Z}^{\varphi}$ is defined almost surely for any fixed $t \in \mathbb{R}$. In other words, it is defined only up to a modification.

REMARK 2.5. Notice that if the random characteristics φ and ψ satisfy condition (A6), then so does any linear combination of them. Further, by the dominated convergence theorem, both the expectation function and the variance function of any linear combination of φ and ψ are càdlàg. This particularly implies that these functions are locally bounded and a.e. continuous. Consequently, if, in addition to (A6), also (A5) holds for φ and ψ, then (A5) also holds for any linear combination of φ and ψ. Indeed, for $\beta_1, \beta_2 \in \mathbb{R}$,

$$\mathrm{Var}[\beta_1\varphi(t) + \beta_2\psi(t)]e^{-\alpha t} \leq 2\beta_1^2 \,\mathrm{Var}[\varphi(t)]e^{-\alpha t} + 2\beta_2^2 \,\mathrm{Var}[\psi(t)]e^{-\alpha t}, \quad t \in \mathbb{R}.$$

By [43, Remark 3.10.5 on p. 237], the function in focus is directly Riemann integrable as a locally Riemann integrable function dominated by a directly Riemann integrable function.

The next proposition gives sufficient conditions for (A4), (A5) and (A6). To formulate it, we introduce the following notation. If $f : \mathbb{R} \to \mathbb{R}$ is a function, we set $f^*(t) := \sup_{|x-t| \leq 1} |f(x)|$. This notation extends immediately (pathwise) to random càdlàg functions such as random characteristics φ.

PROPOSITION 2.6. *Suppose that (A1) holds.*

(a) *If $f:\mathbb{R}\to\mathbb{R}$ is càdlàg and $\int f^*(x)\,\mathrm{d}x<\infty$, then f is directly Riemann integrable. Conversely, if $f:\mathbb{R}\to\mathbb{R}$ is directly Riemann integrable, then so is f^*. ((A1) is not needed.)*

(b) *If a random characteristic φ satisfies*

$$\int \mathbb{E}[\varphi^*](x)e^{-\alpha x}\,\mathrm{d}x<\infty, \tag{2.14}$$

then (A4) holds for φ.

(c) *If a random characteristic φ satisfies*

$$\int \mathbb{E}\big[(\varphi^*)^2\big](x)e^{-\alpha x}\,\mathrm{d}x<\infty, \tag{2.15}$$

then φ also satisfies (A5) and (A6).

By Λ we denote the set of solutions to the equation

$$\mathcal{L}\mu(\lambda)=1 \tag{2.16}$$

such that $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$ and by $\partial\Lambda$ we denote the set of roots on the *critical line* $\mathrm{Re}(\lambda)=\frac{\alpha}{2}$. In the lattice case, $\mathcal{L}\mu$ is $2\pi\mathrm{i}$-periodic, and we define Λ to be the set of λ with $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$ satisfying (2.16) and $\mathrm{Im}(\lambda)\in(-\pi,\pi]$. Analogously, in this case, $\partial\Lambda$ denotes the set of roots λ with $\mathrm{Re}(\lambda)=\frac{\alpha}{2}$ satisfying $\mathrm{Im}(\lambda)\in(-\pi,\pi]$. Finally, in both the non-lattice and the lattice case, we set $\Lambda_\geq:=\Lambda\cup\partial\Lambda$. Notice that $\alpha\in\Lambda$ and that every other element $\lambda\in\Lambda_\geq,\lambda\neq\alpha$ satisfies $\mathrm{Re}(\lambda)\in[\frac{\alpha}{2},\alpha)$ and $\mathrm{Im}(\lambda)\neq 0$. Further, $\lambda=\theta+\mathrm{i}\eta\in\Lambda_\geq$ implies that the complex conjugate $\overline{\lambda}=\theta-\mathrm{i}\eta\in\Lambda_\geq$ except if $\eta=\pi$ in the lattice case.

Although one may consider cases where $\Lambda_\geq$ contains infinitely many elements, in all relevant examples $\Lambda_\geq$ is finite. Therefore, and for simplicity, we assume throughout the paper the following:

(A7) The set of roots $\Lambda_\geq$ is finite.

We stress that if μ has a density with respect to the Lebesgue measure and (A2) holds, then also (A7) holds. This is justified in the proof of Lemma 7.3 in combination with Remark 7.5.

2.3. *Main results.* For each root $\lambda\in\mathbb{C}$ of the function $\mathcal{L}\mu-1$, we denote its multiplicity by $k(\lambda)\in\mathbb{N}$. Then, for any $j=0,\dots,k(\lambda)-1$, we can define

$$W_t^{(j)}(\lambda):=(-1)^j\sum_{u\in\mathcal{C}_t}S(u)^je^{-\lambda S(u)},\qquad t\in\mathbb{R}, \tag{2.17}$$

where $\mathcal{C}_t$ is the coming generation at time t formally defined in (2.11). The Malthusian parameter $\alpha>0$ is a root of multiplicity 1 and gives rise to one martingale, namely, Nerman's martingale $(W_t)_{t\in\mathbb{R}}=(W_t(\alpha))_{t\in\mathbb{R}}$ defined in (2.10), which is of great importance in the law of large numbers for the general branching process. On the other hand, the martingales corresponding to $\lambda\in\Lambda$ are relevant in the central limit theorem.

THEOREM 2.7. *Suppose that (A1) through (A3) hold. Then, for any $\lambda\in\Lambda$ and $j=0,\dots,k(\lambda)-1$, the process $(W_t^{(j)}(\lambda))_{t\geq 0}$ is a martingale and there is a random variable $W^{(j)}(\lambda)\in L^2$ such that*

$$W_t^{(j)}(\lambda)\to W^{(j)}(\lambda)\quad \textit{a. s. and in } L^2 \textit{ as } t\to\infty.$$

There are more technicalities to deal with before the main result (Theorem 2.15 below) can be stated in its most general form. Therefore, we shall first present illustrative special cases through Theorems 2.8, 2.9 and 2.10 (the proofs indeed reveal that they are corollaries of our main result, Theorem 2.15). We start with the non-lattice case. To this end, we need one more piece of notation. For a function $f:\mathbb{R}\mapsto\mathbb{R}$ we define the total variation function $\mathrm{V}f$ by

$$\mathrm{V}f(x) := \sup\left\{\sum_{j=1}^{n} |f(x_j)-f(x_{j-1})| : -\infty < x_0 < \ldots < x_n \leq x,\ n\in\mathbb{N}\right\} \tag{2.18}$$

for $x\in\mathbb{R}$. In some theorems related to the non-lattice case we require the following additional assumption on the characteristic φ:

$$\int (\mathrm{V}\mathbb{E}[\varphi])(x)\big(e^{-\vartheta x}+e^{-\alpha x}\big)\,\mathrm{d}x < \infty \tag{2.19}$$

for some $\vartheta < \frac{\alpha}{2}$. We use the symbol ϑ both in (A2) and in (2.19) to denote some parameter $< \frac{\alpha}{2}$ at which the corresponding condition is satisfied. If we make both assumptions at the same time, there is no harm in assuming that the ϑ's coincide, which is why we do not distinguish them by our notation.

THEOREM 2.8. *Suppose that (A1) through (A3) hold and that the intensity measure μ has a density with respect to the Lebesgue measure. Further, suppose that there are no roots of the equation $\mathcal{L}\mu(z)=1$ in the strip $\vartheta<\mathrm{Re}(z)<\alpha$. Then, for any characteristic φ satisfying (A5), (A6) and (2.19), there exists $\sigma\geq 0$ such that, for $a_\alpha := \beta^{-1}\int\mathbb{E}[\varphi(x)]e^{-\alpha x}\,\mathrm{d}x$ and a standard normal random variable $\mathcal{N}$ independent of W,*

$$e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - a_\alpha e^{\alpha t}W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N} \quad \text{as } t\to\infty.$$

The constant σ can be explicitly computed, see the formula (2.25) given in Theorem 2.15 below.

THEOREM 2.9. *Suppose that (A1) through (A3) hold and that the intensity measure μ has a density with respect to the Lebesgue measure. Then (A7) holds and there are $b_{\lambda,l}$, $l=0,\ldots,k(\lambda)-1$, $\lambda\in\Lambda_\geq$ satisfying $b_{\overline{\lambda},l}=\overline{b_{\lambda,l}}$ such that for any characteristic φ satisfying (A5), (A6) and (2.19) there exists $\sigma\geq 0$ such that, for a standard normal random variable $\mathcal{N}$ independent of W, the following assertions hold.*

(i) *If there are no roots of $\mathcal{L}\mu(z)=1$ on the critical line $\mathrm{Re}(z)=\frac{\alpha}{2}$, then*

$$e^{-\frac{\alpha}{2}t}\left(\mathcal{Z}_t^\varphi - \sum_{\lambda\in\Lambda} e^{\lambda t}\sum_{l=0}^{k(\lambda)-1} b_{\lambda,l}\sum_{j=0}^{l}\binom{l}{j}W^{(j)}(\lambda)\int (t-x)^{l-j}\mathbb{E}[\varphi(x)]e^{-\lambda x}\,\mathrm{d}x\right) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

as $t\to\infty$.

(ii) *Otherwise, let $k\in\mathbb{N}$ be the maximal multiplicity $k(\lambda)$ of a root $\lambda\in\partial\Lambda$. Then*

$$e^{-\frac{\alpha}{2}t}t^{-k+\frac12}\left(\mathcal{Z}_t^\varphi - \sum_{\lambda\in\Lambda} e^{\lambda t}\sum_{l=0}^{k(\lambda)-1} b_{\lambda,l}\sum_{j=0}^{l}\binom{l}{j}W^{(j)}(\lambda)\int (t-x)^{l-j}\mathbb{E}[\varphi(x)]e^{-\lambda x}\,\mathrm{d}x\right) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}$$

as $t\to\infty$.

The lattice analogue of Theorem 2.9 is given next.

THEOREM 2.10. *Suppose that (A1) through (A3) hold and that the intensity measure μ is lattice with span* 1*. Then there are $b_{\lambda,l}$, $l=0,\dots,k(\lambda)-1$, $\lambda\in\Lambda_{\geq}$ satisfying $b_{\overline{\lambda},l}=\overline{b_{\lambda,l}}$ such that for any characteristic φ satisfying*

$$\sum_{n\in\mathbb{Z}}|\mathbb{E}[\varphi(n)]|(e^{-\vartheta n}+e^{-\alpha n})<\infty,$$

for some $\vartheta<\frac{\alpha}{2}$ and

$$\sum_{n\in\mathbb{Z}}\mathrm{Var}[\varphi](n)e^{-\alpha n}<\infty$$

there exists $\sigma\geq 0$ such that, for a standard normal random variable $\mathcal{N}$ independent of W, the following assertions hold.

(i) *If there are no roots of $\mathcal{L}\mu(z)=1$ on the critical line* $\mathrm{Re}(z)=\frac{\alpha}{2}$, *then*

$$e^{-\frac{\alpha}{2}t}\left(\mathcal{Z}_t^{\varphi}-\sum_{\lambda\in\Lambda}e^{\lambda t}\sum_{l=0}^{k(\lambda)-1}b_{\lambda,l}\sum_{j=0}^{l}\binom{l}{j}W^{(j)}(\lambda)\sum_{n\in\mathbb{Z}}(t-n)^{l-j}\mathbb{E}[\varphi(n)]e^{-\lambda n}\right)\xrightarrow{\mathrm{d}}\sigma\sqrt{\frac{W}{\beta}}\mathcal{N}$$

as $t\to\infty$, $t\in\mathbb{N}$.

(ii) *Otherwise, let $k\in\mathbb{N}$ be the maximal multiplicity $k(\lambda)$ of the roots $\lambda\in\partial\Lambda$. Then*

$$e^{-\frac{\alpha}{2}t}t^{-k+\frac{1}{2}}\left(\mathcal{Z}_t^{\varphi}-\sum_{\lambda\in\Lambda}e^{\lambda t}\sum_{l=0}^{k(\lambda)-1}b_{\lambda,l}\sum_{j=0}^{l}\binom{l}{j}W^{(j)}(\lambda)\sum_{n\in\mathbb{Z}}(t-n)^{l-j}\mathbb{E}[\varphi(n)]e^{-\lambda n}\right)\xrightarrow{\mathrm{d}}\sigma\sqrt{\frac{W}{\beta}}\mathcal{N}$$

as $t\to\infty$, $t\in\mathbb{N}$.

REMARK 2.11. In Theorems 2.9 and 2.10, we do not exclude the case $\sigma=0$. There, a more precise limit theorem can be derived with the help of Theorem 2.15. In particular, the expression in parentheses in (i) is just a deterministic function of the order $O(e^{\theta t})$ for some $\theta<\frac{\alpha}{2}$ (cf. Theorem 2.15(i)). In case (ii), we need to additionally subtract a linear combination of $e^{\lambda t}t^l$, where λ runs over the roots on the critical line and $l<k(\lambda)$ with $k(\lambda)$ denoting the order of the root, and to use a different normalization in order to get a nontrivial limit.

REMARK 2.12. The constants $b_{\lambda,l}$, $l=0,\dots,k(\lambda)-1$, $\lambda\in\Lambda_{\geq}$ can be computed using Proposition 7.9.

REMARK 2.13. Note that the fluctuations in Theorem 2.10 and 2.9 are very similar to fluctuations obtain by Janson [29] in the multi-type case where the birth times constitutes a homogeneous Poisson point process. This indicates that there may be a general theorem describing the fluctuations of multitype CMJ process, which covers the aforementioned models (cf. Open Problem 1 in Section 8).

In order to obtain the asymptotic expansion of $\mathcal{Z}_t^\varphi$ as presented in Theorems 2.8, 2.9 and 2.10, we first need an expansion for the mean $m_t^\varphi := \mathbb{E}[\mathcal{Z}_t^\varphi]$, $t \in \mathbb{G}$. It turns out that, under suitable assumptions, the following holds:

$$m_t^\varphi = \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda \in \Lambda_\geq} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l + r(t), \qquad t \in \mathbb{G} \tag{2.20}$$

for some constants $a_{\lambda,l} \in \mathbb{C}$ and a function r satisfying $|r(t)| \leq Ce^{\alpha t/2}/(1+t^2)$ for all $t \in \mathbb{G}$ and some finite constant $C \geq 0$.

We shall provide three different sets of sufficient conditions for (2.20) to hold. The first case is when the characteristic φ is chosen in such a way that $\mathcal{Z}_t^\varphi$ is a rescaled martingale (see Section 5). In this case, (2.20) holds trivially. The second case is when $\mathbb{G} = \mathbb{Z}$. Then expansion (2.20) is obtained in Lemma 7.1 via generating functions. The third case is the non-lattice case where under the additional (technical) assumption (7.7), expansion (2.20) is derived in Lemma 7.6. There might be more examples of $\mathcal{Z}_t^\varphi$ that are not covered by any of the three sufficient conditions, even though the corresponding expansion of $\mathbb{E}[\mathcal{Z}_t^\varphi]$ is of form (2.20). For this reason, we formulate our main result, Theorem 2.15, for processes $\mathcal{Z}_t^\varphi$ for which $\mathbb{E}[\mathcal{Z}_t^\varphi]$ satisfies (2.20). What is more, in some examples, it is in fact more convenient to directly check that (2.20) holds rather than checking the assumptions of Theorem 2.9 or 2.10, see e. g. Sections 3.1 and 3.2.

From (2.20) we can obtain an asymptotic expansion of $\mathcal{Z}_t^\varphi$ where the principal terms are of the form a constant times $e^{\lambda t} t^j W^{(j)}(\lambda)$ for $\lambda \in \Lambda$ and $j = 0, \dots, k(\lambda)-1$. More precisely, the principal terms are given by the expression

$$H_\Lambda(t) := \sum_{\lambda \in \Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} \sum_{j=0}^{l} a_{\lambda,l} \binom{l}{j} t^j W^{(l-j)}(\lambda). \tag{2.21}$$

If, additionally, there are roots $\lambda \in \partial\Lambda$, then the next terms in the expansion are given by the following deterministic sum

$$H_{\partial\Lambda}(t) := \sum_{\lambda \in \partial\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l, \qquad t \in \mathbb{R}. \tag{2.22}$$

(Of course, if $\Lambda_\geq = \partial\Lambda \cup \{\alpha\}$, the terms from $H_{\partial\Lambda}(t)$ are the subleading terms.) We set $H(t) := H_\Lambda(t) + H_{\partial\Lambda}(t)$, $t \in \mathbb{R}$. Further, for any $\lambda \in \partial\Lambda$ and $l = 0, \dots, k(\lambda)-1$, we define a random variable $R_{\lambda,l}$ by

$$R_{\lambda,l} := \sum_{j=l}^{k(\lambda)-1} a_{\lambda,j} \binom{j}{l} \sum_{k=1}^{N} (-X_k)^{j-l} e^{-\lambda X_k}. \tag{2.23}$$

Assumption (A3) guarantees that $R_{\lambda,l} \in L^2$ for all $\lambda \in \partial\Lambda$ and $l = 0, \dots, k(\lambda)-1$. We may thus define

$$\rho_l^2 := \sum_{\substack{\lambda \in \partial\Lambda: \\ k(\lambda) > l}} \mathrm{Var}[R_{\lambda,l}]$$

where $\mathrm{Var}[R_{\lambda,l}] = \mathbb{E}[|R_{\lambda,l}|^2] - |\mathbb{E}[R_{\lambda,l}]|^2$. In general, throughout the paper, if Y is a complex-valued random variable with finite mean, we set $\mathrm{Var}[Y] := \mathbb{E}[|Y - \mathbb{E}[Y]|^2]$.

As a final preparation for our main result, we recall the fact that if a sequence of random variables $(Y_n)_{n\in\mathbb{N}_0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ converges in distribution to some random variable

Y, this convergence is said to be *stable* if for all continuity points y of the distribution function of Y and all $E \in \mathcal{F}$, the limit $\lim_{n\to\infty} \mathbb{P}(\{Y_n \le y\} \cap E)$ exists. In this case, we write $Y_n \overset{\mathrm{st}}{\Rightarrow} Y$. An alternative characterization is the following. There is a copy Y^* of Y defined on some extension of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for every $\mathcal{F}$-measurable random variable X, it holds that

$$(Y_n, X) \overset{\mathrm{d}}{\Rightarrow} (Y^*, X) \text{ as } n \to \infty,$$

see [1, Condition (B')]. Without loss of generality we may and will assume that $Y = Y^*$ whenever we write $Y_n \overset{\mathrm{st}}{\Rightarrow} Y$. The second definition is more convenient as it allows to manipulate the sequence Y_n (e.g. by multiplying with another random variable) without loosing the convergence in distribution. In Remark 2.17 below we prove (2.26) using such an argument.

REMARK 2.14. Although it is not stated explicitly but it follows from the proofs that the convergences in Theorems 2.8, 2.9 and 2.10 are in fact stable.

For a measurable function f and a measure ν (possibly random) on the Borel σ-field, we write $f * \nu = \nu * f$ for the Lebesgue-Stieltjes convolution of f and ν, i.e., $f * \nu(t) = \nu * f(t) = \int f(t-x)\,\nu(\mathrm{d}x)$ whenever the integral exists. In particular,

$$f * \xi(t) = \int f(t-x)\,\xi(\mathrm{d}x) = \sum_{j=1}^{N} f(t - X_j), \quad t \in \mathbb{R}.$$

THEOREM 2.15. *Suppose that ξ satisfies (A1) through (A3) and (A7) and that the real-valued characteristic φ satisfies (A4) through (A6). Further, assume that m_t^φ satisfies (2.20) with $\sup_{t\in\mathbb{G}}(1+t^2)e^{-\frac{\alpha}{2}t}|r(t)| < \infty$, and let $n := \max\{l \in \mathbb{N}_0 : \rho_l > 0\}$ with $n = -1$ if the set is empty; in that case we set $\rho_{-1} := 0$. Then there exists a finite constant $\sigma \ge 0$ such that, with*

$$a_t^2 := \sigma^2 + \tfrac{\rho_n^2}{2n+1} t^{2n+1}, \qquad t > 0,$$

it holds

(i) *if $\sigma^2 = \rho_n^2 = 0$, then, $t \mapsto H(t) + r(t)$ is a càdlàg modification of the process $\mathcal{Z}^\varphi$,*
(ii) *if $\sigma^2 > 0$ or $\rho_n^2 > 0$, then*

$$a_t^{-1} e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - H(t)\big) \overset{\mathrm{st}}{\Rightarrow} \sqrt{\tfrac{W}{\beta}}\mathcal{N} \tag{2.24}$$

as $t \to \infty$, $t \in \mathbb{G}$ where $\mathcal{N}$ is a standard normal random variable independent of $\mathcal{F}$ and β is as defined in (2.7).

If $n = -1$ the constant σ can be explicitly computed, namely,

$$\sigma^2 = \int \mathrm{Var}\big[\varphi(x) + h^\varphi * \xi(x)\big] e^{-\alpha x}\,\ell(\mathrm{d}x) \tag{2.25}$$

where

$$h^\varphi(t) := m_t^\varphi - \sum_{\lambda\in\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l.$$

In the situation of Theorem 2.15, the following remarks are in order.

REMARK 2.16. Observe that, in the non-lattice case, if $\lambda \in \Lambda_{\geq}$, then so is $\overline{\lambda}$ and $k(\overline{\lambda}) = k(\lambda)$. Moreover, as m_t^φ is real for any $t \in \mathbb{R}$, we have

$$m_t^\varphi = \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l e^{\lambda t} + r(t) = \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} a_{\overline{\lambda},l} t^l e^{\overline{\lambda} t} + r(t)$$

$$= \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} \overline{a_{\overline{\lambda},l}} t^l e^{\lambda t} + \overline{r(t)} = \overline{m_t^\varphi}, \quad t \in \mathbb{R},$$

whence

$$0 = m_t^\varphi - \overline{m_t^\varphi} = \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} (a_{\lambda,l} - \overline{a_{\overline{\lambda},l}}) t^l e^{\lambda t} + o(e^{\frac{\alpha}{2} t}) \quad \text{as } t \to \infty,\ t \in \mathbb{R}.$$

Next, we can choose $h > 0$ such that the $e^{\lambda h}$, $\lambda \in \Lambda_{\geq}$ are distinct. Recalling that $\mathrm{Re}(\lambda) \geq \frac{\alpha}{2}$ for $\lambda \in \Lambda_{\geq}$, Lemma A.1 gives that $a_{\lambda,l} - \overline{a_{\overline{\lambda},l}} = 0$, that is, $\overline{a_{\lambda,l}} = a_{\overline{\lambda},l}$. In particular, $r(t)$ is real for any $t \in \mathbb{R}$.

A similar reasoning in the lattice case gives $\overline{a_{\lambda,l}} = a_{\overline{\lambda},l}$ for all $\lambda \in \Lambda$ with $|\mathrm{Im}(\lambda)| < \pi$ and $a_{\lambda,l} \in \mathbb{R}$ if $\mathrm{Im}(\lambda) = \pi$.

REMARK 2.17. Observe that for $N(t) := \mathcal{Z}_t^{\mathbb{1}_{[0,\infty)}}$, the number of individuals born up to and including time t, by (2.13), we have

$$e^{-\alpha t} N(t) \to \frac{1}{\beta} \int\limits_{[0,\infty)} e^{-\alpha x}\, \ell(\mathrm{d}x)\, W = \frac{c_\alpha}{\beta} W \quad \text{a. s. as } t \to \infty,\ t \in \mathbb{G},$$

where $c_\alpha = (1 - e^{-\alpha})^{-1}$ in the lattice case, $c_\alpha = \alpha^{-1}$ in the non-lattice case. The stable convergence in (2.24) yields

$$a_t^{-1} \sqrt{\frac{c_\alpha}{N(t)}} \big(\mathcal{Z}_t^\varphi - H(t)\big) \stackrel{\mathrm{d}}{\to} \mathcal{N} \quad \text{as } t \to \infty \quad \text{conditionally given } \mathcal{S}. \tag{2.26}$$

Indeed, with $G(t) := a_t^{-1} e^{-\frac{\alpha}{2} t} \big(\mathcal{Z}_t^\varphi - H(t)\big)$,
(2.26) is equivalent to

$$\mathbb{P}(\mathcal{S})^{-1} \cdot \mathbb{E}\Big[g\Big(\sqrt{\tfrac{c_\alpha e^{\alpha t}}{N(t)}} G(t)\Big) \mathbb{1}_{\mathcal{S}}\Big] \to \mathbb{E}[g(\mathcal{N})]$$

as $t \to \infty$, for any continuous, nonnegative function g bounded by 1. With $F(t) := \sqrt{\frac{c_\alpha W e^{\alpha t}}{\beta N(t)}} \mathbb{1}_{\mathcal{S}} + \mathbb{1}_{\mathcal{S}^c}$, which is well-defined since $N(t) > 0$ on $\mathcal{S}$, the above convergence can be rewritten as

$$\mathbb{P}(\mathcal{S})^{-1} \cdot \mathbb{E}\Big[g\Big(\sqrt{\beta} \tfrac{F(t) G(t)}{\sqrt{W}}\Big) \mathbb{1}_{\mathcal{S}}\Big] \to \mathbb{E}[g(\mathcal{N})] \quad \text{as } t \to \infty. \tag{2.27}$$

For any fixed $\varepsilon > 0$, there is some $\delta > 0$ such that $\mathbb{P}(\mathcal{S} \cap \{W < \delta\}) = \mathbb{P}(0 < W < \delta) \leq \varepsilon$ and consequently

$$\Big|\mathbb{E}\Big[g\Big(\sqrt{\beta} \tfrac{F(t) G(t)}{\sqrt{W}}\Big) \mathbb{1}_{\mathcal{S}}\Big] - \mathbb{E}\Big[g\Big(\sqrt{\beta} \tfrac{F(t) G(t)}{\sqrt{W \vee \delta}}\Big) \mathbb{1}_{\mathcal{S}}\Big]\Big| \leq \varepsilon$$

as well as

$$\Big|\mathbb{E}[g(\mathcal{N}) \mathbb{1}_{\mathcal{S}}] - \mathbb{E}\Big[g\Big(\tfrac{\sqrt{W} \mathcal{N}}{\sqrt{W \vee \delta}}\Big) \mathbb{1}_{\mathcal{S}}\Big]\Big| \leq \varepsilon.$$

On the other hand, (2.24) yields

$$(G(t), W, \mathbb{1}_{\mathcal{S}}) \xrightarrow{\mathrm{d}} (\sqrt{\tfrac{W}{\beta}}\mathcal{N}, W, \mathbb{1}_{\mathcal{S}}) \quad \text{as } t \to \infty.$$

Since $F(t) \to 1$ almost surely, Slutsky's theorem implies

$$(G(t), W, \mathbb{1}_{\mathcal{S}}, F(t)) \xrightarrow{\mathrm{d}} (\sqrt{\tfrac{W}{\beta}}\mathcal{N}, W, \mathbb{1}_{\mathcal{S}}, 1) \quad \text{as } t \to \infty.$$

Taking advantage of the fact that the function $(u,v,x,y) \mapsto g(\sqrt{\beta}\frac{uy}{\sqrt{v \vee \delta}})(|x| \wedge 1)$ is bounded and continuous, we conclude

$$\mathbb{E}\Big[g\Big(\sqrt{\beta}\tfrac{F(t)G(t)}{\sqrt{W \vee \delta}}\Big)\mathbb{1}_{\mathcal{S}}\Big] \to \mathbb{E}\Big[g\Big(\tfrac{\sqrt{W}\mathcal{N}}{\sqrt{W \vee \delta}}\Big)\mathbb{1}_{\mathcal{S}}\Big]$$

as $t \to \infty$, and consequently

$$\limsup_{t\to\infty} \Big|\mathbb{E}\Big[g\Big(\sqrt{\beta}\tfrac{F(t)G(t)}{\sqrt{W}}\Big)\mathbb{1}_{\mathcal{S}}\Big] - \mathbb{E}[g(\mathcal{N})\mathbb{1}_{\mathcal{S}}]\Big| \leq 2\varepsilon.$$

Letting now ε to 0 and using the independence of $\mathcal{N}$ and $\mathbb{1}_{\mathcal{S}}$ we get (2.27) and thereby (2.26).

REMARK 2.18. (i) Notice that formula (2.25) is not well-defined in the case $n \geq 0$ as then the integral diverges. However, in this case, the exact value of σ is irrelevant and can be set to $\sigma := 0$.

(ii) In the non-lattice case the variance σ^2 given by (2.25) can be calculated using the bilateral Laplace transform $\mathcal{L}h^\varphi$ of h^φ. Indeed, by Plancherel's theorem,

$$\begin{aligned}
\sigma^2 &= \int \mathrm{Var}[\varphi(x) + h^\varphi * \xi(x)]e^{-\alpha x}\,\mathrm{d}x \\
&= \mathbb{E}\Big[\int \big(\big(\varphi(x) - \mathbb{E}[\varphi](x) + (h^\varphi * (\xi - \mu))(x)\big)e^{-\frac{\alpha}{2}x}\big)^2\,\mathrm{d}x\Big] \\
&= \frac{1}{2\pi}\mathbb{E}\Big[\int \big|\mathcal{L}((\varphi(\cdot) - \mathbb{E}[\varphi](\cdot))e^{-\alpha\cdot/2})(\mathrm{i}\eta) + \mathcal{L}\big(((\xi-\mu) * h^\varphi)(\cdot)e^{-\frac{\alpha}{2}\cdot}\big)(\mathrm{i}\eta)\big|^2\,\mathrm{d}\eta\Big] \\
&= \frac{1}{2\pi}\mathbb{E}\Big[\int \big|\mathcal{L}(\varphi - \mathbb{E}[\varphi])(\tfrac{\alpha}{2} + \mathrm{i}\eta) + \mathcal{L}\big(((\xi-\mu) * h^\varphi)(\cdot)\big)(\tfrac{\alpha}{2} + \mathrm{i}\eta)\big|^2\,\mathrm{d}\eta\Big] \\
&= \frac{1}{2\pi}\int_{\mathrm{Re}(z)=\frac{\alpha}{2}} \mathrm{Var}[\mathcal{L}\varphi(z) + \mathcal{L}\xi(z)\mathcal{L}h^\varphi(z)]\,|\mathrm{d}z|,
\end{aligned}$$

where the variance of a complex random variable is defined in terms of absolute squares. An analogous formula holds in the lattice case, see [30]. We refrain from giving further details.

(iii) Suppose now that φ vanishes on $(-\infty, 0)$. Then so do m_t^φ and the remainder function r from the expansion (2.20). Additionally, assume that $r(t) = O(e^{(\frac{\alpha}{2}-\varepsilon)t})$ for some $\varepsilon > 0$, which holds in typical cases (see Section 7). In the non-lattice case of Theorem 2.15, if all the roots in Λ are simple and there are no roots on the critical line $\{\mathrm{Re}(z) = \frac{\alpha}{2}\}$, the bilateral Laplace transform $\mathcal{L}h^\varphi$ of h^φ coincides on a neighborhood of $\{\mathrm{Re}(z) = \frac{\alpha}{2}\}$ with the function

$$z \mapsto \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1 - \mathcal{L}\mu(z)}.$$

To see this, notice that $r(t) = m_t^\varphi - \sum_{\lambda\in\Lambda} a_{\lambda,0}e^{\lambda t}\mathbb{1}_{[0,\infty)}(t)$ and let $h_r(t) := h^\varphi(t) - r(t)$. The bilateral Laplace transform $\mathcal{L}r$ is well-defined on $\{\mathrm{Re}(z) > \frac{\alpha}{2} - \varepsilon\}$. Moreover, for $\mathrm{Re}(z) >$

α,

$$\mathcal{L}r(z) = \mathcal{L}m^{\varphi}(z) - \sum_{\lambda\in\Lambda}\frac{a_{\lambda,0}}{z-\lambda} = \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1-\mathcal{L}\mu(z)} - \sum_{\lambda\in\Lambda}\frac{a_{\lambda,0}}{z-\lambda}.$$

The right-hand side, being holomorphic on $\{\mathrm{Re}(z) > \frac{\alpha}{2}-\varepsilon\}$ (all the singularities are removable), coincides with $\mathcal{L}r$ on that domain. On the other hand, decreasing ε if needed we can and do assume that $\Lambda \subseteq \{\mathrm{Re}(z) > \frac{\alpha}{2}+\varepsilon\}$. In particular, $\mathcal{L}h_r(z)$ is well-defined on $\{\mathrm{Re}(z) < \frac{\alpha}{2}+\varepsilon\}$ and equal to $\sum_{\lambda\in\Lambda}\frac{a_{\lambda,0}}{z-\lambda}$. As a result, on the set $\{\frac{\alpha}{2}-\varepsilon < \mathrm{Re}(z) < \frac{\alpha}{2}+\varepsilon\}$ we have $\mathcal{L}h^{\varphi}(z) = \mathcal{L}r(z) + \mathcal{L}h_r(z) = \frac{\mathcal{L}(\mathbb{E}[\varphi])(z)}{1-\mathcal{L}\mu(z)}$.

REMARK 2.19. Suppose that ξ satisfies (A1) through (A3) and that the real-valued characteristics $\varphi_1,\ldots,\varphi_d$ satisfy (A4) through (A6). Further, assume that each $m_t^{\varphi_j}$ satisfies (2.20) (with coefficients $a^j_{\lambda,l}$ and remainder r_j depending on j). Then Theorem 2.15 gives joint convergence in distribution of the vector $(\mathcal{Z}_t^{\varphi_1},\ldots,\mathcal{Z}_t^{\varphi_d})$. Indeed, by the Cramér-Wold device, convergence in distribution of the vector is equivalent to convergence of all linear combinations of the form

$$\sum_{j=1}^{d} c_j \mathcal{Z}_t^{\varphi_j} = \mathcal{Z}_t^{\sum_{j=1}^d c_j\varphi_j}.$$

A routine verification shows that the characteristic $\sum_{j=1}^d c_j\varphi_j$ satisfies the assumptions of Theorem 2.15.

As a particular case of Remark 2.19 with $\varphi_j(\cdot) = \varphi(\cdot - s_j)$ for $-\infty < s_1 < \ldots < s_d < \infty$ and a given random characteristic φ, we get the following result for the finite-dimensional distributions:

COROLLARY 2.20. *In the situation of Theorem 2.15 suppose that $\sigma^2 \neq 0$ or $\rho_n \neq 0$. Then for $d := (2n+1)\vee 0$,*

$$t^{-\frac{d}{2}}e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}^{\varphi}_{t-s} - H(t-s)\big)_{s\in\mathbb{R}} \stackrel{\text{f.d.}}{\to} \sqrt{\tfrac{W}{\beta}}(G_s)_{s\in\mathbb{R}}$$

where $(G_s)_{s\in\mathbb{R}}$ is a centered Gaussian process with the covariance function

$$\mathbb{E}[G_sG_u] = \int \mathrm{Cov}\left[\varphi(x-s) - h^{\varphi}*\xi(x-s), \varphi(x-u) - h^{\varphi}*\xi(x-u)\right]e^{-\alpha x}\,\ell(\mathrm{d}x)$$

for any $s,u\in\mathbb{R}$ if $d=0$, i.e., $n=-1$ and

$$\mathbb{E}[G_sG_u] = \frac{1}{d}\sum_{\substack{\lambda\in\partial\Lambda:\\ k(\lambda)\geq n+1}} \mathrm{Cov}\Bigg[\sum_{j=n}^{k(\lambda)-1} a_{\lambda,j}\binom{j}{n}\sum_{k=1}^{N}(-X_k-s)^{j-n}e^{-\lambda(X_k+s)},$$
$$\sum_{j=n}^{k(\lambda)-1} a_{\lambda,j}\binom{j}{n}\sum_{k=1}^{N}(-X_k-u)^{j-n}e^{-\lambda(X_k+u)}\Bigg]$$

if $d = 2n+1$ with $n\geq 0$.

We close this section with a figure displaying the way to the proofs of our main results.

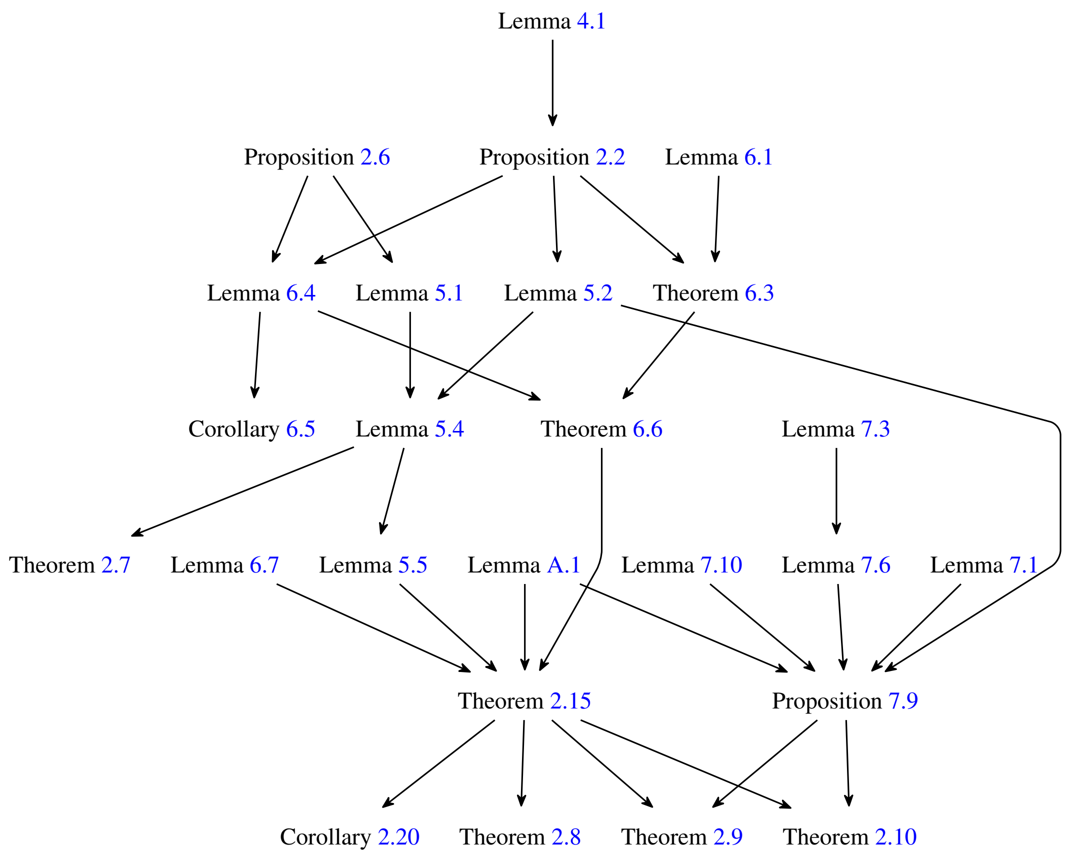

Fig 2: Diagram representing the dependence of the results.

3. Applications.

3.1. *The Galton-Watson process.* Consider a supercritical Galton-Watson branching process, i.e., $\xi = \sum_{k=1}^{N} \delta_1 = N\delta_1$ where N is a random variable taking values in $\mathbb{N}_0$ with $m := \mathbb{E}[N] \in (1, \infty)$ and $\mathbb{E}[N^2] < \infty$. Then ξ is lattice with span 1. Further,

$$\mathcal{L}\mu(\lambda) = \mathbb{E}\left[\sum_{k=1}^{N} e^{-\lambda}\right] = me^{-\lambda}, \quad \lambda \in \mathbb{C}.$$

The equation $\mathcal{L}\mu(z) = 1$ is equivalent to $e^z = m$ and has only one solution in the strip $\mathrm{Im}(z) \in (-\pi, \pi]$. In particular, (A1) holds, i.e., there is a Malthusian parameter $\alpha > 0$, namely, $\alpha = \log m$, and $\Lambda_{\geq} = \{\alpha\}$. In this case the parameter β defined by (2.7) is equal to 1. Then

$$\mathbb{E}\left[\left(\sum_{k=1}^{N} e^{-\theta}\right)^2\right] = e^{-2\theta}\mathbb{E}[N^2] < \infty \quad \text{for all } \theta \in \mathbb{R}.$$

By Remark 2.1, this implies that (A2) and (A3) hold.

Consider the characteristic $\phi(t) := \mathbb{1}_{[0,1)}(t)$. Then for any $n \in \mathbb{N}_0$, $\mathcal{Z}_n^\phi$ is the number of individuals in the n^{th} generation and the corresponding Nerman's martingale (2.10) is the size of n^{th} generation normalized by its expectation m^n, i.e., $W_n = e^{-\alpha n}\mathcal{Z}_n^\phi$. Clearly, ϕ

satisfies (A4), (A5) and (A6). Therefore, we may apply the lattice version of Theorem 2.15 with $\rho_{-1} = 0$, which yields

$$m^{-\frac{n}{2}}\big(m^n W_n - a_\alpha m^n W\big) = e^{-\frac{\alpha}{2}n}\big(\mathcal{Z}_n^\phi - a_\alpha e^{\alpha n} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty$$

where $\mathcal{N}$ is standard normal and independent of W. To calculate $a_\alpha := a_{\alpha,0}$, we use (2.20): $m^n = \mathbb{E}[\mathcal{Z}_n^\phi] = a_\alpha m^n$, i.e., $a_\alpha = 1$. Further, $\sigma > 0$ is given by (2.25), i.e.,

$$\sigma^2 = \sum_{n\in\mathbb{Z}} \mathrm{Var}\,\big[\phi(n) + h * \xi(n)\big] e^{-\alpha n} = \sum_{n\in\mathbb{Z}} \mathrm{Var}\,\big[h * \xi(n)\big] m^{-n},$$

where

$$h(n) = m_n^\phi - a_\alpha e^{\alpha n} = m^n \mathbb{1}_{\{n\geq 0\}} - m^n = -m^n \mathbb{1}_{\{n<0\}}.$$

Now, since $h * \xi(n) = N h(n-1)$ we infer

$$\sigma^2 = \sum_{n<1} \mathrm{Var}[N] m^{2n-2} m^{-n} = \frac{\mathrm{Var}[N]}{m^2 - m}.$$

Consequently,

$$m^{\frac{n}{2}}\big(W_n - W\big) \stackrel{\mathrm{d}}{\to} \Big(\frac{\mathrm{Var}[N] W}{m^2 - m}\Big)^{\frac{1}{2}} \mathcal{N} \quad \text{as } n \to \infty.$$

We have thus just rediscovered Heyde's classical central limit theorem for the martingale in the Galton-Watson process [20].

We can also deal with the total number of individuals in the generations $0, \ldots, n$. Indeed, this number is $\mathcal{Z}_n^f$ for $f(t) := \mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbb{R}$, which satisfies (A4), (A5) and (A6). Invoking once again Theorem 2.15 with $\rho_{-1} = 0$ we obtain

$$e^{-\frac{\alpha}{2}n}\big(\mathcal{Z}_n^f - a_\alpha e^{\alpha n} W\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty.$$

This time a_α can be computed with the help of (2.20) as follows. We have the asymptotic expansion

$$m_n^f = \mathbb{E}[\mathcal{Z}_n^f] = \sum_{k=0}^n m^k = \frac{m^{n+1} - 1}{m - 1} = \frac{m}{m-1} e^{\alpha n} - \frac{1}{m-1},$$

for $n \geq 0$ and 0 otherwise. Consequently, $a_\alpha = \frac{m}{m-1}$ and thereupon

$$m^{-\frac{n}{2}}\Big(\mathcal{Z}_n^f - \frac{m^{n+1}}{m-1} W\Big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{W}\mathcal{N} \quad \text{as } n \to \infty.$$

This time $\sigma > 0$ is given by $\sigma^2 = \sum_{n\in\mathbb{Z}} \mathrm{Var}[N] |h(n-1)|^2 m^{-n}$ with

$$h(n) = m_n^f - \frac{m}{m-1} e^{\alpha n}.$$

Therefore,

$$\begin{aligned}
\sigma^2 &= \mathrm{Var}[N]\Big(\sum_{n<1}\Big(\frac{m}{m-1} m^{n-1}\Big)^2 m^{-n} + \sum_{n\geq 1}\Big(\frac{1}{m-1}\Big)^2 m^{-n}\Big) \\
&= \frac{1}{(m-1)^2}\,\mathrm{Var}[N]\Big(\sum_{n\leq 0} m^n + \sum_{n>0} m^{-n}\Big) = \frac{m+1}{(m-1)^3}\,\mathrm{Var}[N].
\end{aligned}$$

3.2. *Nerman's martingales.* Suppose that ξ is non-lattice and satisfies (A1) through (A3), and let $\lambda = \theta + \mathrm{i}\eta$ be a root to $\mathcal{L}\mu(z) = 1$ with $0 \le \mathrm{Re}(\lambda) = \theta < \frac{\alpha}{2}$. Further, suppose that

$$\mathbb{E}\Big[\Big(\sum_{k=1}^{N} e^{-\theta X_k}\Big)^2\Big] < \infty. \tag{3.1}$$

For simplicity let $Z_1(\lambda) := \sum_{k=1}^{N} e^{-\lambda X_k}$. We can view the complex variable $Z_1(\lambda)$ as a random variable taking values in $\mathbb{R}^2$. We denote by Σ^λ the corresponding covariance matrix. The aforementioned condition guarantees that Σ^λ is well-defined.

Let $(W_t(\lambda))_{t\ge 0}$ be defined by (2.17) for $j = 0$. But, since $\mathrm{Re}(\lambda) < \frac{\alpha}{2}$, we cannot apply Theorem 2.7, though one can still wonder what the long-term behavior of the process is. To analyze this, we shall apply a special case of our main result, Theorem 2.15. Let

$$\phi(t) := e^{\lambda t} \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t) e^{-\lambda X_j}, \quad t \in \mathbb{R}.$$

Then

$$|\phi(t)| \le e^{\theta t} \mathbb{1}_{[0,\infty)}(t) \Big| \sum_{j=1}^{N} e^{-\theta X_j} \Big|,$$

and, by (3.1), we conclude that the functions

$$t \mapsto e^{-(\alpha - p\theta)t} \mathbb{1}_{[0,\infty)}(t) \mathbb{E}\Big[\Big| \sum_{j=1}^{N} e^{-\theta X_j}\Big|^p\Big] \quad \text{with } p = 1, 2$$

are directly Riemann integrable. By [43, Remark 3.10.5], characteristics $\phi, \mathrm{Re}(\phi)$ and $\mathrm{Im}(\phi)$ fulfill (A4), (A5) and (A6). Remark 5.3 below (applied to $\phi = \phi_{\lambda,1}$) gives that

$$\mathcal{Z}_t^{\phi} = e^{\lambda t} W_t(\lambda)$$

and $\mathbb{E}[\mathcal{Z}_t^{\phi}] = e^{\lambda t} \mathbb{1}_{[0,\infty)}(t)$.

By Theorem 6.6, we deduce

$$e^{-\frac{\alpha}{2}t} (\mathcal{Z}_t^{\mathrm{Re}(\phi)}, \mathcal{Z}_t^{\mathrm{Im}(\phi)}) \xrightarrow{\mathrm{d}} \sqrt{\tfrac{W}{\beta}} \mathcal{N}$$

or equivalently

$$(\mathrm{Re}(e^{(\lambda - \frac{\alpha}{2})t} W_t(\lambda)), \mathrm{Im}(e^{(\lambda - \frac{\alpha}{2})t} W_t(\lambda))) \xrightarrow{\mathrm{d}} \sqrt{\tfrac{W}{\beta}} \mathcal{N}$$

where $\mathcal{N}$ is a 2-dimensional centered Gaussian vector with covariance matrix Σ, which can be explicitly computed. Indeed, we have

$$\begin{aligned}
\Sigma_{11} &:= \int \mathrm{Var}\Big[\mathrm{Re}(\phi_\lambda(x)) + \sum_{j=1}^{N} \mathrm{Re}(e^{\lambda(x - X_j)}) \mathbb{1}_{[0,\infty)}(x - X_j)\Big] e^{-\alpha x}\, \ell(\mathrm{d}x) \\
&= \int_0^\infty \mathrm{Var}\big[\mathrm{Re}(e^{\lambda x} Z_1(\lambda))\big] e^{-\alpha x}\, \mathrm{d}x \\
&= \int_0^\infty e^{2\theta x} \Big(\cos^2(\eta x)\Sigma_{11}^\lambda - 2\sin(\eta x)\cos(\eta x)\Sigma_{12}^\lambda + \sin^2(\eta x)\Sigma_{22}^\lambda\Big) e^{-\alpha x}\, \mathrm{d}x \\
&= \frac{\Sigma_{11}^\lambda}{\alpha - 2\theta} \frac{2\eta^2 + (\alpha - 2\theta)^2}{4\eta^2 + (\alpha - 2\theta)^2} - \Sigma_{12}^\lambda \frac{2\eta}{4\eta^2 + (\alpha - 2\theta)^2} + \frac{\Sigma_{22}^\lambda}{\alpha - 2\theta} \frac{2\eta^2}{4\eta^2 + (\alpha - 2\theta)^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\Sigma_{22} &= \int_0^\infty \mathrm{Var}\left[\mathrm{Im}(e^{\lambda x} Z_1(\lambda))\right] e^{-\alpha x}\,\mathrm{d}x \\ &= \frac{\Sigma_{22}^\lambda}{\alpha - 2\theta}\frac{2\eta^2 + (\alpha-2\theta)^2}{4\eta^2+(\alpha-2\theta)^2} + \Sigma_{12}^\lambda \frac{2\eta}{4\eta^2+(\alpha-2\theta)^2} + \frac{\Sigma_{11}^\lambda}{\alpha-2\theta}\frac{2\eta^2}{4\eta^2+(\alpha-2\theta)^2},\end{aligned}$$

and

$$\begin{aligned}\Sigma_{12} &= \int_0^\infty \mathrm{Cov}\left[\mathrm{Re}(e^{\lambda x} Z_1(\lambda)), \mathrm{Im}(e^{\lambda x} Z_1(\lambda))\right] e^{-\alpha x}\,\mathrm{d}x \\ &= \frac{(\Sigma_{11}^\lambda - \Sigma_{22}^\lambda)\eta}{4\eta^2+(\alpha-2\theta)^2} + \Sigma_{12}^\lambda \frac{\alpha-2\theta}{4\eta^2+(\alpha-2\theta)^2}.\end{aligned}$$

3.3. *Epidemic models.* In this section, we consider the epidemic model discussed in [12]. In this model, the role of the ancestor is that of the first person in a community infected by an infectious disease. Birth events become infection events etc.

Suppose that ξ is a Poisson point process on $[0,\infty)$ with intensity measure $R_0 g(x)\,\mathrm{d}x$ where

$$g(x) = \mathbb{1}_{(0,\infty)}(x)\frac{b^a x^{a-1}}{\Gamma(a)} e^{-bx}, \quad x \in \mathbb{R}$$

is the density of the Gamma distribution with parameters $a, b > 0$ and $R_0 > 1$ is the basic reproduction mean. (No additional difficulties would occur if R_0 was replaced by a positive random variable N with mean R_0 and finite variance.) The function g is the infection rate scaled to become a probability density. It models the time delay between the infection of a person and a random person infected by that person. Characteristics of interest are $I(t) = R_0 g(t)$ and $f(t) = \mathbb{1}_{[0,\infty)}(t)$ with $\mathcal{Z}_t^I$ being the incidence at time t and $\mathcal{Z}_t^f$ the number of infections up to time t. In the given situation, the Laplace transform $\mathcal{L}\mu$ can be calculated explicitly in terms of a, b and R_0, namely,

$$\mathcal{L}\mu(\lambda) = \int_0^\infty e^{-\lambda x}\,\mu(\mathrm{d}x) = R_0\Big(\frac{b}{b+\lambda}\Big)^a, \quad \mathrm{Re}(\lambda) > -b.$$

Hence the equation $\mathcal{L}\mu(\lambda) = 1$ takes the form

$$R_0\Big(\frac{b}{b+\lambda}\Big)^a = 1.$$

Write $\frac{b}{b+\lambda} = re^{\mathrm{i}\varphi}$ with $r > 0$ and $|\varphi| < \pi/2$. Then $R_0(re^{\mathrm{i}\varphi})^a = 1$ is equivalent to

$$e^{-\mathrm{i}a\varphi} = r^a R_0.$$

This implies $r = R_0^{-1/a}$ and $\varphi \in (2\pi/a)\mathbb{Z} \cap (-\pi/2, \pi/2)$. Solving for λ yields

$$\lambda = b(R_0^{1/a} e^{-\mathrm{i}\varphi} - 1) \tag{3.2}$$

with $\varphi \in (2\pi/a)\mathbb{Z} \cap (-\pi/2, \pi/2)$, cf. Figure 3. The Malthusian parameter is obtained by setting $\varphi = 0$, i.e.,

$$\alpha = b(R_0^{1/a} - 1).$$

The real part of a root λ as in (3.2) is given by

$$\mathrm{Re}(\lambda) = b(R_0^{1/a}\cos\varphi - 1). \tag{3.3}$$

A second root exists only if $a > 4$ (otherwise $(2\pi/a)\mathbb{Z} \cap (-\pi/2, \pi/2) = \{0\}$), in which case the root $\lambda \neq \alpha$ with largest real part is $\lambda = b(R_0^{1/a} e^{\mathrm{i}2\pi/a} - 1)$ with

$$\mathrm{Re}(\lambda) = b(R_0^{1/a}\cos(\tfrac{2\pi}{a}) - 1).$$

Further

$$\mathrm{Re}(\lambda) = b(R_0^{1/a}\cos(\tfrac{2\pi}{a}) - 1) \geq \frac{\alpha}{2} = \frac{b}{2}(R_0^{1/a} - 1)$$

if and only if $a > 6$ and $R_0 \geq R_0(a) := (2\cos(\frac{2\pi}{a}) - 1)^{-a}$. Notice that $R_0(a) \to \infty$ for $a \downarrow 6$ and $R_0(a) \to 1$ for $a \to \infty$. If $R_0 < R_0(a)$, Theorem 2.8 applies and yields Gaussian fluctuations of $\mathcal{Z}_t^I$ and $\mathcal{Z}_t^f$. That is, for $\mathcal{Z}_t^f$ we have

$$e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^f - a_\alpha e^{\alpha t} W\big) \xrightarrow{\mathrm{d}} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N} \quad \text{as } t \to \infty,$$

with $\beta := R_0 a b^a (b+\alpha)^{-a-1}$ and $a_\alpha := (\alpha\beta)^{-1}$. Left with calculating σ we obtain with the help of Remark 2.18

$$\begin{aligned}
\sigma^2 &= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z)=\frac{\alpha}{2}} \mathrm{Var}\left[\mathcal{L}f(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}f(z)}{1-\mathcal{L}\mu(z)}\right] |\mathrm{d}z| \\
&= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z)=\frac{\alpha}{2}} \left|\frac{1}{z(1-\mathcal{L}\mu(z))}\right|^2 \mathrm{Var}[\mathcal{L}\xi(z)]\, |\mathrm{d}z| \\
&= \frac{1}{2\pi} \int\limits_{\mathrm{Re}(z)=\frac{\alpha}{2}} \left|\frac{1}{z(1-\mathcal{L}\mu(z))}\right|^2 \mathcal{L}\mu(\alpha)\, |\mathrm{d}z| \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{(\alpha^2+4t^2)g(t)}\, \mathrm{d}t,
\end{aligned}$$

with $g(t) := 1 + R_0^2\Big(\frac{4b^2}{(2b+\alpha)^2+4t^2}\Big)^a - 2R_0\Big(\frac{4b^2}{(2b+\alpha)^2+4t^2}\Big)^{\frac{a}{2}} \cos\Big(a\arctan\Big(\frac{2t}{2b+\alpha}\Big)\Big)$.

If $R_0 \geq R_0(a)$, the more general Theorem 2.15 applies and gives additional periodic fluctuations of greater magnitude than the Gaussian fluctuations. We refrain from providing further details.

3.4. *Supercritical binary homogeneous Crump-Mode-Jagers processes.* In this section, we assume that

$$\xi := \sum_{j\geq 1} \mathbb{1}_{[0,\zeta]}(P_j)\delta_{P_j}$$

where $(P_j)_{j\geq 1}$ are the arrival times of a homogeneous Poisson process with intensity $b > 0$, independent of the $[0,\infty]$-valued random variable ζ. We are interested in $\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}}$ the number of individuals alive at time t, see (2.3). Thus, the corresponding characteristic ϕ is given by $\phi(t) := \mathbb{1}_{[0,\zeta)}(t)$ for $t \geq 0$.

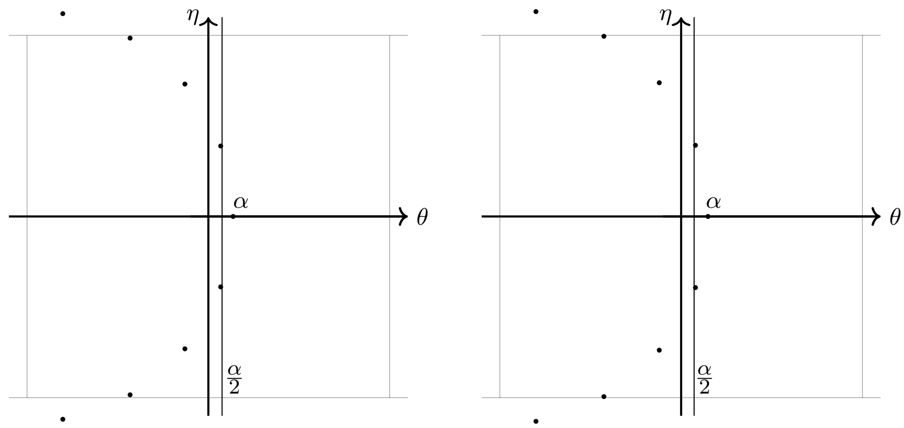

Fig 3: Solutions to $\mathcal{L}\mu(\lambda)=1$ in the cases $a=18$, $b=1$, $R_0=10$ (left figure) and $a=18$, $b=1$, $R_0=12$. In the left figure, the root $\lambda\neq\alpha$ with largest real part has $\mathrm{Re}(\lambda)<\frac{\alpha}{2}$, in the right figure $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$.

We put $\mathcal{L}_\zeta(z):=\mathbb{E}[e^{-z\zeta}]$ for $\mathrm{Re}(z)\geq 0$ and start by noting that

$$\mathcal{L}\mu(z)=\mathbb{E}\Big[\sum_{j\geq 1}\mathbb{1}_{[0,\zeta]}(P_j)e^{-zP_j}\Big]=\mathbb{E}\Big[\int\mathbb{1}_{[0,\zeta]}(x)e^{-zx}b\mathrm{d}x\Big]=\int_0^\infty e^{-zx}b\mathbb{P}(\zeta\geq x)\,\mathrm{d}x$$

$$=\frac{b(1-\mathcal{L}_\zeta(z))}{z},\quad \mathrm{Re}(z)>0.$$

The Malthusian parameter α is the unique real number that satisfies

$$1-\mathcal{L}_\zeta(\alpha)=\frac{\alpha}{b},$$

and the parameter β is given by

$$\beta=\frac{1}{\alpha}(1+b\mathcal{L}_\zeta'(\alpha))=\frac{1}{\alpha}(1-b\mathbb{E}[\zeta e^{-\alpha\zeta}]).$$

Now we shall show that α is the only solution to $\mathcal{L}\mu(z)=1$ with $\mathrm{Re}(z)>0$. Indeed, for positive θ and η

$$-\mathrm{Im}\big(\mathcal{L}\mu(\theta+\mathrm{i}\eta)\big)=\int_0^\infty\sin(\eta x)e^{-\theta x}b\mathbb{P}(\zeta\geq x)\mathrm{d}x$$

$$=b\sum_{l\geq 0}\int_{2\pi l/\eta}^{\pi(2l+1)/\eta}\sin(\eta x)\Big(e^{-\theta x}\mathbb{P}(\zeta\geq x)-e^{-\theta(x+\pi/\eta)}\mathbb{P}(\zeta\geq x+\pi/\eta)\Big)\mathrm{d}x>0$$

which, together with $\mathcal{L}\mu(\overline{\lambda})=\overline{\mathcal{L}\mu(\lambda)}$, shows that $\Lambda_{\geq}\cap\{z:\mathrm{Re}(z)>0\}=\{\alpha\}$. Since $\mathcal{L}(\mathbb{E}[\phi])(\alpha)=b^{-1}\mathcal{L}\mu(\alpha)=b^{-1}$, an application of Theorem 2.8 yields

$$e^{-\frac{\alpha}{2}t}\Big(\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}}-e^{\alpha t}\frac{W}{b\beta}\Big)\xrightarrow{\mathrm{d}}\sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N}.$$

Next, we express the variance σ^2 in terms of the parameters b, α and β. By Remark 2.18

$$\sigma^2=\frac{1}{2\pi}\int_{\mathrm{Re}(z)=\frac{\alpha}{2}}\mathrm{Var}\Big[\mathcal{L}\phi(z)+\mathcal{L}\xi(z)\frac{\mathcal{L}(\mathbb{E}[\phi])(z)}{1-\mathcal{L}\mu(z)}\Big]\,|\mathrm{d}z|.$$

For $\mathrm{Re}(z) > 0$

$$\mathrm{Var}\Big[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}(\mathbb{E}[\phi])(z)}{1-\mathcal{L}\mu(z)}\Big] = \mathbb{E}\Big[\mathrm{Var}\Big[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\Big]\Big] + \mathrm{Var}\Big[\mathbb{E}\Big[\mathcal{L}\phi(z) + \mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\Big]\Big] =: I + II$$

with

$$I = \mathbb{E}\Big[\mathrm{Var}\Big[\mathcal{L}\xi(z)\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|\zeta\Big]\Big] = \mathbb{E}\big[\mathrm{Var}\big[\mathcal{L}\xi(z)\big|\zeta\big]\big]\Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2 = \mathbb{E}\Big[\int_0^\zeta e^{-2\mathrm{Re}(z)x} b\,\mathrm{d}x\Big]\Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2,$$

where we have used properties of the Poisson process, and

$$II = \mathrm{Var}\Big[\frac{1-e^{-z\zeta}}{z} + \frac{b(1-e^{-z\zeta})}{z}\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big] = \Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2 \mathrm{Var}[e^{-z\zeta}].$$

Assuming now that $\mathrm{Re}(z) = \frac{\alpha}{2}$ we arrive at

$$I = \Big|\frac{\mathcal{L}\mu(z)}{b(1-\mathcal{L}\mu(z))}\Big|^2$$

because

$$\mathbb{E}\Big[\int_0^\zeta e^{-2\mathrm{Re}(z)x} b\,\mathrm{d}x\Big] = \mathbb{E}\Big[\int_0^\zeta e^{-\alpha x} b\,\mathrm{d}x\Big] = 1,$$

and

$$II = \Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2\big(\mathcal{L}_\zeta(\alpha) - |\mathcal{L}_\zeta(z)|^2\big).$$

Observing that $\overline{z} = \alpha - z$ and $\overline{\mathcal{L}\mu(z)} = \mathcal{L}\mu(\alpha - z)$ whenever $\mathrm{Re}(z) = \frac{\alpha}{2}$, we further infer

$$\begin{aligned}\Big|\frac{z\mathcal{L}\mu(z)}{b}\Big|^2 + \mathcal{L}_\zeta(\alpha) - |\mathcal{L}_\zeta(z)|^2 &= |1-\mathcal{L}_\zeta(z)|^2 + 1 - \frac{\alpha}{b} - |\mathcal{L}_\zeta(z)|^2 \\ &= 1 - \mathcal{L}_\zeta(z) + \overline{1-\mathcal{L}_\zeta(z)} - \tfrac{\alpha}{b} \\ &= \tfrac{1}{b}\big(z(\mathcal{L}\mu(z)-1) + (\alpha - z)(\mathcal{L}\mu(\alpha - z) - 1)\big),\end{aligned}$$

which in turn gives

$$\begin{aligned} I + II &= \Big|\frac{1}{z(1-\mathcal{L}\mu(z))}\Big|^2\Big(\Big|\frac{z\mathcal{L}\mu(z)}{b}\Big|^2 + \mathcal{L}_\zeta(\alpha) - |\mathcal{L}_\zeta(z)|^2\Big) \\ &= \frac{z(\mathcal{L}\mu(z)-1) + (\alpha - z)(\mathcal{L}\mu(\alpha - z)-1)}{bz(\mathcal{L}\mu(z)-1)(\alpha - z)(\mathcal{L}\mu(\alpha - z)-1)} \\ &= \frac{1}{b(\alpha - z)(\mathcal{L}\mu(\alpha - z)-1)} + \frac{1}{bz(\mathcal{L}\mu(z)-1)} \\ &= \frac{2}{b}\mathrm{Re}\Big(\frac{1}{z(\mathcal{L}\mu(z)-1)}\Big).\end{aligned}$$

We can now compute the variance as follows

$$\begin{aligned}\sigma^2 &= \lim_{R\to\infty} \frac{1}{b\pi}\mathrm{Re}\Bigg(\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R} \frac{1}{z(\mathcal{L}\mu(z)-1)}\,|\mathrm{d}z|\Bigg) = \frac{1}{b\pi}\lim_{R\to\infty}\mathrm{Im}\Bigg(\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R} \frac{1}{z(\mathcal{L}\mu(z)-1)}\,\mathrm{d}z\Bigg)\\
&= \frac{1}{b\pi}\lim_{R\to\infty}\mathrm{Im}\Bigg(\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R} \frac{1}{z(\mathcal{L}\mu(z)-1)}+\frac{1}{z}\,\mathrm{d}z\Bigg) - \lim_{R\to\infty}\frac{1}{b\pi}\int_{-R}^{R}\frac{\frac{\alpha}{2}}{(\frac{\alpha}{2})^2+t^2}\,\mathrm{d}t\\
&= \frac{1}{b\pi}\mathrm{Im}\Bigg(\lim_{R\to\infty}\int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R} \frac{\mathcal{L}\mu(z)}{z(\mathcal{L}\mu(z)-1)}\,\mathrm{d}z\Bigg) - \frac{1}{b}.\end{aligned}$$

To calculate the limit, we use the residue theorem. For $R>\alpha$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{i}Re^{\mathrm{i}\theta}\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})\mathrm{d}\theta}{(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)} - \int_{\frac{\alpha}{2}-\mathrm{i}R}^{\frac{\alpha}{2}+\mathrm{i}R}\frac{\mathcal{L}\mu(z)\mathrm{d}z}{z(\mathcal{L}\mu(z)-1)} = 2\pi\mathrm{i}\operatorname*{Res}_{z=\alpha}\frac{\mathcal{L}\mu(z)}{z(\mathcal{L}\mu(z)-1)}.$$

It suffices to show that the integrand of the first integral decays to zero uniformly in θ as R goes to infinity. In view of the inequality $|\mathcal{L}\mu(z)|\le 2b|z|^{-1}$ and its consequence $|\mathcal{L}\mu(z)-1|\ge 1-2b|z|^{-1}$ (both hold true for $\mathrm{Re}(z)\ge 0$) we conclude that

$$\Bigg|\frac{R\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})}{(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})(\mathcal{L}\mu(\frac{\alpha}{2}+Re^{\mathrm{i}\theta})-1)}\Bigg| \le \frac{R}{|\frac{\alpha}{2}+Re^{\mathrm{i}\theta}|}\cdot\frac{2b}{|\frac{\alpha}{2}+Re^{\mathrm{i}\theta}|-2b} = O(R^{-1})$$

as $R\to\infty$ uniformly in $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Finally,

$$\sigma^2 = \frac{2-\alpha\beta}{\alpha b\beta}$$

and thereupon

$$e^{-\frac{\alpha}{2}t}\Big(\mathcal{Z}_t^{\mathbb{1}_{[0,\zeta)}} - e^{\alpha t}\frac{W}{b\beta}\Big) \xrightarrow{\mathrm{st}} \sqrt{\frac{(2-\alpha\beta)W}{\alpha b\beta^2}}\mathcal{N}.$$

An application of a similar argument as in Remark 2.17 enables us to conclude that the convergence mentioned above holds true conditionally given $\mathcal{S}$. The distribution of W conditionally given $\mathcal{S}$ is exponential with parameter α/b. Therefore, we have just reproved Henry's central limit theorem [19].

3.5. *The conservative fragmentation model.* In this section we consider the conservative fragmentation model as discussed in [31]. Let $b\ge 2$ be integer and $(V_1,V_2,\dots,V_b)$ a vector of nonnegative random variables such that $\sum_{j=1}^b V_j = 1$ a. s. For the sake of simplicity, we assume that $V_1,\dots,V_b$ have Lebesgue densities except possible atoms at 0. (Our theory would allow to cover more general cases, too.) Starting with an object of mass $x\ge 1$, we break it into pieces with masses $(V_1x,V_2x,\dots,V_bx)$. Continue recursively with each piece of mass ≥ 1, using new and independent copies of the random vector $(V_1,V_2,\dots,V_b)$ each time. Once a fragment has mass <1, it is not further crumbled. The process terminates a. s. after a finite number of steps, leaving a finite set of fragments of masses <1.

Denote by $n(x)$ the random number of fragmentation events, i.e., the number of pieces of mass ≥ 1 that appear during the process. Further, let $n_e(x)$ be the final number of fragments, i.e., the number of pieces of mass < 1 that appear. A limit theorem for $n(x)$ has been proved in [31], where it was shown that the asymptotic behavior of $n(x)$ as x goes to infinity depends on the position of the roots of the function $z \mapsto \sum_{j\geq 1} \mathbb{E}[V_j^z]$.

Letting $\xi := \sum_{j=1}^{b} \mathbb{1}_{\{V_j>0\}} \delta_{-\log V_j}$, we conclude that the corresponding Malthusian parameter is 1, i.e., $\alpha = 1$ and the limit of Nerman's martingale satisfies $W = 1$ a. s. Further $\beta = \sum_{j=1}^{b} \mathbb{E}[V_j|\log V_j|] \in (0,\infty)$. Note also that $n(x) = N(\log x)$ corresponds to the number of individuals born up to and including time $\log x$ and similarly, we can represent $n_e(x)$ as a general branching process, namely, $n_e(x) = \mathcal{Z}^{\varphi}_{\log x}$, with

$$\varphi(t) := \sum_{j=1}^{b} \mathbb{1}_{\{V_j>0\}} \mathbb{1}_{[0,-\log V_j)}(t) \quad \text{for } t \in \mathbb{R}.$$

Hence, our main result provides (precise) limit theorems for both n and n_e. For instance, in the case when all root from Λ are simple, we infer from Theorem 2.9 (the constants $b_{\lambda,0}$ in the theorem can easily be seen to equal $b_{\lambda,0} = -1/(\mathcal{L}\mu)'(\lambda)$, $\lambda \in \Lambda$ by Proposition 7.9)

$$x^{-1/2}\Big(n(x) + \sum_{\lambda\in\Lambda} \frac{W(\lambda)}{\lambda(\mathcal{L}\mu)'(\lambda)} x^{\lambda}\Big) \stackrel{\mathrm{d}}{\to} \frac{\sigma}{\sqrt{\beta}}\mathcal{N} \quad \text{if } \partial\Lambda \text{ is empty and}$$

$$x^{-1/2}\big(\log x\big)^{-k+1/2}\Big(n(x) + \sum_{\lambda\in\Lambda} \frac{W(\lambda)}{\lambda(\mathcal{L}\mu)'(\lambda)} x^{\lambda}\Big) \stackrel{\mathrm{d}}{\to} \frac{\rho_{k-1}}{\sqrt{(2k-1)\beta}}\mathcal{N} \quad \text{if } \partial\Lambda \text{ is non-empty,}$$

where k is the largest multiplicity of a root on the critical line $\mathrm{Re}(z) = \frac{\alpha}{2}$, and ρ_{k-1} is as in Theorem 2.15.

4. Preliminaries for the proofs of the main results. In this section we gather facts from the literature, introduce some notation used throughout the paper and perform some basic calculations.

4.1. *Change of measure and the connection to renewal theory.* The existence of the Malthusian parameter (i.e., (2.6)) enables us to use a change-of-measure argument as follows. We define a random walk $(S_n)_{n\in\mathbb{N}_0}$ with $S_0 = 0$ on some probability space with underlying probability measure P and increment distribution given by

$$\mathsf{P}(S_1 \in B) = \mathbb{E}\Big[\sum_{|u|=1} e^{-\alpha S(u)} \mathbb{1}_B(S(u))\Big] = \int_B e^{-\alpha x}\,\mu(\mathrm{d}x), \qquad B \in \mathcal{B}(\mathbb{R}). \tag{4.1}$$

With this definition, the many-to-one formula (see, e.g., [46, Theorem 1.1]) holds:

$$\mathsf{E}[f(S_1,\ldots,S_n)] = \mathbb{E}\Big[\sum_{|u|=n} e^{-\alpha S(u)} f(S(u|_1),\ldots,S(u))\Big] \tag{4.2}$$

for all Borel measurable $f : \mathbb{R}^n \to \mathbb{R}$ such that the expectation on the left- or right-hand side of (4.2) is well-defined, possibly infinite. In particular, under (2.7)

$$\mathsf{E}[S_1] = \beta \in (0,\infty). \tag{4.3}$$

In other words, the increments of the random walk $(S_n)_{n\in\mathbb{N}_0}$ have positive, finite mean. As a consequence, the associated renewal measure

$$\mathsf{U}(\cdot) = \sum_{n\in\mathbb{N}_0} \mathsf{P}(S_n \in \cdot)$$

is uniformly locally finite in the sense that

$$\mathsf{U}([t,t+h]) \leq \mathsf{U}([0,h]) < \infty \qquad \text{for all } t,h \geq 0. \tag{4.4}$$

Indeed, if $\tau := \inf\{n \in \mathbb{N}_0 : S_n \geq t\}$, then

$$\sum_{n\geq 0} \mathbb{1}_{[t,t+h]}(S_n) = \sum_{n\geq 0} \mathbb{1}_{[t,t+h]}(S_{\tau+n}) \leq \sum_{n\geq 0} \mathbb{1}_{[0,h]}(S_{\tau+n} - S_\tau).$$

Now take expectations and use the strong Markov property at τ to infer (4.4).

By the many-to-one formula, (A2) implies that the increments of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$ have a finite exponential moment of order $\alpha - \vartheta > \alpha/2$ since

$$\mathsf{E}[e^{(\alpha-\vartheta)S_1}] = \mathbb{E}\bigg[\sum_{j=1}^{N} e^{-\vartheta X_j}\bigg] = \mathcal{L}\mu(\vartheta) < \infty. \tag{4.5}$$

4.2. *The expectation of the general branching process.* There is a connection between the renewal measure U and the expectation $m_t^\varphi = \mathbb{E}[\mathcal{Z}_t^\varphi]$ of the general branching process counted with characteristic φ provided that φ satisfies suitable assumptions. For instance, if φ is nonnegative and $t \mapsto \mathbb{E}[\varphi(t)]e^{-\alpha t}$ is a directly Riemann integrable function, then we infer from the many-to-one formula

$$\begin{aligned} m_t^\varphi e^{-\alpha t} := \mathbb{E}[\mathcal{Z}_t^\varphi]e^{-\alpha t} &= \sum_{n=0}^{\infty} \mathbb{E}\bigg[\sum_{|u|=n} e^{-\alpha S(u)} \varphi_u(t-S(u)) e^{-\alpha(t-S(u))}\bigg] \\ &= \sum_{n=0}^{\infty} \mathbb{E}\bigg[\sum_{|u|=n} e^{-\alpha S(u)} \mathbb{E}[\varphi](t-S(u)) e^{-\alpha(t-S(u))}\bigg] \\ &= \sum_{n=0}^{\infty} \mathsf{E}\big[\mathbb{E}[\varphi](t-S_n) e^{-\alpha(t-S_n)}\big] \\ &= \int \mathbb{E}[\varphi](t-x) e^{-\alpha(t-x)}\,\mathsf{U}(\mathrm{d}x). \end{aligned} \tag{4.6}$$

By the direct Riemann integrability of $t \mapsto \mathbb{E}[\varphi(t)]e^{-\alpha t}$ and (4.4), the function $t \mapsto m_t^\varphi e^{-\alpha t}$ is bounded and, moreover,

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{G}}} e^{-\alpha t} m_t^\varphi = \frac{1}{\beta} \int \mathbb{E}[\varphi](x) e^{-\alpha x}\,\ell(\mathrm{d}x) = \frac{1}{\beta} (\mathcal{L}\mathbb{E}[\varphi])(\alpha) \tag{4.7}$$

by the key renewal theorem, see [7, Theorem 4.2] in the non-lattice case and [2, Theorem 2.5.3] in the lattice (and non-lattice) case. Recall that, in the lattice case, $\mathcal{L}\mathbb{E}[\varphi]$ denotes the 'discrete' bilateral Laplace transform of $\mathbb{E}[\varphi]$, $\mathbb{G} = \mathbb{Z}$ and ℓ is the counting measure on $\mathbb{Z}$, whereas in the non-lattice case, $\mathcal{L}\mathbb{E}[\varphi]$ is the 'continuous' bilateral Laplace transform, $\mathbb{G} = \mathbb{R}$ and ℓ is the Lebesgue measure on $\mathbb{R}$.

We need a lemma in preparation for the proof of Proposition 2.2.

LEMMA 4.1. *Suppose that (A1) holds and that χ is a centered characteristic, i.e., $\mathbb{E}[\chi(t)] = 0$ for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ and suppose that*

$$\mathbb{E}\bigg[\sum_{u\in\mathcal{I}} \mathrm{Var}[\chi](t - S(u))\bigg] < \infty. \tag{4.8}$$

Let $(u_n)_{n\in\mathbb{N}}$ be an admissible ordering of $\mathcal{I}$ (see the paragraph before Proposition 2.2 for the definition). Define

$$M_n(t) := \sum_{j=1}^{n} \chi_{u_j}(t - S(u_j))$$

for $n \in \mathbb{N}_0$. Then $(M_n(t))_{n\in\mathbb{N}_0}$ is a centered martingale and bounded in L^2. In particular,

$$Z_t^\chi := \sum_{u\in\mathcal{I}} \chi_u(t - S(u)) \tag{4.9}$$

converges unconditionally in L^2 and it is also the almost sure limit of $M_n(t)$ as $n \to \infty$. Further, for any (deterministic) sequence $(\mathcal{I}_n)_{n\in\mathbb{N}_0}$ with $\mathcal{I}_n \uparrow \mathcal{I}$,

$$M_{\mathcal{I}_n}(t) = \sum_{u\in\mathcal{I}_n} \chi_u(t - S(u)) \to \mathcal{Z}_t^\chi \quad \text{in } L^2 \text{ as } n \to \infty.$$

Moreover,

$$\mathrm{Var}[\mathcal{Z}_t^\chi] = \mathbb{E}[(\mathcal{Z}_t^\chi)^2] = \mathbb{E}[\mathcal{Z}_t^{\chi^2}] < \infty. \tag{4.10}$$

Finally, (A5) is sufficient for (4.8) to hold for every $t \in \mathbb{R}$.

PROOF. Let $\mathcal{G}_n = \sigma(\pi_{u_j} : j \le n)$, where it should be recalled that π_u is the projection onto the life space of individual u, in particular, (ξ_u, ζ_u, χ_u) is π_u-measurable. Then $(M_n(t))_{n\in\mathbb{N}_0}$ is adapted with respect to $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ as, for any $u \in \mathcal{I}_n$, both $S(u)$ and χ_u are $\mathcal{G}_n$-measurable. Moreover, (4.8) implies that for any $u \in \mathcal{I}$, $\mathbb{E}[|\chi_u(t - S(u))|] < \infty$. Hence $M_n(t)$ is integrable for any $n \in \mathbb{N}_0$. The martingale property then follows since $S(u_{n+1})$ is $\mathcal{G}_n$-measurable whereas $\chi_{u_{n+1}}$ is independent of $\mathcal{G}_n$ and since $\mathbb{E}[\chi](x) = \mathbb{E}[\chi(x)] = 0$ for all $x \in \mathbb{R}$, so

$$\mathbb{E}[\chi_{u_{n+1}}(t - S(u_{n+1}))|\mathcal{G}_n] = \mathbb{E}[\chi](t - S(u_{n+1})) = 0 \quad \text{almost surely.}$$

Next, we observe that, since the increments of L^2-martingales are uncorrelated,

$$\begin{aligned}
\mathbb{E}[M_n(t)^2] &= \mathbb{E}\bigg[\sum_{j=1}^{n} \chi_{u_j}^2(t - S(u_j))\bigg] = \mathbb{E}\bigg[\sum_{j=1}^{n} \mathbb{E}[\chi^2](t - S(u_j))\bigg] \\
&\le \mathbb{E}\bigg[\sum_{u\in\mathcal{I}} \mathbb{E}[\chi^2](t - S(u))\bigg] = \mathbb{E}\bigg[\sum_{u\in\mathcal{I}} \mathrm{Var}[\chi](t - S(u))\bigg]. \qquad (4.11)
\end{aligned}$$

By (4.8), the martingale $(M_n(t))_{n\in\mathbb{N}_0}$ is bounded in L^2 and thus converges in L^2 and almost surely. We denote the limit by $\mathcal{Z}_t^\chi$ and view it as the limit of the series on the right-hand side of (4.9). This is justified by the following argument. For any subset $\mathcal{J} \subseteq \mathcal{I}$, finite or infinite, since for any $u \in \mathcal{J}$ there is a unique $j \in \mathbb{N}$ with $u = u_j$ and again since martingale increments are uncorrelated, we have

$$\mathbb{E}\bigg[\bigg|\sum_{u\in\mathcal{J}} \chi_u(t - S(u))\bigg|^2\bigg] = \mathbb{E}\bigg[\sum_{u\in\mathcal{J}} \mathrm{Var}[\chi](t - S(u))\bigg].$$

From this and the Cauchy criterion, on the one hand, we infer the unconditional convergence in L^2 of the series in (4.9), thereby justifying to write Z_t^χ for the limit. On the other hand, we conclude the convergence of $M_{\mathcal{I}_n}(t)$ to $\mathcal{Z}_t^\chi$. Since convergence in L^2 implies convergence in L^1, $\mathcal{Z}_t^\chi$ is centered. Using this and again the convergence in L^2, we deduce

$$\begin{aligned}
\mathrm{Var}[\mathcal{Z}_t^\chi] &= \mathbb{E}[(\mathcal{Z}_t^\chi)^2] = \lim_{n\to\infty} \mathbb{E}[M_n(t)^2] \\
&= \lim_{n\to\infty} \mathbb{E}\bigg[\sum_{u\in\mathcal{I}_n} \chi_u^2(t - S(u))\bigg] = \mathbb{E}\bigg[\sum_{u\in\mathcal{I}} \chi_u^2(t - S(u))\bigg],
\end{aligned}$$

i.e., (4.10) holds. Finally, (A5) implies (4.8) since, for any $t \in \mathbb{R}$, by (4.6) and the subsequent arguments,

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathrm{Var}[\chi](t-S(u))\Big] = \mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathbb{E}[\chi^2](t-S(u))\Big]$$
$$= e^{\alpha t}\int \mathbb{E}[\chi^2](t-x)e^{-\alpha(t-x)}\,\mathsf{U}(\mathrm{d}x) \le Ce^{\alpha t}$$

where as before U is the renewal measure of the associated random walk $(S_n)_{n\in\mathbb{N}_0}$ and $C>0$ is some finite constant. □

We are now ready to prove Proposition 2.2.

PROOF OF PROPOSITION 2.2. By (A4), $\mathbb{E}[\varphi](t)$ is finite for every $t \in \mathbb{R}$ and we may write

$$\varphi_u(t-S(u)) = \mathbb{E}[\varphi](t-S(u)) + \big(\varphi_u(t-S(u)) - \mathbb{E}[\varphi](t-S(u))\big)$$

for every $u \in \mathcal{I}$. It is therefore enough to check that both series

$$\sum_{u\in\mathcal{I}} \mathbb{E}[\varphi](t-S(u)) \quad \text{and} \quad \sum_{u\in\mathcal{I}} \big(\varphi_u(t-S(u)) - \mathbb{E}[\varphi](t-S(u))\big) \tag{4.12}$$

converge almost surely over admissible orderings and unconditionally in L^1. For the first series, note that by (A4) the function $f(t) := |\mathbb{E}[\varphi](t)|e^{-\alpha t}$ is directly Riemann integrable as well and by (4.2), we have

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}} |\mathbb{E}[\varphi](t-S(u))|\Big] = \mathsf{E}\Big[\sum_{n\ge 0} e^{\alpha S_n}|\mathbb{E}[\varphi](t-S_n)|\Big] = e^{\alpha t}\cdot f * \mathsf{U}(t),$$

which is finite by (4.4) and the direct Riemann integrability of f. Hence, the series converges unconditionally in L^1 and absolutely almost surely. The same argument as above gives

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathrm{Var}[\varphi](t-S(u))\Big] < \infty,$$

i.e., $\chi(t) := \varphi(t) - \mathbb{E}[\varphi](t)$ is a centered characteristic satisfying (4.8). We may thus apply Lemma 4.1 to conclude that the second series in (4.12) converges almost surely over admissible orderings of $\mathcal{I}$ and unconditionally in L^2. □

We close this subsection with the proof of Proposition 2.6.

PROOF OF PROPOSITION 2.6. (a) Since f is càdlàg, it is locally bounded and continuous Lebesgue-almost everywhere. By (a slightly extended version of) [43, Remark 3.10.4 on p. 236], this together with

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[n,n+1]} |f(x)| \le \int f^*(x)\,\mathrm{d}x < \infty$$

ensures the direct Riemann integrability of f.

Conversely, if f is directly Riemann integrable, then it is locally bounded and continuous Lebesgue-almost everywhere. Local boundedness of f entails that of f^*. Since f^* is continuous on $\{x \in \mathbb{R} : f \text{ is continuous at } x-1 \text{ and } x+1\}$, this implies that also f^* is continuous Lebesgue-almost everywhere. Furthermore, for every $x \in \mathbb{R}$, we have

$$\sum_{n\in\mathbb{Z}} \sup_{n\le x<n+1} f^*(x) = \sum_{n\in\mathbb{Z}} \sup_{n-1\le x<n+2} |f(x)| \le 3\sum_{n\in\mathbb{Z}} \sup_{n\le x<n+1} |f(x)| < \infty$$

since f is directly Riemann integrable. Thus, again by [43, Remark 3.10.4 on p. 236], f^* is directly Riemann integrable.

We prove (b) and (c) at one go. To this end, let $p=1$ in the situation of (b) and $p=2$ in the situation (c). Define $\phi(t) := \varphi(t)^p$ for $t \in \mathbb{R}$. Then we infer

$$\int \mathbb{E}[\phi^*](x) e^{-\alpha x} \, \mathrm{d}x < \infty, \tag{4.13}$$

from (2.14) or (2.15), respectively, where we have used that $(\varphi^2)^* = (\varphi^*)^2$ in the situation of (c). From (4.13) we deduce that $\mathbb{E}[\phi^*](x) < \infty$ for Lebesgue-almost all $x \in \mathbb{R}$ and hence

$$\mathbb{E}\Big[\sup_{|t-x|\leq \frac{1}{2}} |\phi(t)|\Big] < \infty \quad \text{for all } x \in \mathbb{R}. \tag{4.14}$$

In the case of (c), this implies the validity of (A6). In both cases, (4.14) together with the dominated convergence theorem imply that $\mathbb{E}[\phi]$ has càdlàg paths and thus also f defined by $f(t) := \mathbb{E}[\phi(t)]e^{-\alpha t}$. Further, $\int f^*(x)\,\mathrm{d}x < \infty$ by (2.14) and (2.15), respectively, since $(\mathbb{E}[\phi](t)e^{-\alpha t})^* \leq e^{\alpha}\mathbb{E}[\phi]^*(t)e^{-\alpha t} \leq e^{\alpha}\mathbb{E}[\phi^*](t)e^{-\alpha t}$. Part (b) now follows from (a). In the situation of (c), we deduce from (a) that $t \mapsto \mathbb{E}[\varphi^2](t)e^{-\alpha t}$ is directly Riemann integrable. Also, $\mathbb{E}[\varphi]$ has càdlàg paths by (4.14) and the dominated convergence theorem. Therefore, $\mathrm{Var}[\varphi](t) = \mathbb{E}[\varphi^2](t) - \mathbb{E}[\varphi(t)]^2$ is càdlàg and, in particular, locally bounded and continuous Lebesgue-almost everywhere. Since

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[n,n+1]} \mathrm{Var}[\varphi(x)]e^{-\alpha x} \leq \sum_{n\in\mathbb{Z}} \sup_{x\in[n,n+1]} \mathbb{E}[\varphi(x)^2]e^{-\alpha x} < \infty$$

the direct Riemann integrability of $\mathrm{Var}[\varphi](t)e^{-\alpha t}$ follows from [43, Remark 3.10.4], i.e., (A5) holds.

$\square$

4.3. *Matrix notation.* For any $s \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ we define the following lower triangular $k \times k$ matrix

$$\exp(\gamma, s, k) := e^{\gamma s} \times \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ s & 1 & 0 & \dots & 0 \\ s^2 & 2s & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^{k-1} & \binom{k-1}{1}s^{k-2} & \binom{k-1}{2}s^{k-3} & \dots & 1 \end{pmatrix}. \tag{4.15}$$

The $(i,j)^{\mathrm{th}}$ entry of the matrix is $e^{\gamma s}\binom{i-1}{j-1}s^{i-j}$, $i,j = 1,\dots,k$, where $\binom{i-1}{j-1} = 0$ for $j > i$ should be recalled. Matrices of this form will be very useful since they simplify the notation and allow us to deal with polynomial terms with relative ease. Indeed, for any $s,t \in \mathbb{R}$ and $\gamma \in \mathbb{C}$,

$$\exp(\gamma, s, k) \cdot \exp(\gamma, t, k) = \exp(\gamma, s+t, k).$$

This can be seen from elementary but tedious calculations. Alternatively, notice that

$$\exp(\gamma, s, k) = \exp(sJ_{\gamma,k})$$

where the matrix $J_{\gamma,k}$ is defined by

$$J_{\gamma,k} := \begin{pmatrix} \gamma & & & \\ 1 & \ddots & 0 & \\ & \ddots & \ddots & \\ 0 & & k-1 & \gamma \end{pmatrix}.$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(\gamma, x, k) = J_{\gamma,k}\exp(\gamma, x, k). \tag{4.16}$$

With $\|\cdot\|$ denoting the operator norm and $\|\cdot\|_{\mathsf{HS}}$ denoting the Hilbert-Schmidt norm, the following (crude) bound holds for every $\delta > 0$:

$$\|\exp(\gamma, s, k)\| \leq \|\exp(\gamma, s, k)\|_{\mathsf{HS}} \leq C'(1+|s|)^{k-1}e^{\mathrm{Re}(\gamma)s} \leq Ce^{\mathrm{Re}(\gamma)s+\delta|s|} \tag{4.17}$$

for some constant $C' > 0$ depending on k only and another constant $C > 0$ depending on k and $\delta > 0$. For a vector x, we write x^{T} for its transpose. Further, we write $\mathsf{e}_1, \mathsf{e}_2, \ldots$ for the canonical base vectors in Euclidean space. Here, for ease of notation, we are slightly sloppy as we do not specify the dimension of that space (formally, all Euclidean spaces may be embedded into an appropriate infinite-dimensional space such as ℓ^2). Then, for instance,

$$\exp(\gamma, s, k)\cdot\mathsf{e}_1 = e^{\gamma s}\begin{pmatrix}1\\ s\\ s^2\\ \vdots\\ s^{k-1}\end{pmatrix}.$$

Throughout the paper, for $\mathrm{Re}(\lambda) > \vartheta$, $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we denote by $Z_n(\lambda, k)$ the following random matrix

$$Z_n(\lambda, k) := \sum_{|u|=n}\exp(\lambda, -S(u), k). \tag{4.18}$$

We set $Z_n(\lambda) := Z_n(\lambda, 1)$ for $\mathrm{Re}(\lambda) \geq \vartheta$. In particular, $\mu(\theta) = \mathbb{E}[Z_1(\theta)]$ and (A2) becomes $\mathbb{E}[Z_1(\vartheta)] < \infty$.

5. Nerman's martingales as general branching processes. Nerman's martingale and its complex counterparts are crucial for the paper as they constitute the building blocks for the asymptotic expansion of $\mathcal{Z}^{\varphi}$. In the present section, we demonstrate how these martingales can be represented in terms of Crump-Mode-Jagers processes and which characteristics come into play.

Suppose that (A1) holds and that $\mathcal{L}\mu(\vartheta) < \infty$ for some $0 < \vartheta < \alpha$. (Notice that the last condition is implied by (A2).) Further, let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \vartheta$ be a root of multiplicity $k = k(\lambda) \in \mathbb{N}$ of the mapping $z \mapsto \mathcal{L}\mu(z) - 1$, i.e.,

$$\mathcal{L}\mu(\lambda) = \mathbb{E}\bigg[\sum_{j=1}^{N} e^{-\lambda X_j}\bigg] = 1, \tag{5.1}$$

$$\mathcal{L}\mu^{(l)}(\lambda) = (-1)^l\,\mathbb{E}\bigg[\sum_{j=1}^{N} X_j^l e^{-\lambda X_j}\bigg] = 0 \quad \text{for } l = 1, \ldots, k(\lambda)-1, \tag{5.2}$$

$$\mathcal{L}\mu^{(k(\lambda))}(\lambda) \neq 0. \tag{5.3}$$

Conditions (5.1) and (5.2) are equivalent to

$$\mathbb{E}[Z_1(\lambda, k)] = \mathbb{E}\bigg[\sum_{j=1}^{N}\exp(\lambda, -X_j, k)\bigg] = I_k \tag{5.4}$$

where I_k is the $k \times k$ identity matrix.

Define the random matrix

$$Y_u := Z_{u,1}(\lambda,k) - I_k = \int \exp(\lambda,-x,k)\,\xi_u(\mathrm{d}x) - I_k.$$

Notice that $\mathbb{E}[Y_u]$ is the $k\times k$ zero matrix by (5.4). Moreover, if $\mathrm{Re}(\lambda)\geq\frac{\alpha}{2}$ and (A3) is satisfied then, by the penultimate inequality in (4.17), we have $\mathbb{E}[\|Y_u\|^2]\leq C_{\lambda,k}<\infty$ for some constant $C_{\lambda,k}$ that depends only on λ and k.

Now for $\lambda\in\mathbb{C}$ such that $\mathrm{Re}(\lambda)>\vartheta$ and (5.1) through (5.3) holds we define matrix-valued characteristics ϕ_λ and $\chi_{\lambda,I}$, which play a crucial role in the proof of the main theorem. For $t\in\mathbb{R}$, we set

$$\phi_\lambda(t) := \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t)\exp(\lambda,t-X_j,k), \tag{5.5}$$

and for any interval $I=[a,b)\cap\mathbb{R}$ with $-\infty\leq a<b<\infty$

$$\chi_{\lambda,I}(t) := \mathbb{1}_I(t)\exp(\lambda,t,k)Y_\varnothing \quad \text{and} \quad \chi_\lambda(t):=\chi_{\lambda,(-\infty,0)}(t). \tag{5.6}$$

By definition, characteristics take values in $\mathbb{R}^d$ for some $d\in\mathbb{N}$, but here we use an obvious extension to $\mathbb{C}$ by splitting into real and imaginary part. Note also that both ϕ_λ and $\chi_{\lambda,I}$ are $\sigma(\xi)$-measurable and, in particular, the tuples $(\xi_u,\phi_{\lambda,u},\chi_{\lambda,u})$, $u\in\mathcal{I}$ are i. i. d., where $\chi_{\lambda,u}=\mathbb{1}_{(-\infty,0)}(t)\exp(\lambda,t,k)Y_u$, $t\in\mathbb{R}$.

LEMMA 5.1. *Suppose that (A1) through (A3) hold. Let $\lambda\in\Lambda_\geq$, let k denote the multiplicity of λ, and fix $x,y\in\mathbb{R}^k$.*

(a) *The characteristic $x^\mathsf{T}\phi_\lambda y$ satisfies (A4), (A5) and (A6).*
(b) *Let $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$, and $I=[a,b)\cap\mathbb{R}$ be an interval with $-\infty\leq a<b<\infty$. Then the characteristic $x^\mathsf{T}\chi_{\lambda,I}y$ satisfies (A4), (A5) and (A6).*

PROOF. Clearly, both characteristics $x^\mathsf{T}\phi_\lambda y$ and $x^\mathsf{T}\chi_\lambda y$ have càdlàg paths. Without loss of generality, we may assume that $|x|,|y|\leq 1$. In view of Proposition 2.6 it suffices to verify the integrability conditions (2.14) and (2.15). Let us first assume that $\mathrm{Re}(\lambda)>\frac{\alpha}{2}$. Then we take $\gamma\in(\frac{\alpha}{2},\mathrm{Re}(\lambda))$ and from (4.17) we infer the existence of a constant C that depends only on λ,γ and k such that, for $s\leq 0$ and $|t-s|\leq 1$, we have

$$\left|x^\mathsf{T}\exp(\lambda,s,k)y\right| \leq \|\exp(\lambda,s-t,k)\|_{\mathsf{HS}}\|\exp(\lambda,t,k)\|_{\mathsf{HS}} \leq Ce^{\gamma t}.$$

We can thus write for $t\in\mathbb{R}$

$$\begin{aligned}(x^\mathsf{T}\phi_\lambda y)^*(t) &= \sup_{|s-t|\leq 1}\left|x^\mathsf{T}\phi_\lambda(s)y\right| \\ &\leq \sup_{|s-t|\leq 1}\sum_{j=1}^{N}\mathbb{1}_{[0,X_j)}(s)\left|x^\mathsf{T}\exp(\lambda,s-X_j,k)y\right| \\ &\leq C\sum_{j=1}^{N}\mathbb{1}_{[-1,X_j+1)}(t)e^{\gamma(t-X_j)}.\end{aligned}$$

Hence, we have

$$\int \mathbb{E}\Big[(x^\mathsf{T}\phi_\lambda y)^*(t)\Big]e^{-\alpha t}\,\mathrm{d}t \leq C\int_{-\infty}^{\infty}\mathbb{E}\Big[\sum_{j=1}^{N}\mathbb{1}_{[-1,X_j+1)}(t)e^{\gamma(t-X_j)}\Big]e^{-\alpha t}\,\mathrm{d}t$$

$$\leq C\mathbb{E}\Big[\sum_{j=1}^{N} e^{-\gamma X_j}\Big] \int_{-1}^{\infty} e^{(\gamma-\alpha)t}\,\mathrm{d}t < \infty$$

by (A2). Further,

$$\begin{aligned}
&\int \mathbb{E}\Big[\Big((x^{\mathsf{T}}\phi_\lambda y)^*(t)\Big)^2\Big] e^{-\alpha t}\,\mathrm{d}t \\
&\leq C^2 \int_{-\infty}^{\infty} \mathbb{E}\Big[\sum_{1\leq i,j\leq N} \mathbb{1}_{[-1,X_i+1)}(t) e^{\gamma(t-X_i)} \mathbb{1}_{[-1,X_j+1)}(t) e^{\gamma(t-X_j)}\Big] e^{-\alpha t}\,\mathrm{d}t \\
&\leq C^2 \mathbb{E}\Big[\sum_{1\leq i,j\leq N} \int_{-1}^{(X_i\wedge X_j)+1} e^{(2\gamma-\alpha)t}\,\mathrm{d}t\; e^{-\gamma X_i} e^{-\gamma X_j}\Big] \\
&\leq \frac{e^{2\gamma-\alpha}C^2}{2\gamma-\alpha} \mathbb{E}\Big[\sum_{1\leq i,j\leq N} e^{(2\gamma-\alpha)(X_i\wedge X_j)} e^{-\gamma X_i} e^{-\gamma X_j}\Big] \\
&\leq \frac{e^{2\gamma-\alpha}C^2}{2\gamma-\alpha} \mathbb{E}\Big[\sum_{1\leq i,j\leq N} e^{(\gamma-\frac{\alpha}{2})(X_i+X_j)} e^{-\gamma X_i} e^{-\gamma X_j}\Big] \\
&= \frac{e^{2\gamma-\alpha}C^2}{2\gamma-\alpha} \mathbb{E}\Big[\Big(\sum_{j=1}^{N} e^{-\frac{\alpha}{2}X_j}\Big)^2\Big] < \infty
\end{aligned}$$

by (A3). Now assume that $\mathrm{Re}(\lambda) = \frac{\alpha}{2}$. Then for $s \leq 0$ and $|t-s| \leq 1$, we have, again by (4.17),

$$\big|x^{\mathsf{T}} \exp(\lambda, s, k) y\big| \leq C(1+|t|)^{k-1} e^{\frac{\alpha}{2}t}$$

for some C depending on λ and k (not necessarily the exact constant C from (4.17), but a larger, finite one). This, in turn, gives, for arbitrary $t \in \mathbb{R}$,

$$\begin{aligned}
(x^{\mathsf{T}}\phi_\lambda y)^*(t) &= \sup_{|s-t|\leq 1} \big|x^{\mathsf{T}}\phi_\lambda(s) y\big| \\
&\leq \sup_{|s-t|\leq 1} \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(s) \big|x^{\mathsf{T}} \exp(\lambda, s-X_j, k) y\big| \\
&\leq C \sum_{j=1}^{N} \mathbb{1}_{[-1,X_j+1)}(t) (1+|t-X_j|)^{k-1} e^{\frac{\alpha}{2}(t-X_j)},
\end{aligned}$$

and, consequently,

$$\int \mathbb{E}\big[(x^{\mathsf{T}}\phi_\lambda y)^*(t)\big] e^{-\alpha t}\,\mathrm{d}t \leq \frac{C}{\sqrt{2}} \mathbb{E}\Big[\sum_{j=1}^{N} (2+X_j)^{k-\frac{1}{2}} e^{-\frac{\alpha}{2}X_j}\Big] \int_{-1}^{\infty} e^{-\frac{\alpha}{2}t}\,\mathrm{d}t < \infty$$

by (A3) and

$$\begin{aligned}
&\int \mathbb{E}\big[\big((x^{\mathsf{T}}\phi_\lambda y)^*(t)\big)^2\big] e^{-\alpha t}\,\mathrm{d}t \\
&\leq C^2 \int_{-\infty}^{\infty} \mathbb{E}\Big[\sum_{1\leq i,j\leq N} \mathbb{1}_{[-1,X_i+1)}(t) (1+|t-X_i|)^{k-1} e^{\frac{\alpha}{2}(t-X_i)}
\end{aligned}$$

$$\cdot\, \mathbb{1}_{[-1,X_j+1)}(t)(1+|t-X_j|)^{k-1}e^{\frac{\alpha}{2}(t-X_j)}\Big]e^{-\alpha t}\,\mathrm{d}t$$

$$\leq C^2\mathbb{E}\Big[\sum_{1\leq i,j\leq N}\int_{-1}^{X_i\wedge X_j+1}\mathrm{d}t\,(2+X_i)^{k-1}e^{-\frac{\alpha}{2}X_i}(2+X_j)^{k-1}e^{-\frac{\alpha}{2}X_j}\Big]$$

$$\leq C^2\mathbb{E}\Big[\sum_{1\leq i,j\leq N}(X_i\wedge X_j+2)(2+X_i)^{k-1}e^{-\frac{\alpha}{2}X_i}(2+X_j)^{k-1}e^{-\frac{\alpha}{2}X_j}\Big]$$

$$\leq 2^{2k-1}C^2\mathbb{E}\Big[\Big(\sum_{j=1}^{N}(1+\tfrac{X_j}{2})^{k-\frac{1}{2}}e^{-\frac{\alpha}{2}X_j}\Big)^2\Big]<\infty$$

again by (A3), which finish the proof of (a). Regarding part (b), notice that (A4) holds trivially as $\chi_{\lambda,I}$ is centered. Further, observe that

$$\begin{aligned}(x^{\mathsf{T}}\chi_{\lambda,I}y)^*(t) &= \sup_{|s-t|\leq 1}\mathbb{1}_{[a,b)\cap\mathbb{R}}(s)\big|x^{\mathsf{T}}\exp(\lambda,s,k)(Z_1(\lambda,k)-I_k)y\big| \\ &\leq C\mathbb{1}_{(-\infty,b+1)}(t)\|\exp(\lambda,t,k)\|\|Z_1(\lambda,k)-I_k\| \\ &= C^2\mathbb{1}_{(-\infty,b+1)}(t)e^{\gamma t}\Big(1+\sum_{j=1}^{N}e^{-\frac{\alpha}{2}X_j}\Big)\end{aligned}$$

where, as before, $\gamma\in(\frac{\alpha}{2},\mathrm{Re}(\lambda))$. Thus

$$\int\mathbb{E}\big[\big((x^{\mathsf{T}}\chi_{\lambda,I}y)^*(t)\big)^2\big]e^{-\alpha t}\,\mathrm{d}t\leq C^4\mathbb{E}\Big[\Big(1+\sum_{j=1}^{N}e^{-\frac{\alpha}{2}X_j}\Big)^2\Big]\int_{-\infty}^{b+1}e^{(2\gamma-\alpha)t}\,\mathrm{d}t<\infty$$

by (A3), which completes the proof of (b). □

As a consequence of the above lemma we conclude that, under the assumptions (A1) – (A3), for any $t\in\mathbb{R}$ $\mathcal{Z}_t^{\phi_\lambda}$ for $\lambda\in\Lambda_{\geq}$ and $\mathcal{Z}_t^{\chi_{\lambda,I}}$ for $\lambda\in\Lambda$ are well-defined. The first one is so as an unconditional limit in L^1 by Proposition 2.2 and the second as an unconditional limit in L^2 by (the first part of) Lemma 4.1 (by Lemma 5.1, the characteristic $\chi_{\lambda,I}$ satisfies (A5); according to the last part of Lemma 4.1, (A5) entails (4.8), the principal assumption of the first part of Lemma 4.1). In particular,

$$\mathcal{Z}_0^{\chi_\lambda}=\sum_{u\in\mathcal{I}}\exp(\lambda,-S(u),k)Y_u$$

converges unconditionally in L^2 and almost surely along admissible orderings of $\mathcal{I}$.

5.1. *Nerman's martingales with complex parameters.* For $u\in\mathcal{I}$ we define $\mathcal{G}_u:=\sigma(\xi_v: v\preceq u)$ and, for $t\in\mathbb{R}$,

$$\mathcal{F}_t^W:=\sigma(\{A\cap\{S(u)\leq t\}:u\in\mathcal{I}\text{ and }A\in\mathcal{G}_u\}).$$

LEMMA 5.2. *Suppose that (A1) holds and that $\mathcal{L}\mu(\vartheta)<\infty$ for some $0<\vartheta<\alpha$. Let $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>\vartheta$ be a root of $z\mapsto\mathcal{L}\mu(z)-1$ with multiplicity k. If the characteristic ϕ_λ satisfies (A4), (A5) and (A6), then the following process*

$$W_t(\lambda,k):=\exp(\lambda,-t,k)\cdot\mathcal{Z}_t^{\phi_\lambda},\qquad t\in\mathbb{R}$$

is a (matrix-valued) martingale with respect to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$. Moreover, for any $t\in\mathbb{R}$, it holds

$$W_t(\lambda,k) = I_k \mathbb{1}_{[0,\infty)}(t) + \mathcal{Z}_0^{\chi_{\lambda,[-t,1)}} = \sum_{u\in\mathcal{C}_t} \exp(\lambda,-S(u),k) \quad \textit{a. s.} \tag{5.7}$$

where by definition (see (2.11)) $\mathcal{C}_t = \{uj\in\mathcal{T} : S(u)\leq t < S(uj)\}$. In particular,

$$m_t^{\phi_\lambda} = \mathbb{1}_{[0,\infty)}(t)\exp(\lambda,t,k), \quad t\in\mathbb{R}.$$

REMARK 5.3. It is worth mentioning that for $1\leq l\leq k$ and the matrix-valued characteristic $\phi_{\lambda,l}$ obtained by taking the upper left $l\times l$ submatrix, i.e.,

$$\phi_{\lambda,l}(t) := \sum_{j=1}^{N} \mathbb{1}_{[0,X_j)}(t)\exp(\lambda,t-X_j,l),$$

if $\phi_{\lambda,l}$ satisfies (A4), (A5) and (A6), then the proof below carries over and gives that $W_t(\lambda,l)$ is a matrix-valued martingale and

$$W_t(\lambda,l) = \sum_{u\in\mathcal{C}_t}\exp(\lambda,-S(u),l) \quad \text{a. s.}$$

In particular, if the above conditions hold with $l=1$, then we obtain that

$$W_t(\lambda,1) = \sum_{u\in\mathcal{C}_t} e^{-\lambda S(u)}$$

is a martingale.

PROOF OF LEMMA 5.2. (A1), (A4) and (A5) entail that, by Proposition 2.2, $\mathcal{Z}^{\phi_\lambda}$ is well-defined as an unconditional limit in L^1 and that $W_t(\lambda,k)$ is integrable for any $t\in\mathbb{R}$.

We boldly write

$$\begin{aligned}\exp(\lambda,-t,k)\mathcal{Z}_t^{\phi_\lambda} &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u}\mathbb{1}_{[0,X_{u,j})}(t-S(u))\exp(\lambda,-S(uj),k)\\ &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u}\mathbb{1}_{\{S(u)\leq t\}}\mathbb{1}_{\{S(uj)>t\}}\exp(\lambda,-S(uj),k)\\ &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u}\big(\mathbb{1}_{\{S(u)\leq t\}}-\mathbb{1}_{\{S(uj)\leq t\}}\big)\exp(\lambda,-S(uj),k)\\ &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u}\mathbb{1}_{\{S(u)\leq t\}}\exp(\lambda,-S(uj),k) - \sum_{|u|\geq 1}\mathbb{1}_{\{S(u)\leq t\}}\exp(\lambda,-S(u),k)\\ &= I_k\mathbb{1}_{[0,\infty)}(t) + \sum_{u\in\mathcal{I}}\mathbb{1}_{\{S(u)\leq t\}}\exp(\lambda,-S(u),k)\big(Z_1(\lambda,k)\circ\theta_u - I_k\big),\end{aligned} \tag{5.8}$$

where the rearrangements of the infinite series in the last two lines are justified by the fact that there are only finitely many non-zero terms almost surely. Next, note that for any $t\in\mathbb{R}$,

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}}\mathbb{1}_{\{S(u)\leq t\}}\big\|\exp(\lambda,-S(u),k)\big\|\Big(\sum_{j=1}^{N_u}\big\|\exp(\lambda,-X_{u,j},k)\big\|+1\Big)\Big] < \infty. \tag{5.9}$$

Indeed, by (4.17), the expectation in (5.9) can be bounded by a finite, deterministic constant times

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} e^{-\vartheta S(u)}\big(Z_1(\vartheta)\circ\theta_u+1\big)\Big] \le \big(\mathcal{L}\mu(\vartheta)+1\big)\mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} e^{-\vartheta S(u)}\Big] < \infty,$$

where we have used the independence between $S(u)$ and ξ_u. The finiteness of the last expectation follows from the many-to-one lemma (Formula (4.2)), namely,

$$\mathbb{E}\Big[\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} e^{-\vartheta S(u)}\Big] = \mathsf{E}\Big[\sum_{n\ge 0} \mathbb{1}_{\{S_n\le t\}} e^{(\alpha-\vartheta)S_n}\Big] \le e^{(\alpha-\vartheta)t}\mathsf{U}([0,t]) < \infty.$$

Further, $(W_t(\lambda,k))_{t\ge0}$ is adapted to the filtration $(\mathcal{F}_t^W)_{t\ge0}$. In order to show the martingale property note that, for $0\le s<t$,

$$W_t(\lambda,k) - W_s(\lambda,k) = \sum_{u\in\mathcal{I}} \mathbb{1}_{\{s<S(u)\le t\}} \exp(\lambda,-S(u),k)\big(Z_1(\lambda,k)\circ\theta_u - I_k\big),$$

and by (5.9) it suffices to show that for any $u\in\mathcal{I}$

$$\mathbb{E}\big[\mathbb{1}_{\{s<S(u)\le t\}} \exp(\lambda,-S(u),k)\big(Z_1(\lambda,k)\circ\theta_u - I_k\big)\big|\mathcal{F}_s^W\big] = 0 \quad \text{a. s.} \tag{5.10}$$

Let $u,v\in\mathcal{I}$ and note that the fact $S(u)>s$, $S(v)\le s$ implies $u \not\preceq v$. In particular, for such u and v, ξ_u is independent of $\mathcal{G}_v$ and hence for any $A\in\mathcal{G}_v$

$$\mathbb{E}\big[\mathbb{1}_{\{s<S(u)\le t\}} \exp(\lambda,-S(u),k)\big(Z_1(\lambda,k)\circ\theta_u - I_k\big)\mathbb{1}_{A\cap\{S(v)\le s\}}\big] = 0,$$

where $\mathbb{E}[Z_1(\lambda,k)] = I_k$ was used. The argument carries over if we take a finite intersection of sets of the type $A\cap\{S(v)\le s\}$, $A\in\mathcal{G}_v$ for different $v\in\mathcal{I}$. The π-λ-theorem (or monotone class theorem) gives (5.10) and thus proves that $(W_t(\lambda,k))_{t\ge0}$ is a martingale.

It remains to prove (5.7). The first identity of this equation is (5.8). Further, from the calculation leading towards (5.8), we have

$$\begin{aligned} W_t(\lambda,k) &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u} \mathbb{1}_{\{S(u)\le t\}}\mathbb{1}_{\{S(uj)>t\}} \exp(\lambda,-S(uj),k) \\ &= \sum_{u\in\mathcal{I}}\sum_{j=1}^{N_u} \mathbb{1}_{\mathcal{C}_t}(uj) \exp(\lambda,-S(uj),k) = \sum_{u\in\mathcal{C}_t} \exp(\lambda,-S(u),k). \end{aligned}$$

□

The random matrix $W_t(\lambda,k)$ has the following form

$$W_t(\lambda,k) = \begin{pmatrix} W_t^{(0)}(\lambda) & 0 & 0 & \dots & 0 \\ W_t^{(1)}(\lambda) & W_t^{(0)}(\lambda) & 0 & \dots & 0 \\ W_t^{(2)}(\lambda) & 2W_t^{(1)}(\lambda) & W_t^{(0)}(\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_t^{(k-1)}(\lambda) & \binom{k-1}{1}W_t^{(k-2)}(\lambda) & \binom{k-1}{2}W_t^{(k-3)}(\lambda) & \dots & W_t^{(0)}(\lambda) \end{pmatrix}, \tag{5.11}$$

where $W_t^{(j)}(\lambda) = \sum_{u\in\mathcal{C}_t}(-S(u))^j e^{-\lambda S(u)}$ as in (2.17).

5.2. *Convergence of Nerman's martingales.* The following lemma implies Theorem 2.7.

LEMMA 5.4. *Suppose that (A1) through (A3) hold and let λ be a solution to (2.16) with multiplicity k and $\mathrm{Re}(\lambda) > \frac{\alpha}{2}$. Then the process $W_t(\lambda,k)$ is an L^2-bounded martingale with limit given by*

$$W(\lambda,k) := I_k + \sum_{u\in\mathcal{I}} \exp(\lambda,-S(u),k) Y_u = I_k + \mathcal{Z}_0^{\chi_{\lambda,(-\infty,1)}},$$

where the series above converges unconditionally in L^2. In particular, for every $0 \le j \le k-1$, the martingale $(W_r^{(j)}(\lambda))_{t\ge 0}$ converges a. s. and in L^2.

PROOF. Fix $x, y \in \mathbb{R}^k$. It suffices to show the corresponding result for the martingale $x^\mathsf{T} W_t(\lambda,k) y$. By Lemma 5.1 the centered characteristics $x^\mathsf{T}\chi_{\lambda,(-\infty,1)}y$, $x^\mathsf{T}\chi_{\lambda,(-\infty,-t)}y$ and $x^\mathsf{T}\chi_{\lambda,[-t,1)}y$ satisfy (A5) and (A6). In particular, by Lemma 4.1, the general branching processes counted with these characteristics are well-defined as unconditional limits in L^2, and, for $t \ge 0$,

$$\sum_{u\in\mathcal{I}} x^\mathsf{T}\exp(\lambda,-S(u),k)Y_u y = \mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,(-\infty,1)}y} = \mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,(-\infty,-t)}y} + \mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,[-t,1)}y}$$

$$= \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)>t\}} x^\mathsf{T}\exp(\lambda,-S(u),k)Y_u y + \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} x^\mathsf{T}\exp(\lambda,-S(u),k)Y_u y.$$

Taking into account that, for any $s\in\mathbb{R}$, $t\ge -1$,

$$\mathrm{Var}[x^\mathsf{T}\chi_{\lambda,(-\infty,-t)}y](s) \le \mathrm{Var}[x^\mathsf{T}\chi_{\lambda,(-\infty,1)}y](s),$$

and applying the identity (4.10) (by splitting the characteristics into real and imaginary part) the dominated convergence theorem yields

$$\lim_{t\to\infty} \mathbb{E}\Big[\Big|\mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,(-\infty,-t)}y}\Big|^2\Big] = 0.$$

In particular, in view of (5.7), we infer that

$$x^\mathsf{T} W_t(\lambda,k) y = x^\mathsf{T} y + \mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,(-\infty,1)}y} - \mathcal{Z}_0^{x^\mathsf{T}\chi_{\lambda,(-\infty,-t)}y}$$

converges in L^2 as $t\to\infty$. □

5.3. *Limits of Nerman's martingales as general branching processes.* Suppose now that the martingale $(W_t(\lambda,k))_{t\ge 0}$ is uniformly integrable. Then it converges in L^1 as $t\to\infty$ to some random matrix $W(\lambda,k)$ of the form

$$W(\lambda,k) = \begin{pmatrix} W^{(0)}(\lambda) & 0 & 0 & \dots & 0 \\ W^{(1)}(\lambda) & W^{(0)}(\lambda) & 0 & \dots & 0 \\ W^{(2)}(\lambda) & 2W^{(1)}(\lambda) & W^{(0)}(\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W^{(k-1)}(\lambda) & \binom{k-1}{1} W^{(k-2)}(\lambda) & \binom{k-1}{2} W^{(k-3)}(\lambda) & \dots & W^{(0)}(\lambda) \end{pmatrix}. \tag{5.12}$$

By uniform integrability, $\mathbb{E}[W(\lambda,k)] = I_k$.

LEMMA 5.5. *Suppose that (A1) through (A3) hold and let λ be a solution to (2.16) with multiplicity k and $\mathrm{Re}(\lambda) > \frac{\alpha}{2}$. Then, for $t\ge 0$,*

$$\exp(\lambda,-t,k)\mathcal{Z}_t^{\chi_\lambda} = W(\lambda,k) - W_t(\lambda,k).$$

In particular, for $t \ge 0$,

$$\mathcal{Z}_t^{\phi_\lambda+\chi_\lambda} = \exp(\lambda,t,k) W(\lambda,k). \tag{5.13}$$

PROOF. Note that for any $s \in \mathbb{R}$, $t \geq 0$, we have

$$\exp(\lambda, -t, k)\chi_\lambda(s) = \chi_{\lambda,(-\infty,-t)}(s-t).$$

In particular,

$$\exp(\lambda, -t, k)\mathcal{Z}_t^{\chi_\lambda} = \mathcal{Z}_0^{\chi_{\lambda,(-\infty,-t)}},$$

which equals $W(\lambda, k) - W_t(\lambda, k)$ by Lemma 5.4. This together with Lemma 5.2 implies

$$\mathcal{Z}_t^{\phi_\lambda + \chi_\lambda} = \exp(\lambda, t, k)W(\lambda, k),$$

for any $t \geq 0$. □

6. Proofs of the main results. In this section we provide a proof of our main Theorem 2.15. We begin by outlining the main ideas and steps.

The basic step is to decompose a given general branching process $\mathcal{Z}_t^\varphi$

$$\mathcal{Z}_t^\varphi = H_\Lambda(t) + H_{\partial\Lambda}(t) + \mathcal{Z}_t^{\varrho - \phi_{\partial\Lambda}} + \mathcal{Z}_t^\chi.$$

into $H_\Lambda(t)$ and $H_{\partial\Lambda}(t)$, see (2.21) and (2.22), the leading terms in the expansion, plus two general branching processes $\mathcal{Z}_t^{\varrho - \phi_{\partial\Lambda}}$ and $\mathcal{Z}_t^\chi$, the first one with mean roughly of the order $o(e^{\frac{\alpha}{2}t})$ as $t \to \pm\infty$ and the second one with centered characteristic, i.e., $\mathbb{E}[\chi(t)] = 0$ for all $t \in \mathbb{R}$.

General branching processes with centered characteristics are studied in Section 6.1. Theorem 6.3 provides the fluctuations of $\mathcal{Z}_t^\chi$ for a centered characteristic χ. There are two different cases of interest. First, when χ satisfies (A5) and second when $\int_0^t \mathrm{Var}[\chi(x)]e^{-\alpha x}\,\mathrm{d}x \sim ct^\theta$ for some $\theta \geq 0$ and $c > 0$. In both cases, $\mathcal{Z}_t^\chi$, appropriately rescaled, is asymptotically normal. The main tools to prove this are the martingale central limit theorem and Nerman's strong law of large numbers for supercritical general branching processes. The second case requires the additional auxiliary Lemma 6.1.

Characteristics such that the corresponding general branching process has mean function roughly of the order $o(e^{\frac{\alpha}{2}t})$ as $t \to \pm\infty$ are treated in Section 6.3. Theorem 6.6 of this section yields asymptotic normality for such processes.

Section 6.2 provides a connection between the cases studied in Sections 6.1 and 6.3. Roughly speaking, Lemma 6.4 enables us to rewrite the process $\mathcal{Z}_t^f$ for a deterministic characteristic f in the form

$$\mathcal{Z}_t^f = \mathcal{Z}_t^{\chi_f} + m_t^f,$$

for an appropriately chosen centered characteristic χ_f. This enables us to reduce the case of general branching processes with mean function roughly of the order $o(e^{\frac{\alpha}{2}t})$ as $t \to \pm\infty$ to the case of centered characteristics.

In Section 6.4, we put all the pieces together and prove the main Theorem 2.15.

We investigate the asymptotic behavior of the general branching process $\mathcal{Z}_t^\varphi$ counted with characteristic φ as $t \to \infty$ in several steps. In the first step, we prove convergence of Nerman's martingales at complex parameters.

6.1. *Centered characteristics.* In this section we study the fluctuations of $\mathcal{Z}_t^\chi$ as $t \to \infty$ for centered characteristics, that is, for characteristics χ satisfying $\mathbb{E}[\chi(t)] = 0$ for all $t \in \mathbb{R}$. Theorem 6.3 below plays a key role in the proof of our main result Theorem 2.15. Before we state it, we give a preparatory lemma.

LEMMA 6.1. *Suppose that (A1) through (A3) hold. Let $\theta \ge 0$ and $f:[0,\infty)\to[0,\infty)$ be a continuous function with $f(x) = O(x^\theta)$ as $x\to\infty$ such that $x^{-\theta}f(x)$ is uniformly continuous on $[1,\infty)$ and the limit*

$$\lim_{t\to\infty,\, t\in\mathbb{G}} \frac{1}{t^{\theta+1}} \int_{[0,t]} f(x)\,\ell(\mathrm{d}x) =: c \in (0,\infty) \tag{6.1}$$

exists. Then, for $\varphi(t) := e^{\alpha t} f(t)\mathbb{1}_{[0,\infty)}(t)$, we have $\sup_{t\ge 1} e^{-\alpha t} t^{-\theta-1}\mathbb{E}[\mathcal{Z}_t^\varphi] < \infty$ and

$$\frac{e^{-\alpha t}}{t^{\theta+1}} \mathcal{Z}_t^\varphi \to \frac{cW}{\beta} \quad \text{as } t\to\infty,\ t\in\mathbb{G} \quad \text{a. s.} \tag{6.2}$$

PROOF. For any $t\ge 0$, we have $e^{-\alpha t}N((t-1,t]) = e^{-\alpha t}\mathcal{Z}_t^{\mathbb{1}_{[0,1)}}$. Taking expectations and using (4.6) gives

$$e^{-\alpha t}\mathbb{E}[N((t-1,t])] = e^{-\alpha t}\mathbb{E}[\mathcal{Z}_t^{\mathbb{1}_{[0,1)}}] = \int \mathbb{1}_{[0,1)}(t-x) e^{-\alpha(t-x)}\,\mathsf{U}(\mathrm{d}x),$$

which converges to a finite constant as $t\to\infty$, $t\in\mathbb{G}$ by the key renewal theorem and (A1). Hence,

$$\frac{e^{-\alpha t}}{t^{\theta+1}}\mathbb{E}[\mathcal{Z}_t^\varphi] \le C_\theta\left(\frac{\mathbb{E}[N(\{0\})]}{t} + \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{n=0}^{\lfloor t\rfloor} e^{\alpha(t-n)}(t-n)^\theta \mathbb{E}[N((n,n+1])]\right)$$

is bounded for $t\ge 1$. It remains to show (6.2). To this end, first assume that $|t^{-\theta}f(t)| \le C_\theta < \infty$ for all $t>0$. In particular, $f(0)=0$ if $\theta>0$. First notice that for any fixed $r>0$ and $t\ge r$,

$$\frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{\substack{u\in\mathcal{I}:\\ S(u)\le r}} e^{\alpha(t-S(u))} f(t-S(u)) \le \frac{C_\theta}{t}\cdot N([0,r]) \to 0$$

almost surely as $t\to\infty$. Hence, almost surely, the limiting behavior of $e^{-\alpha t}t^{-\theta-1}\mathcal{Z}_t^\varphi$ as $t\to\infty$, $t\in\mathbb{G}$, is the same as that of

$$e^{-\alpha t}t^{-\theta-1} \sum_{\substack{u\in\mathcal{I}:\\ r<S(u)\le t}} e^{\alpha(t-S(u))} f(t-S(u)). \tag{6.3}$$

Now first consider the lattice case and notice that by [36, Corollary 3.1(b)], for given $\varepsilon>0$, with probability 1 we may choose (a random) $r\in\mathbb{N}$ so large that

$$(1-\varepsilon)e^{\alpha k}\frac{W}{\beta} \le N(\{k\}) \le (1+\varepsilon)e^{\alpha k}\frac{W}{\beta}$$

for all $k\in\mathbb{N}$, $k\ge r$. Then, for $t\in\mathbb{N}$ with $t>r$,

$$\begin{aligned} e^{-\alpha t}t^{-\theta-1} \sum_{\substack{u\in\mathcal{I}:\\ r\le S(u)\le t}} e^{\alpha(t-S(u))} f(t-S(u)) &= \frac{1}{t^{\theta+1}} \sum_{k=r}^{t} f(t-k)e^{-\alpha k}N(\{k\}) \\ &\le \frac{(1+\varepsilon)W}{\beta t^{\theta+1}} \sum_{k=0}^{t-r} f(k) \to (1+\varepsilon)\frac{cW}{\beta} \quad \text{as } t\to\infty \end{aligned}$$

by (6.1). The corresponding lower bound can be obtained analogously. Now (6.2) follows by letting $\varepsilon\to 0$.

Next, we turn to the non-lattice case and fix small $\varepsilon, \delta > 0$. By [36, Corollary 3.1(a)], with probability 1, we may choose (a random) $r \in \delta\mathbb{N}$, $r \geq 1$ so large that

$$(1-\varepsilon)e^{\alpha t}\frac{e^{\alpha\delta}-1}{\alpha}\frac{W}{\beta} \leq N((t,t+\delta]) \leq (1+\varepsilon)e^{\alpha t}\frac{e^{\alpha\delta}-1}{\alpha}\frac{W}{\beta}$$

for all $t \geq r-\delta$. For $t \geq r$, define $I_k^\delta := [k\delta,(k+1)\delta)$ for $k = 0,\ldots,t_\delta - 1$ where $t_\delta := \lfloor\frac{t-r}{\delta}\rfloor$, and $I_{t_\delta}^\delta := [t_\delta\delta, t-r)$. Notice that $t - t_\delta\delta \geq r$. Hence, almost surely,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{\substack{u\in\mathcal{I}:\\ r<S(u)\leq t}} e^{\alpha(t-S(u))} f(t-S(u)) \\
&\qquad\leq \limsup_{t\to\infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{k=0}^{t_\delta} \sum_{\substack{u\in\mathcal{I}:\\ t-S(u)\in I_k^\delta}} e^{\alpha \sup I_k^\delta} f(t-S(u)) \\
&\qquad\leq \limsup_{t\to\infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{k=0}^{t_\delta} e^{\alpha \sup I_k^\delta} N(t-I_k^\delta) \sup_{x\in I_k^\delta} f(x) \\
&\qquad\leq (1+\varepsilon)\frac{e^{\alpha\delta}-1}{\alpha\delta}\frac{W}{\beta} \limsup_{t\to\infty} \frac{1}{t^{\theta+1}} \sum_{k=0}^{t_\delta} \delta \sup_{x\in I_k^\delta} f(x). \qquad (6.4)
\end{aligned}$$

Write $w(\delta) := \sup_{x,y\geq 1, |x-y|\leq\delta} |x^{-\theta}f(x) - y^{-\theta}f(y)|$ for the modulus of continuity of $x^{-\theta}f(x)$ on $[1,\infty)$. By uniform continuity, $w(\delta) \to 0$ as $\delta \to 0$. We now estimate the $\limsup$ in (6.4):

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{\theta+1}} \sum_{k=0}^{t_\delta} \delta \sup_{x\in I_k^\delta} f(x) \\
&\leq \limsup_{t\to\infty} \left(\frac{1}{t^{\theta+1}} \int_0^{t-r+\delta} f(x)\,\mathrm{d}x + \frac{1}{t^{\theta+1}} \int_0^{t-r+\delta} ((x+\delta)^\theta(x^{-\theta}f(x)+w(\delta)) - x^\theta x^{-\theta} f(x))\,\mathrm{d}x \right) \\
&= c + \limsup_{t\to\infty} \left(\frac{C_\theta}{t^{\theta+1}} \int_0^{t-r+\delta} \left((x+\delta)^\theta - x^\theta\right)\mathrm{d}x + \frac{w(\delta)}{t^{\theta+1}} \int_0^{t-r+\delta} (x+\delta)^\theta \mathrm{d}x \right) \\
&\leq c + \frac{w(\delta)}{\theta+1}.
\end{aligned}$$

Using this in (6.4) gives

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{e^{-\alpha t}}{t^{\theta+1}} \sum_{\substack{u\in\mathcal{I}:\\ r\leq S(u)\leq t}} e^{\alpha(t-S(u))} f(t-S(u)) \\
&\qquad\leq (1+\varepsilon)\frac{e^{\alpha\delta}-1}{\alpha\delta}\frac{W}{\beta}\Big(c + \frac{w(\delta)}{\theta+1}\Big)
\end{aligned}$$

almost surely. Letting $\varepsilon, \delta \to 0$ yields the upper bound of (6.2). The lower bound can be obtained analogously.

For the general case, we split $f = f_1 + f_2$ with

$$f_2(x) = \begin{cases} f(1)x^\theta & \text{for } 0 \leq x \leq 1, \\ f(x) & \text{for } x \geq 1 \end{cases}$$

and $f_1 := f - f_2$. Then f_1, f_2 are continuous and the previous part of the proof applies to f_2. Further, the limit in (6.1) is the same if f is replaced by f_2. Define $\varphi_i(t) := e^{\alpha t} f_i(t)\mathbb{1}_{[0,\infty)}(t)$ for $i = 1, 2$ so that $\varphi = \varphi_1 + \varphi_2$. We conclude

$$\frac{e^{-\alpha t}}{t^{\theta+1}} \mathcal{Z}_t^{\varphi_2} \to \frac{cW}{\beta} \quad \text{as } t \to \infty,\ t \in \mathbb{G} \quad \text{a. s.}$$

On the other hand, as φ_1 is bounded and supported on $[0, 1)$, we have $e^{-\alpha t}\mathcal{Z}_t^{\varphi_1}$ converges a. s. to an a. s. finite limit by [39, Theorem 5.4] and [16, Theorem 3.2], which finishes the proof. □

REMARK 6.2. Notice that in the proof of Lemma 6.1, we actually do not use the full power of assumptions (A2) and (A3). Indeed, we only need the assumptions regarding ξ that allow us to apply [36, Corollary 3.1]. What is more, we could replace the application of [36, Corollary 3.1(a)] in the non-lattice case by an application of [39, Theorem 5.4] and the application of [36, Corollary 3.1(b)] in the lattice case by an application of [16, Theorem 3.2] to get the assertion of the lemma under the even weaker assumptions of [39, Theorem 5.4] and [16, Theorem 3.2], respectively.

The following theorem gives the central limit theorem in the case of a centered characteristic χ. Recall that $\mathbb{G} = \mathbb{Z}$ in the lattice case and $\mathbb{G} = \mathbb{R}$ in the non-lattice case and that $\mathcal{F} = \sigma(\pi_u : u \in \mathcal{I})$ where π_u is the projection onto the life space of individual u (in particular, (ξ_u, ζ_u, χ_u) is $\sigma(\pi_u)$-measurable).

THEOREM 6.3. *Suppose that (A1) through (A3) hold. Let χ be a real-valued, centered characteristic, and let $\mathcal{N}$ be a standard normal random variable independent of $\mathcal{F}$.*

(i) *Suppose that (A5) holds for the characteristic χ. Then*

$$e^{-\frac{\alpha}{2}t}\mathcal{Z}_t^{\chi} \stackrel{\text{st}}{\to} \left(\frac{W}{\beta}\int_{\mathbb{G}} \mathbb{E}[\chi^2](x)e^{-\alpha x}\,\ell(\mathrm{d}x)\right)^{1/2}\mathcal{N} \quad \text{as } t \to \infty,\ t \in \mathbb{G}. \tag{6.5}$$

(ii) *Suppose that there are $\theta \geq 0$ and a function f not vanishing identically on $\mathbb{G}$ and satisfying the conditions of Lemma 6.1, $\mathbb{E}[\chi^2(t)] = e^{\alpha t} f(t)\mathbb{1}_{[0,\infty)}(t)$ and*

$$\mathbb{E}\big[\chi^2(t)\mathbb{1}_{\{\chi^2(t) > \varepsilon e^{\alpha t} t^{\theta+1}\}}\big] = o(t^{\theta} e^{\alpha t}) \quad \text{as } t \to \infty \tag{6.6}$$

for every $\varepsilon > 0$. Then

$$\left(e^{\alpha t}\int_{[0,t]} \mathbb{E}[\chi^2(x)]e^{-\alpha x}\,\ell(\mathrm{d}x)\right)^{-1/2}\mathcal{Z}_t^{\chi} \stackrel{\text{st}}{\to} \left(\tfrac{W}{\beta}\right)^{1/2}\mathcal{N} \text{ as } t \to \infty,\ t \in \mathbb{G}. \tag{6.7}$$

PROOF OF THEOREM 6.3. Consider an admissible ordering $v_1, v_2, \ldots$ of $\mathcal{I}$ and put $\mathcal{I}_n := \{v_1, \ldots, v_n\}$ and $\mathcal{G}_n := \sigma(\pi_{v_j} : j = 1, \ldots, n)$. Now we set

$$a_t := \begin{cases} \frac{1}{\beta}\int_{\mathbb{G}} \mathbb{E}[\chi^2](x)e^{-\alpha x}\,\ell(\mathrm{d}x)\cdot e^{\alpha t} & \text{in case (i),} \\ \frac{1}{\beta}\int_0^t \mathbb{E}[\chi^2](x)e^{-\alpha x}\,\ell(\mathrm{d}x)\cdot e^{\alpha t} & \text{in case (ii)} \end{cases}$$

for all $t \in \mathbb{G}$. If $\|\chi\|_{L^2(\mathrm{d}\mathbb{P}\otimes e^{-\alpha t}\ell(\mathrm{d}t))} = \int \mathbb{E}[\chi^2(x)]e^{-\alpha x}\,\ell(\mathrm{d}x) = 0$ in case (i), then the assertion is trivial. Hence, we exclude this case and may thus assume that $a_t > 0$ for all sufficiently large $t \in \mathbb{G}$. The latter is automatic in case (ii) in view of the assumption that f does not vanish identically on $\mathbb{G}$ and is uniformly continuous. For t with $a_t > 0$, we define

$$M_n(t) := a_t^{-1/2} \sum_{u \in \mathcal{I}_n} \chi_u(t - S(u)).$$

Then $(M_n(t), \mathcal{G}_n)_{n \in \mathbb{N}_0}$ is a centered, martingale and bounded in L^2 by Lemma 4.1 We write $M(t)$ for its limit (almost sure and in L^2). Let $(t_n)_{n \in \mathbb{N}}$ be an increasing sequence in $\mathbb{G}$ that diverges to infinity. Then there exists an increasing sequence $(k_n)_{n \in \mathbb{N}}$ such that $\mathbb{E}[(M(t_n) - M_{k_n}(t_n))^2] \leq 2^{-n}$ for every $n \in \mathbb{N}$ and, therefore, $M(t_n) - M_{k_n}(t_n)$ converges to 0 almost surely as $n \to \infty$. In view of Slutsky's theorem [44, Theorem 8.6.1], in order to prove the convergence in distribution of $M(t_n) = M(t_n) - M_{k_n}(t_n) + M_{k_n}(t_n)$ as $n \to \infty$, it suffices to prove convergence in distribution of $M_{k_n}(t_n)$ as $n \to \infty$. For the latter, we rely on the martingale central limit theorem [18, Corollary 3.1 on p. 58]. To apply the cited theorem, it suffices to verify that

$$a_{t_n}^{-1} \sum_{j=1}^{k_n} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j)) \Big| \mathcal{G}_{j-1}\Big] \stackrel{\mathbb{P}}{\to} W \quad \text{as } n \to \infty \tag{6.8}$$

$$a_{t_n}^{-1} \sum_{j=1}^{k_n} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j)) \mathbb{1}_{\{|\chi_{v_j}(t_n - S(v_j))| > \varepsilon a_{t_n}^{1/2}\}} \Big| \mathcal{G}_{j-1}\Big] \stackrel{\mathbb{P}}{\to} 0 \quad \text{as } n \to \infty \tag{6.9}$$

for every $\varepsilon > 0$. To prove (6.8) observe that

$$a_{t_n}^{-1} \mathbb{E}\Big[\sum_{j=k_n+1}^{\infty} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j)) \Big| \mathcal{G}_{j-1}\Big]\Big] = \mathbb{E}\big[(M(t_n) - M_{k_n}(t_n))^2\big] \leq 2^{-n}$$

and hence (6.8) is equivalent to

$$a_{t_n}^{-1} \sum_{j=1}^{\infty} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j)) \Big| \mathcal{G}_{j-1}\Big] = a_{t_n}^{-1} \sum_{u \in \mathcal{I}} \mathbb{E}[\chi^2](t_n - S(u)) \stackrel{\mathbb{P}}{\to} W. \tag{6.10}$$

In case (i), (6.10) is equivalent to

$$e^{-\alpha t_n} \sum_{u \in \mathcal{I}} \mathbb{E}[\chi^2](t_n - S(u)) = e^{-\alpha t_n} \mathcal{Z}_{t_n}^{\mathbb{E}[\chi^2]} \stackrel{\mathbb{P}}{\to} \frac{W}{\beta} \int e^{-\alpha x} \mathbb{E}[\chi^2](x)\, \ell(\mathrm{d}x),$$

which follows from [28, Theorem 6.1] in the non-lattice case. The lattice case is analogous. Lemma 6.1 gives (6.10) in case (ii).

Now we show (6.9). Let $v_2(t,s) := \mathbb{E}[\chi^2(t) \mathbb{1}_{\{|\chi(t)| > s\}}]$ for $t \in \mathbb{R}$ and $s \geq 0$. In case (i), for any $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{n \to \infty} e^{-\alpha t_n} \sum_{j=1}^{k_n} \mathbb{E}\big[\chi_{v_j}^2(t_n - S(v_j)) \mathbb{1}_{\{|\chi_{v_j}(t_n - S(v_j))| > \varepsilon e^{\alpha t_n/2}\}} \,\big|\, \mathcal{G}_{j-1}\big]& \\
\leq \limsup_{n \to \infty} e^{-\alpha t_n} \sum_{j=1}^{\infty} v_2(t_n - S(v_j), \varepsilon e^{\alpha t_n/2})& \\
\leq \liminf_{s \to \infty} \limsup_{n \to \infty} e^{-\alpha t_n} \mathcal{Z}_{t_n}^{v_2(\cdot,s)}& \\
= \liminf_{s \to \infty} \frac{W}{\beta} \int v_2(x,s) e^{-\alpha x}\, \ell(\mathrm{d}x) = 0 \qquad \text{a. s.}&
\end{aligned}$$

by [28, Theorem 6.1] in the non-lattice case and the dominated convergence theorem. The lattice case is analogous.
We turn to case (ii) and fix $\varepsilon > 0$. We infer from (6.6) that for any $\varepsilon, \delta > 0$ there is a $T \geq 0$ such that, for all $t \geq T$,

$$v_2(t, \varepsilon e^{\alpha t/2} t^{\frac{\theta+1}{2}}) = \mathbb{E}\Big[\chi(t)^2 \mathbb{1}_{\{|\chi(t)| > \varepsilon e^{\alpha t/2} t^{\frac{\theta+1}{2}}\}}\Big] \leq \delta e^{\alpha t} t^{\theta}.$$

Therefore, with $\|\mathbb{E}[\chi^2]\|_{[0,T]} := \sup_{x\in[0,T]} \mathbb{E}[\chi^2](x)$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \sum_{j=1}^{k_n} \mathbb{E}\Big[\chi_{v_j}^2(t_n - S(v_j)) \mathbb{1}_{\{|\chi_{v_j}(t_n - S(v_i))| > \varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}\}} \Big| \mathcal{G}_{j-1}\Big] \\
&\leq \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \sum_{u\in\mathcal{I}} v_2(t_n - S(u), \varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}) \\
&= \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \Bigg(\sum_{\substack{u\in\mathcal{I}:\\ S(u)\leq t_n - T}} v_2(t_n - S(u), \varepsilon e^{\alpha t_n/2} t_n^{\frac{\theta+1}{2}}) + \|\mathbb{E}[\chi^2]\|_{[0,T]} \cdot N((t_n - T, t_n]) \Bigg) \\
&\leq \limsup_{n\to\infty} \frac{e^{-\alpha t_n}}{t_n^{\theta+1}} \Bigg(\delta \sum_{\substack{u\in\mathcal{I}:\\ S(u)\leq t_n - T}} e^{\alpha(t_n - S(u))} (t_n - S(u))^{\theta} + \|\mathbb{E}[\chi^2]\|_{[0,T]} \cdot N((t_n - T, t_n]) \Bigg) \\
&\leq \frac{\delta W}{\beta(\theta+1)} \qquad \text{a. s.}
\end{aligned}$$

by Lemma 6.1 with $f(t) = t^{\theta}$, $t \geq 0$ and the fact that $e^{-\alpha t_n} N((t_n - T, t_n])$ converges a. s. by [39, Theorem 5.4] in the non-lattice case and by [16, Theorem 3.2] in the lattice case. Since $\delta > 0$ was arbitrary, we conclude that the limit is zero and, therefore, (6.9) holds true in both cases.

It remains to justify that the convergence is stable and that limiting random variable $\mathcal{N}$ is independent of $\mathcal{F}$. Although, this is not stated explicitly in [18, Theorem 3.2], it follows from the proof of the preceding Lemma 3.1 of [18], cf. Eq. (3.15) there, that is, for any $E \in \mathcal{F}$, we have

$$\mathbb{E}\Big[e^{\mathrm{i}\theta M_{k_n}(t_n)} \mathbb{1}_E\Big] \to \mathbb{E}\Big[e^{-W\frac{\theta^2}{2}} \mathbb{1}_E\Big]$$

for every $\theta \in \mathbb{R}$. The latter is equivalent, by a standard approximation argument, to say that for any $\mathcal{F}$-measurable random variable Y

$$\mathbb{E}\Big[e^{\mathrm{i}\theta M_{k_n}(t_n)} e^{\mathrm{i}\eta Y}\Big] \to \mathbb{E}\Big[e^{-W\frac{\theta^2}{2}} e^{\mathrm{i}\eta Y}\Big] = \mathbb{E}\Big[e^{\mathrm{i}\theta\sqrt{W}\mathcal{N}} e^{\mathrm{i}\eta Y}\Big]$$

for a standard normal variable $\mathcal{N}$ independent of (W, Y). This also implies the stable convergence by [1, Proposition 1].

□

6.2. *Deterministic characteristics.* Let f be a deterministic characteristic, i.e., a càdlàg function $f : \mathbb{R} \to \mathbb{R}$. We investigate the behavior of $\mathcal{Z}_t^f$ as $t \to \infty$ by means of an auxiliary centered random characteristic χ_f defined by

$$\chi_f(t) := f * \xi * V(t) - f * \mu * V(t) = m^f * \xi(t) - m^f * \mu(t), \tag{6.11}$$

where $V(\cdot) = \sum_{n=0}^{\infty} \mu^{*n}(\cdot) = \mathbb{E}[\sum_{u\in\mathcal{I}} \delta_{S(u)}(\cdot)]$ and $*$ denotes Lebesgue-Stieltjes convolution. For instance, for every $t \in \mathbb{R}$,

$$f * V(t) = \int f(t-x)\, V(\mathrm{d}x) = \mathbb{E}\Big[\sum_{u\in\mathcal{I}} f(t-S(u))\Big] = m_t^f$$

if the integrals are well-defined. However, the latter is not guaranteed a priori. The following lemma provides a sufficient condition along with an important connection between $\mathcal{Z}_t^f$ and $\mathcal{Z}_t^{\chi_f}$.

LEMMA 6.4. *Assume that (A1) holds. Let $f : \mathbb{R} \to \mathbb{R}$ be a deterministic càdlàg function such that $t \mapsto f(t)e^{-\alpha t}$ is directly Riemann integrable.*

(a) *The characteristic χ_f given by (6.11) is well-defined and has almost surely càdlàg paths.*
(b) *If (A3) holds and the function $t \mapsto m_t^f e^{-\frac{\alpha}{2}t}(1+t^2)$ is bounded, then the characteristic χ_f satisfies (A5) and (A6).*
(c) *If (A3) holds and the function $t \mapsto m_t^f e^{-\frac{\alpha}{2}t}(1+t^2)$ is bounded, then for any $t \in \mathbb{R}$, $\mathbb{E}[\chi_f(t)] = 0$ and*

$$\mathcal{Z}_t^f - m_t^f = \mathcal{Z}_t^{\chi_f} \qquad a.s. \quad \textit{for all } t \in \mathbb{R}. \tag{6.12}$$

(d) *If f is supported on $[0,\infty)$, i.e., if $f(t) = 0$ for all $t < 0$, then (6.12) also holds.*

PROOF. A function $g : R \to \mathbb{R}$ is directly Riemann integrable if and only if g_+ and g_-, the positive and negative part of g, respectively, are. Hence, if $t \mapsto f(t)e^{-\alpha t}$ is directly Riemann integrable, then so is $t \mapsto |f(t)|e^{-\alpha t}$.

(a) In order to see that χ_f is well-defined, it suffices to check that $|f| * \mu * V(t)$ is finite for all $t \in \mathbb{R}$. Indeed, as $\mu * V \le \delta_0 + \mu * V = V$, putting $g(t) := |f(t)|e^{-\alpha t}$ we obtain

$$|f| * \mu * V(t) \le |f| * V(t) = \sum_{n=0}^{\infty} \mathbb{E}\Big[\sum_{|u|=n} |f|(t-S(u))\Big]$$

$$= e^{\alpha t} \sum_{n=0}^{\infty} \mathbb{E}\Big[\sum_{|u|=n} e^{-\alpha S(u)} g(t-S(u))\Big] = e^{\alpha t} g * \mathsf{U}(t) < \infty, \tag{6.13}$$

where we have used the many-to-one formula (4.2) in the next-to-last step and the direct Riemann integrability of g in combination with (4.4) in the last. To prove that χ_f has càdlàg paths almost surely, it suffices to show that $f^* * \mu * V$ is finite. This is justified by the fact that f has càdlàg paths together with the dominated convergence theorem. Since $f^* * \mu * V \le f^* * V$, we have to check that the latter is finite. Further, $e^{-\alpha t} f^*(t) \le e^{\alpha} g^*(t)$. Therefore, by a calculation analogous to (6.13), it is enough to show that $g^* * \mathsf{U}$ is finite, which, in view of (4.4), is true if g^* is directly Riemann integrable. This however follows from the converse part of Proposition 2.6(a) since g is directly Riemann integrable.

(b) By Proposition 2.6(c), it suffices to show that

$$\int \mathbb{E}\big[(\chi_f^*)^2\big](x) e^{-\alpha x}\, \mathrm{d}x < \infty, \tag{6.14}$$

To this end, note that, since $\chi_f = m^f * \xi - m^f * \mu$, we have, for any $x \in \mathbb{R}$,

$$\mathbb{E}\big[|\chi_f^*(x)|^2\big] \le 2\mathbb{E}\Big[\sup_{|t-x|\le 1} |m^f * \xi(t)|^2\Big] + 2\sup_{|t-x|\le 1} |m^f * \mu(t)|^2 \le 4\mathbb{E}\Big[\sup_{|t-x|\le 1} |m^f * \xi(t)|^2\Big],$$

where we have used Jensen's inequality. For $|t-x|\leq 1$, we obtain

$$\begin{aligned}1+(x-X_j)^2 &= 1+(t-X_j+x-t)^2 \leq 1+2(t-X_j)^2+2(x-t)^2 \leq 3+2(t-X_j)^2\\ &\leq 3(1+(t-X_j)^2)\end{aligned}$$

for $j=1,\ldots,N$, and, therefore, with $C:=3e^{\frac{\alpha}{2}}\sup_{t\in\mathbb{R}} e^{-\frac{\alpha}{2}t}(1+t^2)|m_t^f|$,

$$|m^f*\xi(t)|^2 = \Big|\sum_{j=1}^N m_{t-X_j}^f\Big|^2 \leq C^2 \sum_{1\leq i,j\leq N} \frac{e^{\frac{\alpha}{2}(x-X_i)}}{1+(x-X_i)^2}\frac{e^{\frac{\alpha}{2}(x-X_j)}}{1+(x-X_j)^2}.$$

Thus, since

$$\int \frac{1}{1+(x-X_i)^2}\frac{1}{1+(x-X_j)^2}\,\mathrm{d}x \leq \int \frac{\mathrm{d}x}{1+(x-X_i)^2} = \pi,$$

we conclude

$$\int \mathbb{E}\big[|\chi_f^*(x)|^2\big]e^{-\alpha x}\,\mathrm{d}x \leq 4\pi C^2\mathbb{E}\bigg[\sum_{1\leq i,j\leq N} e^{-\frac{\alpha}{2}X_i}e^{-\frac{\alpha}{2}X_j}\bigg] < \infty$$

from assumption (A3).

(c) By part (a), $|f|*\mu*V(t)$ is finite for all $t\in\mathbb{R}$ and, hence, $\mathbb{E}[\chi_f](t)=\mathbb{E}[f*\xi*V(t)]-f*\mu*V(t)=f*\mu*V(t)-f*\mu*V(t)=0$ for all $t\in\mathbb{R}$. Further, since f satisfies (A4) by assumption and trivially also (A5), Proposition 2.2 implies that $\mathcal{Z}_t^f$ converges in L^1 for every $t\in\mathbb{R}$. The characteristic χ_f on the other hand satisfies (A5) by part (b) and trivially also (A4) because it is centered. Thus, Proposition 2.2 yields that also the series defining $\mathcal{Z}_t^{\chi_f}$ converges unconditionally in L^1 for all $t\in\mathbb{R}$.

In particular, for any $n\in\mathbb{N}$, the infinite series

$$\sum_{0\leq|u|\leq n}\chi_{f,u}(t-S(u))$$

also converges unconditionally in L^1 and so is well-defined and converges to $\mathcal{Z}_t^{\chi_f}$ as $n\to\infty$ in L^1. Moreover, due to the fact that f is deterministic, χ_f is ξ-measurable. For $u\in\mathcal{I}$, we have $\chi_{f,u}(t)=f*\xi_u*V(t)-f*\mu*V(t)$. Using this and $V=\delta_0+\mu*V$, we infer

$$\begin{aligned}\sum_{0\leq|u|\leq n}\chi_{f,u}(t-S(u)) &= \sum_{0\leq|u|\leq n}\big(f*\xi_u*V(t-S(u))-f*\mu*V(t-S(u))\big)\\ &= \sum_{1\leq|u|\leq n+1} f*V(t-S(u)) - \sum_{0\leq|u|\leq n} f*\mu*V(t-S(u))\\ &= \sum_{1\leq|u|\leq n} f(t-S(u)) - f*\mu*V(t) + \sum_{|u|=n+1} f*V(t-S(u))\\ &= \sum_{0\leq|u|\leq n} f(t-S(u)) - f*V(t) + \sum_{|u|=n+1} f*V(t-S(u)).\end{aligned}$$

In the last line, $f*V(t)=m_t^f$. The manipulations in the above chain of equalities are justified by the fact that

$$\mathbb{E}\bigg[\sum_{0\leq|u|\leq n+1}|f*V(t-S(u))|\bigg] < \infty.$$

The finiteness of the above expectation follows from

$$\mathbb{E}\Big[\sum_{|u|=k} |f * V(t-S(u))|\Big] \leq \mathbb{E}\Big[\sum_{|u|=k} |f| * V(t-S(u))\Big] = \mathbb{E}\Big[\sum_{|u|\geq k} |f|(t-S(u))\Big]$$
$$\leq \mathbb{E}\Big[\sum_{u\in\mathcal{I}} |f|(t-S(u))\Big] = m_t^{|f|} < \infty.$$

The dominated convergence theorem yields

$$\sum_{|u|=n+1} f * V(t-S(u)) \to 0 \quad \text{and} \sum_{0\leq|u|\leq n} f(t-S(u)) \to \mathcal{Z}_t^f$$

as $n\to\infty$ in L^1.

(d) The last calculation carries over if $f(t)=0$ for all $t<0$. Indeed, the latter condition implies $\chi_f(t)=0$ for all $t<0$ and hence with probability one, all sums have only finitely many non-vanishing terms (since only finitely many individuals are born before any fixed time almost surely by [25, Theorem 6.2.3]). □

Lemma 6.4 has the following corollary.

COROLLARY 6.5. *If the assumptions of Lemma 6.4(b) are satisfied, then, for every $t\in\mathbb{R}$,*

$$\mathrm{Var}[\mathcal{Z}_t^f] = m_t^{\chi_f^2} < \infty.$$

PROOF. By Lemma 6.4, we have $\mathcal{Z}_t^f = \mathcal{Z}_t^{\chi_f} + m_t^f$ where $\mathcal{Z}_t^{\chi_f}$ is centered and

$$e^{-\alpha t}\mathrm{Var}[\chi_f](t) = e^{-\alpha t}\mathbb{E}[\chi_f^2](t)$$

is directly Riemann integrable. Using $\mathrm{Var}[\mathcal{Z}_t^f] = \mathrm{Var}[\mathcal{Z}_t^{\chi_f} + m_t^f] = \mathrm{Var}[\mathcal{Z}_t^{\chi_f}] = \mathbb{E}[(\mathcal{Z}_t^{\chi_f})^2]$ we infer with the help of (4.10) that

$$\mathbb{E}[(\mathcal{Z}_t^{\chi_f})^2] = \mathbb{E}[\mathcal{Z}_t^{\chi_f^2}] \quad \text{for all } t\in\mathbb{R}.$$

□

6.3. *Slowly growing mean process with signed characteristics.* We now treat the case where m_t^φ grows relatively slowly as $|t|\to\infty$. Later, we shall reduce the general case to this one.

THEOREM 6.6. *Suppose that (A1) through (A3) hold, the random characteristic φ satisfies (A4), (A5) and that the function $t\mapsto e^{-\frac{\alpha}{2}t}(1+t^2)m_t^\varphi$ is bounded. If $\mathbb{G}=\mathbb{R}$, assume in addition that (A6) holds. Then, with $\mathcal{N}$ a standard normal random variable independent of $\mathcal{F}$,*

$$e^{-\frac{\alpha}{2}t}\mathcal{Z}_t^\varphi \stackrel{\mathrm{st}}{\to} \sigma_\varphi\sqrt{\frac{W}{\beta}}\mathcal{N},$$

with

$$\sigma_\varphi^2 := \int \mathrm{Var}[\varphi(x) + \xi * m^\varphi(x)]e^{-\alpha x}\,\ell(\mathrm{d}x).$$

Moreover, $\sigma_\varphi^2 = 0$ if and only if the mean m^φ is a version of the process $\mathcal{Z}^\varphi$.

PROOF. Clearly, $m^{\mathbb{E}[\varphi]} = m^{\varphi}$ where $\mathbb{E}[\varphi]$ denotes the function $t \mapsto \mathbb{E}[\varphi(t)]$. In view of Lemma 6.4, we can write

$$\begin{aligned} \mathcal{Z}_t^{\varphi} &= \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]} + \mathcal{Z}_t^{\mathbb{E}[\varphi]} = \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]} + \mathcal{Z}_t^{\chi_{\mathbb{E}[\varphi]}} + m_t^{\varphi} = \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]+\chi_{\mathbb{E}[\varphi]}} + m_t^{\varphi} \\ &= \mathcal{Z}_t^{\varphi+m^{\varphi}*\xi-\mathbb{E}[\varphi+m^{\varphi}*\xi]} + m_t^{\varphi} \quad \text{a. s.} \end{aligned} \tag{6.15}$$

By assumption, $e^{-\alpha t/2} m_t^{\varphi} \to 0$ as $t \to \infty$. Hence, it suffices to show that $e^{-\frac{\alpha}{2}t} \mathcal{Z}_t^{\varphi-\mathbb{E}[\varphi]+\chi_{\mathbb{E}[\varphi]}}$ converges in distribution to the claimed distribution. Since the characteristic

$$t \mapsto \varphi(t) - \mathbb{E}[\varphi(t)] + \chi_{\mathbb{E}[\varphi]}(t) \tag{6.16}$$

is centered, it is reasonable to apply Theorem 6.3(i). To this end, we need to check that (A5) holds for the characteristic in (6.16), i.e., that the function

$$\begin{aligned} t &\mapsto e^{-\alpha t}\operatorname{Var}[\varphi(t) - \mathbb{E}[\varphi(t)] + \chi_{\mathbb{E}[\varphi]}(t)] \\ &= e^{-\alpha t}\operatorname{Var}[\varphi(t) + \chi_{\mathbb{E}[\varphi]}(t)] \text{ is directly Riemann integrable.} \end{aligned} \tag{6.17}$$

From Lemma 6.4 we conclude that $\chi_{\mathbb{E}[\varphi]}(t)$ satisfies (A5) and (A6). This is also true for φ. Hence, (6.17) holds by Remark 2.5.

Finally, if $\sigma_{\varphi} = 0$, then $\varphi(x) + m^{\varphi} * \xi(x)$ is equal to its expectation a. s. for ℓ-almost every $x \in \mathbb{G}$, i.e., $\varphi(x) + m^{\varphi} * \xi(x) - (\mathbb{E}[\varphi](x) + m^{\varphi} * \mu(x)) = 0$ a. s. for ℓ-almost every $x \in \mathbb{G}$. On the other hand, Lemma 6.4(a) implies that $\chi_{\mathbb{E}[\varphi]} = m^{\varphi} * \xi - m^{\varphi} * \mu$ has càdlàg paths a. s. and by Remark 2.5 the same holds true for the characteristic $\mathbb{E}[\varphi]$, which in turn implies that, except on a $\mathbb{P}$-null set, $\varphi(x) + m^{\varphi} * \xi(x) - (\mathbb{E}[\varphi](x) + m^{\varphi} * \mu(x)) = 0$ for every $x \in \mathbb{G}$. Consequently, by (6.15), for every fixed $t \in \mathbb{G}$,

$$\mathcal{Z}_t^{\varphi} = \mathcal{Z}_t^{\varphi+m^{\varphi}*\xi-\mathbb{E}[\varphi+m^{\varphi}*\xi]} + m_t^{\varphi} = m_t^{\varphi} \quad \text{a. s.,}$$

i.e., for every fixed $t \in \mathbb{G}$, $\mathcal{Z}_t^{\varphi}$ is a. s. deterministic. □

6.4. *Proof of Theorem 2.15.* In the proof of Theorem 2.15, we use the following fact.

LEMMA 6.7. *Let $\eta_1, \dots, \eta_m$ be distinct real numbers. In the lattice case, we additionally assume that $\eta_i \in (-\pi, \pi]$. Consider a collection of centered, square-integrable random variables $(Y_{j,l})_{1 \le j \le m, 0 \le l \le n}$. Then for*

$$\chi(t) := \mathbb{1}_{[0,\infty)}(t) e^{\frac{\alpha}{2}t} \sum_{l=0}^{n} \sum_{j=1}^{m} t^l e^{\mathrm{i}\eta_j t} Y_{j,l}$$

it holds that

$$\frac{1}{t^{2n+1}} \int_{[0,t]} \operatorname{Var}\left[\chi(x)\right] e^{-\alpha x}\,\ell(\mathrm{d}x) \to \frac{1}{2n+1} \sum_{j=1}^{m} \operatorname{Var}[Y_{j,n}] \quad \text{as } t \to \infty,\ t \in \mathbb{G} \tag{6.18}$$

and, for any $\varepsilon > 0$,

$$\mathbb{E}\big[|\chi(t)|^2 \mathbb{1}_{\{|\chi(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\big] = o(t^{2n} e^{\alpha t}) \quad \text{as } t \to \infty,\ t \in \mathbb{G}. \tag{6.19}$$

In other words, $\chi(t)$ fulfills the assumption of Theorem 6.3(ii) with $\theta = 2n$.

PROOF. Expanding the variance gives

$$\begin{aligned} \operatorname{Var}\left[\sum_{l=0}^{n} \sum_{j=1}^{m} x^l e^{\mathrm{i}\eta_j x} Y_{j,l}\right] &= x^{2n} \sum_{j=1}^{m} \operatorname{Var}[Y_{j,n}] \\ &\quad + x^{2n} \sum_{j \ne k} e^{\mathrm{i}(\eta_j - \eta_k)x} \operatorname{Cov}[Y_{j,n}, Y_{k,n}] + O(x^{2n-1}) \end{aligned} \tag{6.20}$$

as $x \to \infty$. Further notice that, for $\eta \in \mathbb{R}$, with $|\eta| < 2\pi$ in the lattice case,

$$\frac{1}{t^{2n+1}} \int_{[0,t]} x^{2n} e^{\mathrm{i}\eta x}\, \ell(\mathrm{d}x) \to \begin{cases} \frac{1}{2n+1}, & \text{if } \eta = 0, \\ 0, & \text{if } \eta \neq 0. \end{cases} \tag{6.21}$$

This follows from the fundamental theorem of calculus in the non-lattice case and integration by parts if $\eta \neq 0$, whereas in the lattice case, it follows from Faulhaber's formula if $\eta = 0$ and from summation by parts if $\eta \neq 0$. Relation (6.18) now follows from (6.20) and (6.21).

In order to prove that (6.19) holds, for $t \geq 0$ we set $\chi_j(t) := e^{\frac{\alpha}{2}t} \sum_{l=0}^{n} t^l e^{\mathrm{i}\eta_j t} Y_{j,l}$. Since for any complex numbers $c_1, \ldots, c_m$ and $y > 0$

$$|c_1 + \cdots + c_m|^2 \mathbb{1}_{\{|c_1+\cdots+c_m|>y\}} \leq m^2\big(|c_1|^2 \mathbb{1}_{\{|c_1|>y/m\}} + \cdots + |c_m|^2 \mathbb{1}_{\{|c_m|>y/m\}}\big),$$

it suffices to prove (6.19) for χ_j instead of χ. By Markov's inequality, we have

$$\mathbb{P}(|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}) \leq \frac{(n+1)^2}{\varepsilon t^{2n+1}} \sum_{l=0}^{n} t^{2l} \mathbb{E}[|Y_{j,l}|^2] \to 0 \quad \text{as } t \to \infty$$

as the sum is of the order t^{2n} as $t \to \infty$. Consequently,

$$\begin{aligned} &\mathbb{E}\big[|\chi_j(t)|^2 \mathbb{1}_{\{|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\big] \\ &\leq 2t^{2n} e^{\alpha t} \mathbb{E}\big[|Y_{j,n}|^2 \mathbb{1}_{\{|\chi_j(t)|^2 > \varepsilon t^{2n+1} e^{\alpha t}\}}\big] + O(t^{2n-1} e^{\alpha t}) = o(t^{2n} e^{\alpha t}) \end{aligned}$$

as $t \to \infty$. This proves (6.19). □

We now turn to the proof of Theorem 2.15.

PROOF OF THEOREM 2.15. Suppose that φ is a random characteristic satisfying

$$m_t^{\varphi} = \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} t^l e^{\lambda t} + r(t), \quad t \in \mathbb{G} \tag{2.20}$$

for some constants $a_{\lambda,l} \in \mathbb{R}$ and a function r such that $|r(t)| \leq C e^{\frac{\alpha}{2}t}/(1+t^2)$ for a finite constant $C > 0$. For any $\lambda \in \Lambda_{\geq}$, we put

$$\vec{a}_\lambda := \sum_{l=1}^{k(\lambda)} a_{\lambda,l-1} \mathsf{e}_l$$

and consider the following characteristic

$$\psi_\Lambda(t) = \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} (\phi_\lambda(t) + \chi_\lambda(t)) \mathsf{e}_1$$

for ϕ_λ and χ_λ defined in (5.5) and (5.6), respectively. Then, by Lemma 5.5, for $t \geq 0$,

$$\mathcal{Z}_t^{\psi_\Lambda} = \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} \mathcal{Z}_t^{\phi_\lambda + \chi_\lambda} \mathsf{e}_1 = \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t, k(\lambda)) W(\lambda, k(\lambda)) \mathsf{e}_1 = H_\Lambda(t), \tag{6.22}$$

where the definition of H_Λ should be recalled from (2.21). Further, by Lemma 5.2

$$\begin{aligned} m_t^{\psi_\Lambda} &= \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} \mathbb{E}[\mathcal{Z}_t^{\phi_\lambda + \chi_\lambda}] \mathsf{e}_1 = \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} \mathbb{E}[\mathcal{Z}_t^{\phi_\lambda}] \mathsf{e}_1 \\ &= \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda \in \Lambda} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t, k(\lambda)) \mathsf{e}_1 = \mathbb{1}_{[0,\infty)}(t) \mathbb{E}[H_\Lambda(t)]. \end{aligned} \tag{6.23}$$

Similarly, putting

$$\psi_{\partial\Lambda}(t) = \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \mathbb{E}[\phi_\lambda(t)]\mathsf{e}_1,$$

again by Lemma 5.2, we obtain, for any $t \in \mathbb{R}$,

$$\begin{aligned} m_t^{\psi_{\partial\Lambda}} = \mathbb{E}[\mathcal{Z}_t^{\psi_{\partial\Lambda}}] &= \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \mathbb{E}[\mathcal{Z}_t^{\phi_\lambda}]\mathsf{e}_1 \\ &= \mathbb{1}_{[0,\infty)}(t) \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t, k(\lambda))\mathsf{e}_1 = \mathbb{1}_{[0,\infty)}(t) H_{\partial\Lambda}(t), \end{aligned} \tag{6.24}$$

where the definition of $H_{\partial\Lambda}$ should be recalled from (2.22). In view of Remark 2.16, we have $\overline{\vec{a}_\lambda} = \vec{a}_{\overline{\lambda}}$, $\overline{\phi_\lambda} = \phi_{\overline{\lambda}}$ and $\overline{\chi_\lambda} = \chi_{\overline{\lambda}}$. We thus conclude that both characteristics ψ_Λ and $\psi_{\partial\Lambda}$ are in fact real-valued.

Now write

$$\mathcal{Z}_t^{\varphi} = \mathcal{Z}_t^{\psi_\Lambda} + \mathcal{Z}_t^{\psi_{\partial\Lambda}} + \mathcal{Z}_t^{\varrho} = H_\Lambda(t) + \mathcal{Z}_t^{\psi_{\partial\Lambda}} + \mathcal{Z}_t^{\varrho}, \tag{6.25}$$

where $\varrho := \varphi - \psi_\Lambda - \psi_{\partial\Lambda}$. Next, since $\psi_{\partial\Lambda}$ is deterministic, we may consider the associated centered characteristic $\chi_{\psi_{\partial\Lambda}}$ defined in (6.11), namely,

$$\begin{aligned} \chi_{\psi_{\partial\Lambda}}(t) &= \xi * m_t^{\psi_{\partial\Lambda}} - \mu * m_t^{\psi_{\partial\Lambda}} \\ &= \sum_{\lambda\in\partial\Lambda} \sum_{j=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1 \mathbb{1}_{[0,\infty)}(t - X_j) - \mu * m_t^{\psi_{\partial\Lambda}}. \end{aligned}$$

If we set now

$$\begin{aligned} \psi_\lambda(t) &:= \mathbb{1}_{[0,\infty)}(t) \sum_{j=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1 \\ &\qquad - \mathbb{1}_{[0,\infty)}(t)\mathbb{E}\Big[\sum_{j=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1\Big], \\ \chi(t) &:= \sum_{\lambda\in\partial\Lambda} \psi_\lambda(t) \end{aligned}$$

and

$$\begin{aligned} \phi_{\partial\Lambda}(t) &:= \sum_{\lambda\in\partial\Lambda} \sum_{j=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1 \mathbb{1}_{[0,X_j)}(t) \\ &\qquad - \sum_{\lambda\in\partial\Lambda} \mathbb{E}\Big[\sum_{j=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_j, k(\lambda))\mathsf{e}_1 \mathbb{1}_{[0,X_j)}(t)\Big] \\ &= \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \phi_\lambda(t)\mathsf{e}_1 - \sum_{\lambda\in\partial\Lambda} \mathbb{E}\big[\vec{a}_\lambda^{\mathsf{T}} \phi_\lambda(t)\mathsf{e}_1\big]. \end{aligned}$$

we get the following decomposition

$$\chi_{\psi_{\partial\Lambda}}(t) = \chi(t) - \phi_{\partial\Lambda}(t). \tag{6.26}$$

The fact that all the expectations above are finite and thus the characteristics are well-defined follows from (A2) and (A7). Note also that, for every $\lambda \in \partial\Lambda$,

$$\begin{aligned}
\sum_{i=1}^{N} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t - X_i, k(\lambda))\mathsf{e}_1 &= \sum_{i=1}^{N} e^{\lambda(t-X_i)} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l}(t-X_i)^l \\
&= \sum_{i=1}^{N} e^{\lambda(t-X_i)} \sum_{l=0}^{k(\lambda)-1} a_{\lambda,l} \sum_{j=0}^{l} \binom{l}{j} t^j (-X_i)^{l-j} \\
&= \sum_{j=0}^{k(\lambda)-1} \sum_{l=j}^{k(\lambda)-1} \sum_{i=1}^{N} e^{-\lambda X_i} a_{\lambda,l} \binom{l}{j} (-X_i)^{l-j} t^j e^{\lambda t} \\
&= \sum_{j=0}^{k(\lambda)-1} R_{\lambda,j} t^j e^{\lambda t}
\end{aligned}$$

by (2.23). With this at hand, we infer

$$\psi_\lambda(t) = \mathbb{1}_{[0,\infty)}(t) \sum_{l=0}^{k(\lambda)-1} (R_{\lambda,l} - \mathbb{E}[R_{\lambda,l}]) t^l e^{\lambda t}, \quad t \in \mathbb{R}. \tag{6.27}$$

Using Lemma 6.4(d) with $f = \psi_{\partial\Lambda}$ (note here that such f fulfills the assumptions as for any $\lambda \in \partial\Lambda$ the characteristic ϕ_λ vanishes on $(-\infty, 0)$ and satisfies (A4) by Lemma 5.1), we get that $\mathcal{Z}_t^{\psi_{\partial\Lambda}} - m_t^{\psi_{\partial\Lambda}} = \mathcal{Z}_t^{\chi_{\psi_{\partial\Lambda}}}$ and, hence, $\mathcal{Z}_t^{\psi_{\partial\Lambda}} = \mathcal{Z}_t^{\chi_{\psi_{\partial\Lambda}}} + m_t^{\psi_{\partial\Lambda}} = \mathcal{Z}_t^{\chi_{\psi_{\partial\Lambda}}} + H_{\partial\Lambda}(t)$ for $t \geq 0$. Therefore, from (6.25) and (6.26) we obtain the following decomposition,

$$\mathcal{Z}_t^{\varphi} = H_\Lambda(t) + H_{\partial\Lambda}(t) + \mathcal{Z}_t^{\varrho - \phi_{\partial\Lambda}} + \mathcal{Z}_t^{\chi}, \quad t \geq 0. \tag{6.28}$$

It suffices to prove the limit theorem for $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$ and $\mathcal{Z}_t^{\chi}$. To this end, we invoke Theorem 6.6 for the first process and Theorem 6.3(ii) for the second as both characteristics are real-valued as a consequence of Remark 2.16. We begin with $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$ and first notice that, in view of Lemma 5.1 and Remark 2.5, the characteristic $\varrho - \phi_{\partial\Lambda}$ has càdlàg paths and satisfies (A4), (A5) and (A6). Moreover, using the fact that $\phi_{\partial\Lambda}$ is centered in combination with (6.23), (6.24) and (2.20) we infer

$$\begin{aligned}
m_t^{\varrho-\phi_{\partial\Lambda}} &= m_t^{\varrho} = m_t^{\varphi - \psi_\Lambda - \psi_{\partial\Lambda}} = m_t^{\varphi} - m_t^{\psi_\Lambda} - m_t^{\psi_{\partial\Lambda}} \\
&= m_t^{\varphi} - \mathbb{1}_{[0,\infty)}(t)\big(\mathbb{E}\big[H_\Lambda(t)\big] + H_{\partial\Lambda}(t)\big) = r(t).
\end{aligned} \tag{6.29}$$

We may thus apply Theorem 6.6 to conclude that

$$e^{-\frac{\alpha}{2}t} \mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}} \xrightarrow{\text{st}} \sigma \sqrt{\frac{W}{\beta}} \mathcal{N}, \tag{6.30}$$

where

$$\sigma^2 := \int v(x) e^{-\alpha x} \, \ell(\mathrm{d}x), \tag{6.31}$$

and $v(t) := \mathrm{Var}\big[\varrho(t) - \phi_{\partial\Lambda}(t) + r * \xi(t)\big]$. The function v can be further simplified in the following way

$$\begin{aligned}
v(t) &= \mathrm{Var}\big[\varphi(t) - \psi_\Lambda(t) - \psi_{\partial\Lambda}(t) - \phi_{\partial\Lambda}(t) + r * \xi(t)\big] \\
&= \mathrm{Var}\Big[\varphi(t) - \sum_{\lambda\in\Lambda} \vec{a}_\lambda^{\mathsf{T}} (\phi_\lambda(t) + \chi_\lambda(t))\mathsf{e}_1 - \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \phi_\lambda(t)\mathsf{e}_1 + r * \xi(t)\Big]
\end{aligned}$$

$$= \operatorname{Var}\Bigg[\varphi(t) + \sum_{j=1}^{N}\bigg(-\sum_{\lambda\in\Lambda}\vec{a}_\lambda^{\mathsf{T}}\mathbb{1}_{(-\infty,X_j)}(t)\exp(\lambda,t-X_j,k(\lambda))\mathsf{e}_1$$
$$-\sum_{\lambda\in\partial\Lambda}\vec{a}_\lambda\mathbb{1}_{[0,X_j)}(t)\exp(\lambda,t-X_j,k(\lambda))\mathsf{e}_1 + r(t-X_j)\bigg)\Bigg]$$
$$= \operatorname{Var}\Bigg[\varphi(t) + \sum_{j=1}^{N}\bigg(-\sum_{\lambda\in\Lambda}\vec{a}_\lambda^{\mathsf{T}}\exp(\lambda,t-X_j,k(\lambda))\mathsf{e}_1$$
$$-\mathbb{1}_{[0,\infty)}(t)\sum_{\lambda\in\partial\Lambda}\vec{a}_\lambda^{\mathsf{T}}\exp(\lambda,t-X_j,k(\lambda))\mathsf{e}_1 + m^{\varphi}(t-X_j)\bigg)\Bigg].$$

This proves the theorem under the assumption that $\rho_l = 0$ for all $l \geq 0$. Indeed, in this case $\mathcal{Z}_t^{\chi} = 0$ and $\sum_{j=1}^{N}\vec{a}_\lambda^{\mathsf{T}}\exp(\lambda,t-X_j,k(\lambda))\mathsf{e}_1$ is a. s. constant for any $\lambda\in\partial\Lambda, t\in\mathbb{G}$ and (2.25) follows.

Now, combining (6.28) and (6.30), we arrive at (2.24) if $\sigma^2 > 0$. However, if $\sigma^2 = 0$, then for all $t\in\mathbb{G}$, $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$ equals its expectation, which is $r(t)$ a. s., as shown by (6.29). This establishes (i).

It remains to prove the theorem in the case where $\rho_l > 0$ for some $l\geq 0$. First notice that by (6.30), the already established central limit theorem for $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}}$, we have $\mathcal{Z}_t^{\varrho-\phi_{\partial\Lambda}} = o(t^{\frac{1}{2}}e^{\frac{\alpha}{2}t})$ as $t\to\infty$ in probability. Let $n\in\mathbb{N}_0$ be maximal with $\rho_n > 0$. We show that the characteristic χ satisfies the assumptions of Theorem 6.3(ii) with $\theta = 2n$. Observe that, for any $\lambda\in\partial\Lambda$, $l\leq k(\lambda)-1$ and some constant $C_{\lambda,\vec{a}_\lambda}$ depending on λ, l and $\vec{a}_\lambda$,

$$|R_{\lambda,l}| \leq C_{\lambda,\vec{a}_\lambda}\sum_{j=1}^{N}(1+X_j^{k(\lambda)-1})e^{-\frac{\alpha}{2}X_j}.$$

In view of assumption (A3) the random variable $R_{\lambda,l}$ is square integrable. Setting $R_{\lambda,l} := 0$ for $l\geq k(\lambda)$, we write

$$\chi(t) = \mathbb{1}_{[0,\infty)}(t)e^{\frac{\alpha}{2}t}\cdot\sum_{\lambda\in\partial\Lambda}\sum_{l=0}^{n}(R_{\lambda,l}-\mathbb{E}[R_{\lambda,l}])t^l e^{\mathrm{i}\operatorname{Im}(\lambda)t}.$$

An application on Lemma 6.7 gives

$$\frac{1}{t^{2n+1}}\int_0^t\mathbb{E}[\chi^2(x)]e^{-\alpha x}\,\ell(\mathrm{d}x) \to \frac{\sum_{\lambda\in\partial\Lambda}\operatorname{Var}[R_{\lambda,n}]}{2n+1} = \frac{\rho_n^2}{2n+1}$$

and

$$\mathbb{E}\big[|\chi(t)|^2\mathbb{1}_{\{|\chi(t)|^2>\varepsilon t^{2n+1}e^{\alpha t}\}}\big] = o(t^{2n}e^{\alpha t}) \quad \text{as } t\to\infty,\ t\in\mathbb{G}.$$

Finally, by Theorem 6.3(ii),

$$\Big(\frac{\rho_n^2 t^{2n+1}}{2n+1}e^{\alpha t}\Big)^{-\frac{1}{2}}\mathcal{Z}_t^{\chi} \stackrel{\mathrm{st}}{\Rightarrow} \sqrt{\tfrac{W}{\beta}}\mathcal{N},$$

which finishes the proof. □

REMARK 6.8. The proofs of Theorems 2.15 and 6.6 reveal that for any characteristic φ satisfying the assumptions (A4) through (A6), there exists a decomposition $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where each term also satisfies (A4) through (A6). Furthermore, for the corresponding Crump-Mode-Jagers processes, it is established that $\mathcal{Z}_t^{\varphi_1} = H(t)$ for $t\geq 0$, $\mathcal{Z}_t^{\varphi_2}$ is centered, and $\mathcal{Z}_t^{\varphi_3}$ is a deterministic function equal to $r(t)$. In particular, for the characteristic $\tilde{\varphi} := \varphi_1 + \varphi_3$, one obtains that for $t \geq 0$, $\mathcal{Z}_t^{\tilde{\varphi}} = H(t) + r(t)$ almost surely, indicating the lack of Gaussian fluctuations as t goes to infinity.

PROOF OF COROLLARY 2.20. First observe that linear combinations as well as the translations $\varphi(\cdot) \mapsto \varphi(\cdot - s)$ preserve the conditions (A4), (A5) and (A6). Moreover, for the characteristic $\psi(t) := \varphi(t-s)$ the mean function m_t^{ψ} has expansion (2.20) with coefficients given by vectors $\big(\exp(\lambda, -s, k(\lambda))^{\mathsf{T}} \vec{a}_\lambda\big)$. According to the Cramér–Wold device the convergence in distribution of

$$t^{-\frac{d}{2}} e^{-\frac{\alpha}{2}t} \big(\mathcal{Z}_{t-s_1}^{\varphi} - H(t-s_1), \ldots, \mathcal{Z}_{t-s_n}^{\varphi} - H(t-s_n)\big)$$

is equivalent to the convergence in distribution of

$$t^{-\frac{d}{2}} e^{-\frac{\alpha}{2}t} \sum_{j=1}^{n} c_j \big(\mathcal{Z}_{t-s_j}^{\varphi} - H(t-s_j)\big),$$

for all choices $c_1, \ldots, c_n \in \mathbb{R}$, and the latter convergence follows from Theorem 2.15. The covariance can be obtained by the polarization identity applied to the variance and the fact that $m_t^{\psi} = m_{t-s}^{\varphi}$, $h^{\psi}(t) = h^{\varphi}(t-s)$. □

7. Asymptotic expansion of the mean. In this section we are concerned with the asymptotic expansion of the mean $m_t^{\varphi} = \mathbb{E}[\mathcal{Z}_t^{\varphi}]$ of a supercritical general branching process $(\mathcal{Z}_t^{\varphi})_{t \geq 0}$ as $t \to \infty$. Throughout the section, we assume that (A1) and (A2) hold. We fix some notation throughout the section. By θ we denote a parameter from $(0, \frac{\alpha}{2})$ such that

$$\mathcal{L}\mu(\theta) < \infty \text{ and } \mathcal{L}\mu(z) \neq 1 \quad \text{whenever } \theta \leq \mathrm{Re}(z) < \tfrac{\alpha}{2}. \tag{7.1}$$

Such a θ exists in all particular cases considered in this section and may sometimes be enlarged in order to ensure the validity of additional conditions. We also fix $\gamma > \frac{\alpha}{2}$ such that $\gamma < \min\{\mathrm{Re}(\lambda) : \lambda \in \Lambda\}$.

It's worth noting that the mean m_t^{φ} depends on the underlying point process ξ only through its intensity measure μ. Therefore, when analyzing m_t^{φ}, without loss of generality, we can assume that, additionally to (A1) and (A2), the condition (A3) holds true. Otherwise, we can always replace ξ with the Poisson point process whose intensity measure is μ.

In the non-lattice case we work with the corresponding bilateral Laplace transforms whereas in the lattice case, we use generating functions.

7.1. *The lattice case.* In the present subsection, we assume that μ is concentrated on the lattice $\mathbb{Z}$ (and not on a smaller lattice). We set

$$\mathcal{G}\mu(z) := \sum_{k=0}^{\infty} \mu(\{k\}) z^k = \int z^x \, \mu(\mathrm{d}x) \tag{7.2}$$

for all $z \in \mathbb{C}$ for which the series is absolutely convergent. In particular, $\mathcal{L}\mu(z) = \mathcal{G}\mu(e^{-z})$. Note that, due to assumption (A2), $\mathcal{G}\mu(e^{-\vartheta}) < \infty$ and hence the power series (7.2) defines a holomorphic function on $\{|z| < e^{-\vartheta}\}$. Further, by slightly increasing the value of ϑ if necessary, we may assume without loss of generality that there are only finitely many solutions of the equation $\mathcal{G}\mu(z) = 1$ in the disc $\{|z| < e^{-\vartheta}\}$.

LEMMA 7.1. *Assume that (A1) and (A2) hold. Let $\theta \in (\vartheta, \frac{\alpha}{2})$ be such that there are no solutions to $\mathcal{G}\mu(z) = 1$ in $\{z : e^{-\alpha/2} < |z| \leq e^{-\theta}\}$. Then there are constants $b_{\lambda,l}$, $\lambda \in \Lambda_{\geq}$, $l = 0, \dots, k(\lambda) - 1$ such that, for any characteristic φ with*

$$\sum_{n \in \mathbb{Z}} |\mathbb{E}[\varphi(n)]| (e^{-\theta n} + e^{-\alpha n}) < \infty, \tag{7.3}$$

it holds that, for $t \in \mathbb{Z}$,

$$m_t^{\varphi} = \begin{cases} \sum_{\lambda \in \Lambda_{\geq}} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{n \in \mathbb{Z}} \mathbb{E}[\varphi(n)] (t-n)^l e^{\lambda(t-n)} + O(e^{\theta t}) & \text{as } t \to \infty \\ O(e^{\gamma t}) & \text{as } t \to -\infty. \end{cases} \tag{7.4}$$

REMARK 7.2. Defining $\vec{b}_\lambda := (b_{\lambda,l-1})_{l=1,\dots,k(\lambda)}$, we may write (7.4) more compactly in the form

$$m_t^{\varphi} = \sum_{\lambda \in \Lambda_{\geq}} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \sum_{n \in \mathbb{Z}} \mathbb{E}[\varphi(n)] \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, t-n, k(\lambda)) \mathsf{e}_1 + O(e^{\theta t} \wedge e^{\gamma t})$$

as $t \to \pm\infty$, $t \in \mathbb{Z}$.

PROOF OF LEMMA 7.1. For $r > 0$, let $B_r = \{|z| < r\}$ and $\partial B_r = \{|z| = r\}$. Now fix $r < e^{-\alpha}$. As $\mathcal{G}(\mu^{*l}) = (\mathcal{G}\mu)^l$, for any $l \in \mathbb{N}$ and since $\mathcal{G}\mu$ is holomorphic on $B_{e^{-\vartheta}}$, we infer from Cauchy's integral formula that

$$\mu^{*l}(\{n\}) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(\mathcal{G}\mu)^l(z)}{z^{n+1}} \, \mathrm{d}z.$$

In particular,

$$\mathbb{E}[N(\{n\})] = \sum_{l=0}^{\infty} \mu^{*l}(\{n\}) = \sum_{l=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(\mathcal{G}\mu)^l(z)}{z^{n+1}} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{\mathrm{d}z}{(1 - \mathcal{G}\mu(z)) z^{n+1}}$$

where the last equality follows by Fubini's theorem. For $\lambda \in \Lambda_{\geq}$, let

$$\sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z - e^{-\lambda})^j$$

be the principle part of the Laurent expansion of the meromorphic function $(1 - \mathcal{G}\mu(z))^{-1}$ around $e^{-\lambda}$. Then the function

$$H(z) := \frac{1}{1 - \mathcal{G}\mu(z)} - \sum_{\lambda \in \Lambda_{\geq}} \sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z - e^{-\lambda})^j$$

is holomorphic on $B_{e^{-\theta}}$. On the other hand, for any $d \in \mathbb{N}_0$,

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(z - e^{-\lambda})^{-d}}{z^{n+1}} \, \mathrm{d}z = (-e^{\lambda})^d e^{\lambda n} \binom{n+d-1}{d-1}$$

by the residue theorem. Therefore,

$$G(z) := \sum_{\lambda \in \Lambda_{\geq}} \sum_{j=-k(\lambda)}^{-1} b_j(\lambda)(z - e^{-\lambda})^j$$

satisfies

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z)}{z^{n+1}}\,\mathrm{d}z = \sum_{\lambda\in\Lambda_\geq} p_\lambda(n) e^{\lambda n}$$

where p_λ, for $\lambda \in \Lambda_\geq$, is a polynomial with complex coefficients of degree $k(\lambda)-1$. From the analyticity of H, we infer

$$\left| \int_{\partial B_{e^{-\theta}}} \frac{H(z)}{z^{n+1}}\,\mathrm{d}z \right| = O(e^{\theta n}) \quad \text{as } n \to \infty,$$

which in turn gives

$$\begin{aligned}\mathbb{E}[N(\{n\})] &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z)+H(z)}{z^{n+1}}\,\mathrm{d}z \\ &= \sum_{\lambda\in\Lambda_\geq} \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, n, k(\lambda))\mathsf{e}_1 + O(e^{\theta n}) \quad \text{as } n\to\infty\end{aligned}$$

for some $\vec{b}_\lambda = \sum_{l=1}^{k(\lambda)} b_{\lambda,l-1}\mathsf{e}_l \in \mathbb{R}^{k(\lambda)}$. In other words, there exists a constant $C>0$ such that, for any $n\in\mathbb{Z}$,

$$\left| \mathbb{E}[N(\{n\})] - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{n\geq 0 \text{ or } \lambda\in\Lambda\}} \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda,n,k(\lambda))\mathsf{e}_1 \right| \leq C(e^{\theta n}\wedge e^{\gamma n}). \tag{7.5}$$

Now we are ready to investigate the asymptotic behavior of m_t^φ as $t\to\pm\infty$, $t\in\mathbb{Z}$. Since $m_t^\varphi = m_t^{\mathbb{E}[\varphi]}$, we assume without loss of generality that $\varphi = f$ is a deterministic function satisfying

$$\sum_{n\in\mathbb{Z}} |f(n)|(e^{-\theta n} + e^{-\alpha n}) < \infty,$$

Then, for $t\in\mathbb{Z}$, we have $m_t^f = \sum_{n\in\mathbb{Z}} f(n)\mathbb{E}[N(\{t-n\})]$. We write

$$\begin{aligned}&\left| m_t^f - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t\geq 0 \text{ or } \lambda\in\Lambda\}} \sum_{n\in\mathbb{Z}} f(n)\vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1 \right| \\ &\leq \sum_{n\in\mathbb{Z}} |f(n)| \left| \mathbb{E}[N(\{t-n\})] - \sum_{\lambda\in\Lambda_\geq} \mathbb{1}_{\{t\geq n \text{ or } \lambda\in\Lambda\}} \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1 \right| \\ &\quad + \sum_{n\in\mathbb{Z}} |f(n)| \left| \sum_{\lambda\in\partial\Lambda} |\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}| \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, t-n, k(\lambda))\mathsf{e}_1 \right|.\end{aligned} \tag{7.6}$$

We use (7.5) to estimate the first sum on the right-hand side of (7.6) by

$$\begin{aligned}C\sum_{n\in\mathbb{Z}} |f(n)|(e^{\theta(t-n)} \wedge e^{\gamma(t-n)}) &\leq C\left(\sum_{n\in\mathbb{Z}} |f(n)| e^{\theta(t-n)}\right) \wedge \left(\sum_{n\in\mathbb{Z}} |f(n)| e^{\gamma(t-n)}\right) \\ &\leq (e^{\theta t}\wedge e^{\gamma t})\, C \sum_{n\in\mathbb{Z}} |f(n)|(e^{-\theta n} + e^{-\alpha n}).\end{aligned}$$

On the other hand, we use (4.17) to conclude that for any $0<\epsilon<\alpha/2-\theta$ and $\lambda\in\partial\Lambda$ there is a constant $C_\epsilon \geq 0$ such that $\|\exp(\lambda,n,k(\lambda))\| \leq C_\epsilon e^{\frac{\alpha}{2}n + \epsilon|n|}$. Hence the second sum on the right-hand side of (7.6) is bounded by

$$C_\epsilon \sum_{\lambda\in\partial\Lambda} |\vec{b}_\lambda| \sum_{n\in\mathbb{Z}} |f(n)||\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}|e^{\frac{\alpha}{2}(t-n)+\epsilon|t-n|}.$$

The latter sum can be estimated as follows

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} |f(n)||\mathbb{1}_{\{t\geq 0\}} - \mathbb{1}_{\{t\geq n\}}|e^{\frac{\alpha}{2}(t-n)+\epsilon|t-n|}\\
&= \mathbb{1}_{\mathbb{N}_0}(t)\sum_{n>t}|f(n)|e^{(\frac{\alpha}{2}-\epsilon)(t-n)} + \mathbb{1}_{\mathbb{Z}\setminus\mathbb{N}_0}(t)\sum_{n\leq t}|f(n)|e^{(\frac{\alpha}{2}+\epsilon)(t-n)}\\
&\leq \mathbb{1}_{\mathbb{N}_0}(t)\sum_{n\in\mathbb{Z}}|f(n)|e^{\theta(t-n)} + \mathbb{1}_{\mathbb{Z}\setminus\mathbb{N}_0}(t)\sum_{n\leq t}|f(n)|e^{\alpha(t-n)} = O(e^{\theta t}\wedge e^{\alpha t}),
\end{aligned}$$

as $t\to\pm\infty$. □

7.2. *The non-lattice case.* We again work under the conditions (A1) and (A2) as in Section 7.1, but now we assume that μ is non-lattice.

Similar to the lattice case, first we study the behavior of $\mathbb{E}[N(t)]$. This was already done in [31, Theorem 3.1] in the special case where $\mathcal{L}\mu(z)-1$ has only simple roots. However, the proof given in the cited source can be adapted to the more general setting here. In order to make this paper self-contained and for the reader's convenience, we include the proof.

LEMMA 7.3. *Suppose that, besides (A1) and (A2), the following condition holds:*

$$\limsup_{\eta\to\infty} |\mathcal{L}\mu(\tfrac{\alpha}{2}-\delta+\mathrm{i}\eta)| < 1 \tag{7.7}$$

for some $\delta\in(0,\frac{\alpha}{2}-\vartheta]$. Then $\Lambda_\geq$ is finite. In fact, the function $\mathcal{L}\mu$ takes the value 1 only at finitely many points in the strip $\frac{\alpha}{2}-\delta<\mathrm{Re}(z)<\alpha$. Then, for any root $\lambda\in\Lambda_\geq$ of multiplicity $k(\lambda)\in\mathbb{N}$, there exist constants $c_{\lambda,l}$, $l=0,\ldots,k(\lambda)-1$ such that, for any $\theta\in(\frac{\alpha}{2}-\delta,\frac{\alpha}{2})$ satisfying (7.1) *it holds that*

$$\mathbb{E}[N(t)] = \sum_{\lambda\in\Lambda_\geq} e^{\lambda t}\sum_{l=0}^{k(\lambda)-1} c_{\lambda,l}t^l + O(e^{\theta t}) \quad \text{as } t\to\infty. \tag{7.8}$$

REMARK 7.4. Note that (7.8) can be rewritten in the form

$$\mathbb{E}[N(t)] = \sum_{\lambda\in\Lambda_\geq} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda,t,k(\lambda))\mathsf{e}_1 + O(e^{\theta t})$$

as $t\to\infty$ with $\vec{c}_\lambda := \sum_{l=1}^{k(\lambda)} c_{\lambda,l-1}\mathsf{e}_l$.

REMARK 7.5. Suppose that (A1) and (A2) hold and that the intensity measure μ has a density with respect to the Lebesgue measure. Then one can check using the Riemann-Lebesgue lemma that (7.7) holds for any $\delta\in(0,\frac{\alpha}{2}-\vartheta]$. Hence, in this case, Lemma 7.3 applies.

PROOF OF LEMMA 7.3. First, condition (7.7) implies that

$$\limsup_{\eta\to\infty} |\mathcal{L}\mu(\theta+\mathrm{i}\eta)| < 1$$

for all $\theta \geq \frac{\alpha}{2} - \delta$ and that there are only finitely many roots of the equation $\mathcal{L}\mu(z) = 1$ in the strip $\frac{\alpha}{2} - \delta \leq \mathrm{Re}(z) < \alpha$, see Lemmas 2.1 and 2.3 in [31] (note that although the setup in [31] is slightly different, the proofs carry over without changes).

Now let $f = \mathbb{1}_{[0,\infty)}$ and recall that $N(t) = \mathcal{Z}_t^f$, hence $V(t) := \mathbb{E}[N(t)] = m_t^f$ for $t \in \mathbb{R}$. In analogy to the derivation of [31, Eq. (3.11)], we use the recursive structure of $\mathcal{Z}_t^f$ to obtain a renewal equation for $V(t)$ as follows. We start with

$$\mathcal{Z}_t^f = f(t) + \sum_{j=1}^{N} \mathcal{Z}_{j,t-X_j}^f$$

where $\mathcal{Z}_{1,t}^f, \mathcal{Z}_{2,t}^f \ldots$ are independent copies of $\mathcal{Z}_t^f$. Taking expectations, then conditioning with respect to ξ, the reproduction point process of the ancestor, we infer

$$m_t^f = f(t) + \int m_{t-x}^f \, \mu(\mathrm{d}x) = f(t) + \mu * m_t^f, \quad t \in \mathbb{R}. \tag{7.9}$$

Our subsequent proof relies on a smoothing technique. So let $\rho := \mathbb{1}_{[0,1]}$. For any $\varepsilon > 0$, we set

$$\rho_\varepsilon(t) := \tfrac{1}{\varepsilon}\rho(\tfrac{t}{\varepsilon}) = \tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}(t), \quad t \in \mathbb{R}.$$

Then for $f_\varepsilon := f * \rho_\varepsilon$ (Lebesgue convolution), we have

$$f_\varepsilon(t) \leq f(t) \leq f_\varepsilon(t+\varepsilon)$$

for all $t \in \mathbb{R}$, which in turn gives

$$m_t^{f_\varepsilon} \leq m_t^f \leq m_{t+\varepsilon}^{f_\varepsilon}. \tag{7.10}$$

Also, one can check that $t \mapsto m_t^{f_\varepsilon}$ is a continuous function. First, we find the asymptotic expansion of this function and then, we let ε tend to 0 in a controlled way while letting $t \to \infty$ to deduce the asymptotic behavior of $V(t) = m_t^f$ from that of $m_t^{f_\varepsilon}$. From the renewal equation (7.9) we conclude that for $\mathrm{Re}(z) > \alpha$ it holds

$$\mathcal{L}m^{f_\varepsilon}(z) = \mathcal{L}(\rho_\varepsilon * m_t^f)(z) = \mathcal{L}f_\varepsilon(z) + \mathcal{L}\mu(z)\mathcal{L}m^{f_\varepsilon}(z),$$

hence,

$$\mathcal{L}m^{f_\varepsilon}(z) = \frac{\mathcal{L}f_\varepsilon(z)}{1 - \mathcal{L}\mu(z)} \quad \text{for } \mathrm{Re}(z) > \alpha.$$

The function

$$\frac{\mathcal{L}f_\varepsilon(z)}{1 - \mathcal{L}\mu(z)} = \frac{\mathcal{L}\rho_\varepsilon(z)\mathcal{L}f(z)}{1 - \mathcal{L}\mu(z)} = \frac{1 - e^{-\varepsilon z}}{\varepsilon z^2(1 - \mathcal{L}\mu(z))}$$

defines a meromorphic extension of $\mathcal{L}m^{f_\varepsilon}$ on $\mathrm{Re}(z) > \vartheta$. This function decays as $|\mathrm{Im}(z)|^{-2}$ as $\mathrm{Im}(z) \to \pm\infty$ and $\mathrm{Re}(z)$ is constant, hence, it is integrable along vertical lines. Thus, for any $\tau > \alpha$, the Laplace inversion formula (see, for instance, [48, Theorem 7.3 on p. 66]) gives

$$m_t^{f_\varepsilon} = \frac{m_{t+}^{f_\varepsilon} + m_{t-}^{f_\varepsilon}}{2} = \frac{1}{2\pi \mathrm{i}} \int_{\tau - \mathrm{i}\infty}^{\tau + \mathrm{i}\infty} e^{tz} \mathcal{L}m^{f_\varepsilon}(z) \, \mathrm{d}z, \quad t > 0.$$

To simplify notation, we assume without loss of generality that $\vartheta = \frac{\alpha}{2} - \delta$ and that $\mathcal{L}\mu$ is holomorphic on a neighborhood of $\mathrm{Re}(z) \geq \vartheta$. Then, for large enough R, an application of the residue theorem gives

$$\int_{\tau-\mathrm{i}R}^{\tau+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{\lambda\in\Lambda_\geq}\mathrm{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) + \int_{\vartheta-\mathrm{i}R}^{\vartheta+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z + \int_{\vartheta+\mathrm{i}R}^{\tau+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z - \int_{\vartheta-\mathrm{i}R}^{\tau-\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z.$$

Here,

$$\left|\int_{\vartheta+\mathrm{i}R}^{\tau+\mathrm{i}R} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z\right| \leq e^{t\tau}\int_{\vartheta}^{\tau}\left|\frac{1-e^{-\varepsilon(x+\mathrm{i}R)}}{\varepsilon(x+\mathrm{i}R)^2(1-\mathcal{L}\mu(x+\mathrm{i}R))}\right|\mathrm{d}x \leq Ce^{t\tau}\int_{\vartheta}^{\tau}\left|\frac{1}{\varepsilon(x+\mathrm{i}R)^2}\right|\mathrm{d}x \underset{R\to\infty}{\longrightarrow} 0$$

for some constant C that depends only on μ. Here we used the fact that, by Lemma 2.1 of [31], $\mathcal{L}\mu(x+\mathrm{i}R)$ for $x \geq \vartheta$ and $R \geq R_0$ is uniformly bounded away from 1 for some sufficiently large $R_0 > 0$. The same bound holds for the second horizontal integral. Therefore, by letting R tend to infinity we conclude

$$m_t^{f_\varepsilon} = \sum_{\lambda\in\Lambda_\geq}\mathrm{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) + \frac{1}{2\pi\mathrm{i}}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\,\mathrm{d}z. \tag{7.11}$$

Next, denoting by $\{b_j(\lambda)\}_{j\in\mathbb{Z}}$ the coefficients in the Laurent expansion of the function $(1-\mathcal{L}\mu(z))^{-1}$ at $z=\lambda\in\Lambda_\geq$ (hence, in particular, $b_j(\lambda)=0$ for $j<-k(\lambda)$), we have

$$\begin{aligned}
\mathrm{Res}_{z=\lambda}\left(e^{tz}\mathcal{L}m^{f_\varepsilon}(z)\right) &= \mathrm{Res}_{z=\lambda}\left(e^{tz}\frac{\mathcal{L}f_\varepsilon(z)}{1-\mathcal{L}\mu(z)}\right)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}}\frac{t^l}{l!}\frac{(\mathcal{L}f_\varepsilon)^{(n)}(\lambda)}{n!}b_{-1-n-l}(\lambda)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}}\frac{t^l}{l!}\frac{\int(-x)^n f_\varepsilon(x)e^{-\lambda x}\,\mathrm{d}x}{n!}b_{-1-n-l}(\lambda)\\
&= e^{\lambda t}\sum_{\substack{n,l\geq 0\\ n+l<k(\lambda)}}\frac{t^l}{l!}\frac{\int(-x)^n f(x)e^{-\lambda x}\mathrm{d}x}{n!}b_{-1-n-l}(\lambda)+\varepsilon O(e^{\alpha t})\\
&=: e^{\lambda t}\sum_{0\leq l<k(\lambda)}c_{\lambda,l}t^l+\varepsilon O(e^{\alpha t}), \qquad (7.12)
\end{aligned}$$

where the implicit constant depends only on λ, not on ε. It remains to estimate the second term in (7.11). For $\varepsilon \le \vartheta^{-1}$, using that $|1-e^{-z}| \le |z| \wedge 2$ for $\mathrm{Re}(z) \ge 0$, we infer

$$
\begin{aligned}
\Bigg| \int_{\vartheta - \mathrm{i}\infty}^{\vartheta + \mathrm{i}\infty} e^{tz} \mathcal{L}m^{f_\varepsilon}(z) \, \mathrm{d}z \Bigg| &\le e^{\vartheta t} \int_{\vartheta - \mathrm{i}\infty}^{\vartheta + \mathrm{i}\infty} \bigg| \frac{1-e^{-\varepsilon z}}{\varepsilon z^2 (1-\mathcal{L}\mu(z))} \bigg| \, |\mathrm{d}z| \\
&\le C e^{\vartheta t} \int_{\vartheta - \mathrm{i}\infty}^{\vartheta + \mathrm{i}\infty} \frac{|\varepsilon z|^{-1} \wedge 1}{|z|} \, |\mathrm{d}z| \\
&= C e^{\vartheta t} \int_{\varepsilon\vartheta - \mathrm{i}\infty}^{\varepsilon\vartheta + \mathrm{i}\infty} (|z|^{-1} \wedge |z|^{-2}) \, |\mathrm{d}z| \\
&\le C e^{\vartheta t} \int_{-\infty}^{\infty} x^{-1} \wedge x^{-2} \wedge (\varepsilon\vartheta)^{-1} \, \mathrm{d}x \\
&\le C' e^{\vartheta t} (|\log \varepsilon| + 1) \qquad (7.13)
\end{aligned}
$$

for some constant C' that depends neither on t nor on ε. Using (7.12) with $t+\varepsilon$ instead of t, we conclude that

$$\big| \mathrm{Res}_{z=\lambda} \big(e^{(t+\varepsilon)z} \mathcal{L}m^{f_\varepsilon}(z) \big) - \mathrm{Res}_{z=\lambda} \big(e^{tz} \mathcal{L}m^{f_\varepsilon}(z) \big) \big| = \varepsilon O(e^{\alpha t}),$$

where we used $k(\lambda) = 1$ for $\lambda = \alpha$, and thereupon, by (7.11) and (7.13),

$$m_{t+\varepsilon}^{f_\varepsilon} - m_t^{f_\varepsilon} = O(\varepsilon e^{\alpha t} + |\log \varepsilon| e^{\vartheta t}).$$

Setting now $\varepsilon := e^{-\alpha t}$, by (7.10), we infer

$$m_t^f = \sum_{\lambda \in \Lambda_\ge} e^{\lambda t} \sum_{0 \le l \le k(\lambda)-1} c_{\lambda,l} t^l + O(t e^{\vartheta t}),$$

which completes the proof of the lemma. □

Now we are ready to provide the asymptotic expansion for the expectation function of a general branching process counted with a random characteristic φ.

LEMMA 7.6. *Suppose that, besides (A1) and (A2), condition (7.7) holds. Then $\Lambda_\ge$ is finite and there are constants $b_{\lambda,l}$, $\lambda \in \Lambda_\ge$, $0 \le l < k(\lambda)$ such that, for any $\vartheta < \theta < \frac{\alpha}{2}$ fulfilling (7.1) and any random characteristic φ satisfying (2.19), we have*

$$m_t^\varphi = \begin{cases} \sum_{\lambda \in \Lambda_\ge} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \int (t-x)^l e^{\lambda(t-x)} \mathbb{E}[\varphi(x)] \, \mathrm{d}x + O(e^{\theta t}) & \text{as } t \to \infty \\ O(e^{\gamma t}) & \text{as } t \to -\infty. \end{cases} \qquad (7.14)$$

REMARK 7.7. If we set $\vec{b}_\lambda := (b_{\lambda,l})_{l=0,\ldots,k(\lambda)-1}$, then formula (7.14) can be written in the form

$$m_t^\varphi = \sum_{\lambda \in \Lambda_\ge} \mathbb{1}_{\{t \ge 0 \text{ or } \lambda \in \Lambda\}} \int \vec{b}_\lambda^{\mathsf{T}} \exp(\lambda, t-x, k(\lambda)) \mathsf{e}_1 \mathbb{E}[\varphi(x)] \, \mathrm{d}x + O(e^{\theta t} \wedge e^{\gamma t})$$

as $t \to \pm\infty$.

PROOF. Without loss of generality we assume that the characteristic $\varphi = f$ is a deterministic function. By Lemma 7.3 there exist constants $c_{\lambda,l}$, $l = 0, \dots, k(\lambda) - 1$, $\theta \in (\vartheta, \frac{\alpha}{2})$ and a constant C such that, for any $t \in \mathbb{R}$,

$$\Big| \mathbb{E}[N(t)] - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t, k(\lambda)) \mathsf{e}_1 \Big| \leq C(e^{\theta t} \wedge e^{\gamma t}) \tag{7.15}$$

and hence for the characteristic $f(t) = \mathbb{1}_{[x,\infty)}(t) = \mathbb{1}_{[0,\infty)}(t - x)$, we find

$$\Big| m_t^f - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t - x \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) \mathsf{e}_1 \Big| \leq C(e^{\theta(t-x)} \wedge e^{\gamma(t-x)}).$$

Suppose now that $f \geq 0$ is a càdlàg, nondecreasing function with

$$\int f(x)(e^{-\alpha x} + e^{-\vartheta x}) \, \mathrm{d}x < \infty. \tag{7.16}$$

Then f is the measure-generating function of a locally finite measure ν on the Borel sets of $\mathbb{R}$, namely, for any $y \in \mathbb{R}$,

$$f(y) = \nu((-\infty, y]) = \int \mathbb{1}_{[x,\infty)}(y) \, \nu(\mathrm{d}x).$$

For any $t \in \mathbb{R}$, by an application of Fubini's theorem, we infer

$$m_t^f = \int \mathbb{E}[N(t - x)] \, \nu(\mathrm{d}x).$$

By (4.16) we have $\frac{\mathrm{d}}{\mathrm{d}x} \exp(\lambda, x, k) = J_{\lambda,k} \exp(\lambda, x, k)$. We show that (7.14) holds with $\vec{b}_\lambda := J_{\lambda,k}^{\mathsf{T}} \vec{c}_\lambda$, $\lambda \in \Lambda_\geq$. To this end, first notice that, as f fulfills (7.16), another application of Fubini's theorem yields

$$\int \exp(\lambda, -x, k) \, \nu(\mathrm{d}x) = \int J_{\lambda,k} \exp(\lambda, -x, k) f(x) \, \mathrm{d}x.$$

We now write

$$\begin{aligned}
&\Big| m_t^f - \int \Big(\sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{b}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) f(x) \mathsf{e}_1 \Big) \, \mathrm{d}x \Big| \\
&= \Big| m_t^f - \int \Big(\sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} J_{\lambda,k(\lambda)} \exp(\lambda, t - x, k(\lambda)) f(x) \mathsf{e}_1 \Big) \, \mathrm{d}x \Big| \\
&= \Big| \int \Big(\mathbb{E}[N(t - x)] - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) \mathsf{e}_1 \Big) \, \nu(\mathrm{d}x) \Big| \\
&\leq \int \Big| \mathbb{E}[N(t - x)] - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t - x \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) \mathsf{e}_1 \Big| \, \nu(\mathrm{d}x) \\
&\quad + \sum_{\lambda \in \partial\Lambda} \int \big| \mathbb{1}_{\{t \geq 0\}} - \mathbb{1}_{\{t - x \geq 0\}} \big| \big| \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) \mathsf{e}_1 \big| \, \nu(\mathrm{d}x).
\end{aligned}$$

For the first term, by (7.15), we have

$$\int \Big| \mathbb{E}[N(t - x)] - \sum_{\lambda \in \Lambda_\geq} \mathbb{1}_{\{t - x \geq 0 \text{ or } \lambda \in \Lambda\}} \vec{c}_\lambda^{\,\mathsf{T}} \exp(\lambda, t - x, k(\lambda)) \mathsf{e}_1 \Big| \, \nu(\mathrm{d}x)$$

$$\leq C\int (e^{\theta(t-x)}\wedge e^{\gamma(t-x)})\,\nu(\mathrm{d}x)$$
$$\leq C\Big(\int e^{\theta(t-x)}\nu(\mathrm{d}x)\wedge\int e^{\gamma(t-x)}\,\nu(\mathrm{d}x)\Big)$$
$$\leq C(e^{\theta t}\wedge e^{\gamma t})\Big(\theta\int f(x)e^{-\theta x}\,\mathrm{d}x+\gamma\int f(x)e^{-\gamma x}\,\mathrm{d}x\Big).$$

Next, for $\lambda\in\partial\Lambda$, we estimate

$$\int\big|\mathbb{1}_{\{t\geq 0\}}-\mathbb{1}_{\{t-x\geq 0\}}\big|\big|\vec{c}_\lambda^{\,\mathsf{T}}\exp(\lambda,t-x,k(\lambda))\mathsf{e}_1\big|\,\nu(\mathrm{d}x)$$
$$=\mathbb{1}_{(-\infty,0)}(t)\int_{(-\infty,t]}\big|\vec{c}_\lambda^{\,\mathsf{T}}\exp(\lambda,t-x,k(\lambda))\mathsf{e}_1\big|\,\nu(\mathrm{d}x)$$
$$+\mathbb{1}_{[0,\infty)}(t)\int_{(t,\infty)}\big|\vec{c}_\lambda^{\,\mathsf{T}}\exp(\lambda,t-x,k(\lambda))\mathsf{e}_1\big|\,\nu(\mathrm{d}x)$$
$$\leq C\mathbb{1}_{(-\infty,0)}(t)\int e^{\alpha(t-x)}\,\nu(\mathrm{d}x)+C\mathbb{1}_{[0,\infty)}(t)\int e^{\vartheta(t-x)}\,\nu(\mathrm{d}x)$$
$$\leq C'\big(e^{\alpha t}\wedge e^{\vartheta t}\big),$$

where we have used (4.17) in the penultimate step. This completes the proof of the theorem for non-decreasing $f\geq 0$.

Now let f be an arbitrary càdlàg function satisfying the integrability condition (2.19) (with f in place of $\mathbb{E}[\varphi]$). Define

$$f_\pm(x):=\sup\Big\{\sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))^\pm:-\infty<x_0<\ldots<x_n\leq x,\ n\in\mathbb{N}\Big\}$$

for $x\in\mathbb{R}$. Clearly, $f_+,f_-:\mathbb{R}\to\mathbb{R}$ are nondecreasing with $f_\pm\geq 0$. It is known that $f=f_+-f_-$. (This is the Jordan decomposition of f on $\mathbb{R}$.) It is further known that f_+ and f_- are càdlàg since f is. Further, $\mathrm{V}f(x)=f_+(x)+f_-(x)$ and hence (2.19) implies that both, f_+ and f_- satisfy (7.16). The previous part of the proof thereby applies to f_+ and f_- and, by linearity, extends to f. □

REMARK 7.8. In the situations of Lemmas 7.1 and 7.6, m_t^φ has a representation as in (2.20). Indeed, in both cases, m_t^φ can be written as

$$m_t^\varphi=\mathbb{1}_{[0,\infty)}(t)\sum_{\lambda\in\Lambda_\geq}\int_{\mathbb{G}}\vec{b}_\lambda^{\,\mathsf{T}}\exp(\lambda,t-x,k(\lambda))\mathsf{e}_1\mathbb{E}[\varphi](x)\,\ell(\mathrm{d}x)+O(e^{\theta t}\wedge e^{\gamma t})$$
$$=\mathbb{1}_{[0,\infty)}(t)\sum_{\lambda\in\Lambda_\geq}\vec{a}_\lambda^{\,\mathsf{T}}\exp(\lambda,t,k(\lambda))\mathsf{e}_1+O(e^{\theta t}\wedge e^{\gamma t}),$$

where $\vec{a}_\lambda:=\int_{\mathbb{G}}\exp(\lambda,-x,k(\lambda))^{\mathsf{T}}\vec{b}_\lambda\mathbb{E}[\varphi](x)\,\ell(\mathrm{d}x)$. Consequently (cf. (6.22), (2.22) and (6.24)), we have

$$H_\Lambda(t)=\sum_{\lambda\in\Lambda}\vec{a}_\lambda^{\,\mathsf{T}}\exp(\lambda,t,k(\lambda))W(\lambda,k(\lambda))\mathsf{e}_1$$
$$=\sum_{\lambda\in\Lambda}\vec{b}_\lambda^{\,\mathsf{T}}\int_{\mathbb{G}}\exp(\lambda,t-x,k(\lambda))W(\lambda,k(\lambda))\mathsf{e}_1\mathbb{E}[\varphi](x)\ell(\mathrm{d}x)$$

$$= \sum_{\lambda\in\Lambda} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \int_{\mathbb{G}} e^{\lambda(t-x)} \sum_{j=0}^{l} \binom{l}{j} (t-x)^{l-j} W^{(j)}(\lambda) \mathbb{E}[\varphi](x) \ell(\mathrm{d}x)$$

$$= \sum_{\lambda\in\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \sum_{j=0}^{l} \binom{l}{j} W^{(j)}(\lambda) \int_{\mathbb{G}} (t-x)^{l-j} \mathbb{E}[\varphi](x) e^{-\lambda x} \ell(\mathrm{d}x)$$

and, similarly,

$$\begin{aligned} H_{\partial\Lambda}(t) &= \sum_{\lambda\in\partial\Lambda} \vec{a}_\lambda^{\mathsf{T}} \exp(\lambda, t, k(\lambda)) \mathsf{e}_1 \\ &= \sum_{\lambda\in\partial\Lambda} \vec{b}_\lambda^{\mathsf{T}} \int_{\mathbb{G}} \exp(\lambda, t-x, k(\lambda)) \mathsf{e}_1 \mathbb{E}[\varphi](x)\, \ell(\mathrm{d}x) \\ &= \sum_{\lambda\in\partial\Lambda} e^{\lambda t} \sum_{l=0}^{k(\lambda)-1} b_{\lambda,l} \int_{\mathbb{G}} (t-x)^l \mathbb{E}[\varphi](x) e^{-\lambda x}\, \ell(\mathrm{d}x). \end{aligned}$$

7.3. *Proofs of Theorems 2.8, 2.9 and 2.10.*

PROOF OF THEOREM 2.8. Theorem 2.8 is a consequence of Theorem 2.15. To see this, we first check that the assumptions of Theorem 2.8 imply those of Theorem 2.15. In a second step, we show how the conclusion of Theorem 2.8 follows from that of Theorem 2.15.

So suppose the assumptions of Theorem 2.8, in particular, (A1) through (A3), hold. Also (A7) holds because $\Lambda_{\geq} = \{\alpha\}$ and, by assumption, there are no roots of the equation $\mathcal{L}\mu(z) = 1$ in the strip $\vartheta < \mathrm{Re}(z) < \alpha$. Let φ be a characteristic satisfying (A5), (A6) and (2.19). Notice that (2.19) implies (A4). To see this, decompose $\mathbb{E}[\varphi] = f_1 - f_2$ for two non-negative, non-decreasing functions f_1 and f_2 such that $\mathrm{V}\mathbb{E}[\varphi](x) = f_1(x) + f_2(x)$. Then

$$\int \Big(f_i(x)e^{-\alpha x}\Big)^* \,\mathrm{d}x \leq \int f_i(x+1)e^{-\alpha(x-1)}\,\mathrm{d}x \leq e^{2\alpha} \int \mathrm{V}\mathbb{E}[\varphi](x)e^{-\alpha x}\,\mathrm{d}x < \infty.$$

We conclude from Proposition 2.6 that both $x \mapsto f_1(x)e^{-\alpha x}$ and $x \mapsto f_2(x)e^{-\alpha x}$ are directly Riemann integrable and hence so is their difference, i.e., (A4) is satisfied. We have to check that m_t^φ has a representation of the form (2.20). This follows from Lemma 7.6 once we have shown that (7.7) holds (cf. Remark 7.8). However, the latter follows from the existence of a Lebesgue density for μ and the Riemann-Lebesgue lemma (cf. Remark 7.5).

Since $\Lambda_{\geq} = \{\alpha\}$ and $\mathcal{L}\mu'(\alpha) = -\beta \neq 0$, i.e., $k(\alpha) = 1$, Lemma 7.6 gives

$$m_t^\varphi = \begin{cases} e^{\alpha t} b_{\alpha,0} \int \mathbb{E}[\varphi(x)]e^{-\alpha x}\,\mathrm{d}x + O(e^{\theta t}) & \text{as } t\to\infty \\ O(e^{\gamma t}) & \text{as } t \to -\infty \end{cases}$$

for some constant $b_{\alpha,0} \in \mathbb{R}$. From Nerman's law of large numbers [28, Theorem 6.1, see the proof on p. 246] or alternatively Proposition 7.9, we know that $b_{\alpha,0} = \beta^{-1}$, so

$$m_t^\varphi = \mathbb{1}_{[0,\infty)}(t)\beta^{-1}\mathcal{L}(\mathbb{E}[\varphi])(\alpha)e^{\alpha t} + O(e^{\theta t} \wedge e^{\gamma t})$$

as $t \to \pm\infty$, i.e., m_t^φ indeed has a representation of the form (2.20). Hence, Theorem 2.15 applies. Let $a_\alpha := \beta^{-1}\mathcal{L}(\mathbb{E}[\varphi])(\alpha)$. Then $H(t) = e^{\alpha t}a_\alpha W$. Notice that $n = -1$ and thus $\rho_{-1} = 0$ in Theorem 2.15. With $\sigma \geq 0$ as in Theorem 2.15, we now infer that in both cases, $\sigma = 0$ and $\sigma > 0$, that

$$e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - a_\alpha e^{\alpha t}W\big) = e^{-\frac{\alpha}{2}t}\big(\mathcal{Z}_t^\varphi - H(t)\big) \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\tfrac{W}{\beta}}\mathcal{N} \quad \text{as } t\to\infty$$

for a standard normal random variable $\mathcal{N}$ independent of W. □

PROOF OF THEOREM 2.9. As before we shall prove that the assumptions of Theorem 2.15 are fulfilled. As in the proof of Theorem 2.8, we conclude that (2.19) implies (A4). Lemma 7.6 yields (A7) and that m_t^φ has an expansion of the form (2.20) with $r(t) = O(e^{\theta t} \wedge e^{\gamma t})$ as $t \to \infty$, see also Remark 7.8. The assertion now follows from Theorem 2.15 and Remark 7.8. Note, that $H_{\partial\Lambda}(t) = O(t^{k-1} e^{\frac{\alpha}{2} t}) = o(t^{k-\frac{1}{2}} e^{\frac{\alpha}{2} t})$ by the definition of $H_{\partial\Lambda}$ (cf. (2.22)) and hence this term can be neglected in the limit theorem. □

PROOF OF THEOREM 2.10. Again, we first check that the assumptions of Theorem 2.10 imply those of Theorem 2.15. So suppose that the assumptions of Theorem 2.10, in particular, (A1) through (A3), hold. Regarding (A7), i.e., the finiteness of $\Lambda_\geq$, notice that since μ is lattice, we have

$$\Lambda_\geq = \{\lambda \in \mathbb{C} : \tfrac{\alpha}{2} \leq \mathrm{Re}(\lambda) \leq \alpha,\ -\pi < \mathrm{Im}(\lambda) \leq \pi,\ \mathcal{L}\mu(\lambda) = 1\}.$$

By (A2), $\mathcal{L}\mu$ is holomorphic on $\{z \in \mathbb{C} : \mathrm{Re}(z) > \vartheta\}$ and non-constant by (A1). Hence, $\mathcal{L}\mu(z) = 1$ can hold for only finitely many z in the compact box $\frac{\alpha}{2} \leq \mathrm{Re}(z) \leq \alpha$, $|\mathrm{Im}(z)| \leq \pi$, that is, $\Lambda_\geq$ is finite. Now suppose that φ is a characteristic satisfying $\sum_{n\in\mathbb{Z}} |\mathbb{E}[\varphi(n)]|(e^{-\theta n} + e^{-\alpha n}) < \infty$ and (A5). Then φ satisfies (A4) and (A6). Moreover, by Lemma 7.1 and Remark 7.8 we conclude (2.20) hence, Theorem 2.15 applies. As before, the assertion follows from Theorem 2.15 in combination with Remark 7.8. By the same argument as in the proof of Theorem 2.9, the term $H_{\partial\Lambda}$ can be neglected. □

7.4. *Determining the coefficients.* Note that although the constants $\vec{c}_\lambda$ and $\vec{b}_\lambda$ are not given explicitly it is not hard to follow the proofs and provide explicit expressions for them. However, even for small $k(\lambda)$, this approach may lead to tedious calculations, not to mention that there can also be several roots in the relevant strip. It seems that a more efficient way to determine the constants b_l^λ is an application of Lemma 7.1 or 7.6, respectively, to a characteristic for which we explicitly know the asymptotic behavior of the expectation of the associated general branching process.

PROPOSITION 7.9. *Let $\lambda \in \Lambda_\geq$ be a root of $\mathcal{L}\mu(z) = 1$ of multiplicity k (in the lattice case we also assume that $\mathrm{Im}(\lambda) \in (-\pi, \pi]$). Then the vector $\vec{b}_\lambda$ appearing in Lemma 7.1 or 7.6, respectively, is given by $M^\lambda \vec{b}_\lambda = \mathsf{e}_k$, where M^λ is the $k \times k$ upper triangular matrix such that for $j \geq i$*

$$(M^\lambda)_{i,j} := -\frac{(j-1)!(k-1)!}{(i-1)!(j-i+k)!} (\mathcal{L}\mu)^{(k+j-i)}(\lambda)$$

in the non-lattice case. In contrast, in the lattice case,

$$(M^\lambda)_{i,j} := \binom{j-1}{i-1} P_{k,j-i}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big) \mathcal{L}\mu(z)_{|z=\lambda},$$

where the polynomials $P_{k,l}$ are given by

$$P_{k,l}(y) := (-1)^l \sum_{m=1}^{k} \binom{k-1}{m-1} y^{k-m} \frac{B_{l+m}(-y) - B_{l+m}(0)}{l+m},$$

and B_n is the n^{th} Bernoulli polynomial. In particular, in both cases, as $(\mathcal{L}\mu)^{(j)}(\lambda) = 0$ for $j = 1, \ldots, k-1$,

$$\det(M^\lambda) = \Big(\frac{-\mathcal{L}\mu^{(k)}(\lambda)}{k}\Big)^k \neq 0$$

and the matrix M^λ is invertible.

For the proof, we need a lemma which essentially is Jensen's inequality for the total variation operator V defined in (2.18).

LEMMA 7.10. *Let $\phi = (\phi(t))_{t\in\mathbb{R}}$ be a stochastic process with càdlàg paths such that $\phi(t) \in L^1$ for every $t \in \mathbb{R}$ and $t \mapsto \mathbb{E}[\phi](t)$ is again càdlàg. Then, finite or not,*

$$\mathrm{V}\mathbb{E}[\phi](t) \leq \mathbb{E}[\mathrm{V}\phi(t)] \tag{7.17}$$

for every $t \in \mathbb{R}$.

PROOF. First notice that $\mathrm{V}\phi(t)$ is a random variable. Indeed, since the paths of ϕ are cádlág, we have

$$\mathrm{V}\phi(t) = \sup\left\{\sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})| : -\infty < t_0 < \ldots < t_n \leq t,\ t_0, \ldots, t_n \in \mathbb{Q},\ n \in \mathbb{N}\right\},$$

which is measurable as the supremum of a family of random variables indexed by a countable set. Since $\mathbb{E}[\phi]$ is also càdlàg, we infer

$$\begin{aligned}
\mathrm{V}\mathbb{E}[\phi](t) &= \sup\left\{\sum_{j=1}^{n} |\mathbb{E}[\phi(t_j) - \phi(t_{j-1})]| : t_0 < \ldots < t_n \leq t,\ t_0, \ldots, t_n \in \mathbb{Q},\ n \in \mathbb{N}\right\} \\
&\leq \sup\left\{\mathbb{E}\left[\sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})|\right] : t_0 < \ldots < t_n \leq t,\ t_0, \ldots, t_n \in \mathbb{Q},\ n \in \mathbb{N}\right\} \\
&\leq \mathbb{E}\left[\sup\left\{\sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})| : t_0 < \ldots < t_n \leq t,\ t_0, \ldots, t_n \in \mathbb{Q},\ n \in \mathbb{N}\right\}\right] \\
&= \mathbb{E}[\mathrm{V}\phi](t).
\end{aligned}$$

□

PROOF OF PROPOSITION 7.9. For $\lambda \in \Lambda_{\geq}$, consider the characteristic

$$\begin{aligned}
\phi(t) &= \mathsf{e}_k^{\mathsf{T}}\mathbb{E}[\phi_\lambda(t)]\mathsf{e}_1 \\
&= \mathbb{1}_{[0,\infty)}(t)\mathbb{E}\left[\sum_{j=1}^{N} \mathsf{e}_k^{\mathsf{T}}\mathbb{1}_{[0,X_j)}(t)\exp(\lambda, t - X_j, k)\mathsf{e}_1\right] \\
&= \mathbb{1}_{[0,\infty)}(t)\mathbb{E}\left[\sum_{j=1}^{N} f_{X_j}(t)\right], \quad t \in \mathbb{R}
\end{aligned}$$

where, for any $x \geq 0$,

$$f_x(t) := \mathsf{e}_k^{\mathsf{T}}\mathbb{1}_{[0,x)}(t)\exp(\lambda, t - x, k)\mathsf{e}_1 = \mathbb{1}_{[0,x)}(t)(t-x)^{k-1}e^{\lambda(t-x)}.$$

By Lemma 5.2 (the assumption are fulfilled by Lemma 5.1), the following holds for all $t \geq 0$

$$m_t^\phi = \mathsf{e}_k^{\mathsf{T}}\exp(\lambda, t, k)\mathsf{e}_1. \tag{7.18}$$

We now aim to apply either Lemma 7.6 in the lattice case or Lemma 7.1 in the non-lattice case to the characteristic ϕ. To do this, we need to verify that ϕ satisfies either (2.19) or (7.3) respectively.

Using equations (4.16) and (4.17), we get

$$\begin{aligned}\mathrm{V}f_x(t) &\le |f_x(0)|\mathbb{1}_{[0,\infty)}(t) + \int_0^{t\wedge x} \big|\frac{\mathrm{d}}{\mathrm{d}s} f_x(s)\big| \,\mathrm{d}s + \mathbb{1}_{\{t\ge x\}} \\ &\le C \times \begin{cases} 0 & \text{for } t<0, \\ e^{(\mathrm{Re}(\lambda)-\delta)(t-x)} & \text{for } 0\le t<x, \\ 1 & \text{for } x \le t \end{cases}\end{aligned}$$

for some $\delta \in (0, \mathrm{Re}(\lambda)-\vartheta)$ and a constant C that does not depend on $x \ge 0$. Thus,

$$\int \mathrm{V}f_x(t)\big(e^{-\vartheta t} + e^{-\alpha t}\big)\,\mathrm{d}t \le 2\int \mathrm{V}f_x(t) e^{-\vartheta t}\,\mathrm{d}t \le C' e^{-\vartheta x}$$

for some other constant C' that also does not depend on x. Now, observe that we can apply (7.17) to $\mathsf{e}_k^{\mathsf{T}} \phi_\lambda \mathsf{e}_1$ since both this characteristic and its expectation have càdlàg paths. Only the latter requires a proof. By (4.17),

$$|\mathsf{e}_k^{\mathsf{T}} \phi_\lambda(t) \mathsf{e}_1| \le C\mathbb{1}_{[0,\infty)}(t) e^{\vartheta t} \sum_{j=1}^N e^{-\vartheta X_j}$$

for any $t\in\mathbb{R}$. Hence, (A2) and the dominated convergence theorem imply that $\mathbb{E}[\mathsf{e}_k^{\mathsf{T}}\phi_\lambda \mathsf{e}_1]$ is càdlàg. Using the subadditivity of V, (7.17) and assumption (A2), we get

$$\begin{aligned}\int \mathrm{V}\phi(t)\big(e^{-\vartheta t}+e^{-\alpha t}\big)\,\mathrm{d}t &= \int_0^\infty \mathrm{V}\Big(\mathbb{E}\Big[\sum_{j=0}^N f_{X_j}\Big]\Big)(t)\big(e^{-\vartheta t}+e^{-\alpha t}\big)\,\mathrm{d}t \\ &\le \mathbb{E}\Big[\sum_{j=0}^N \int \mathrm{V}f_{X_j}(t)\big(e^{-\vartheta t}+e^{-\alpha t}\big)\,\mathrm{d}t\Big] \le C'\mathbb{E}\Big[\sum_{j=0}^N e^{-\vartheta X_j}\Big] < \infty.\end{aligned}$$

A similar argument gives (7.3) in the lattice case. An application of either Lemma 7.6 or Lemma 7.1 (as mentioned in Remark 7.8), respectively, to the characteristic ϕ, yields

$$\begin{aligned} m_t^\phi &= \sum_{\lambda'\in\Lambda_\ge} \vec{b}_{\lambda'}^{\mathsf{T}} \int \exp(\lambda', t-x, k)\phi(x)\,\ell(\mathrm{d}x)\mathsf{e}_1 + O(e^{\theta t}) \\ &= \sum_{\lambda'\in\Lambda_\ge} \bigg(\Big(\int \exp(\lambda', -x, k)\phi(x)\,\ell(\mathrm{d}x)\Big)^{\mathsf{T}} \vec{b}_{\lambda'}\bigg)^{\mathsf{T}} \exp(\lambda', t, k)\mathsf{e}_1 + O(e^{\theta t}), \end{aligned} \tag{7.19}$$

as t goes to infinity. Next take the difference of the two asymptotic expansions (7.18) and (7.19) for m_t^ϕ and then apply Lemma A.1 to infer

$$\Big(\int \exp(\lambda, -x, k)\phi(x)\,\ell(\mathrm{d}x)\Big)^{\mathsf{T}} \vec{b}_\lambda = \mathsf{e}_k.$$

It now suffices to evaluate the coefficients of the matrix

$$M^\lambda := \int \exp(\lambda, -x, k)\phi(x)\,\ell(\mathrm{d}x).$$

First, we deal with the non-lattice case. For this purpose, recalling basic properties of the beta function B, we infer

$$\int (-x)^l e^{-\lambda x}\phi(x)\,\mathrm{d}x = (-1)^l \mathbb{E}\Big[\sum_{j=1}^N \int_0^{X_j} x^l (x-X_j)^{k-1} e^{-\lambda X_j}\,\mathrm{d}x\Big]$$

$$
\begin{aligned}
&= (-1)^l \iint_0^y x^l (x-y)^{k-1}\,\mathrm{d}x e^{-\lambda y}\,\mu(\mathrm{d}y)\\
&= (-1)^{l+k-1} \int B(l+1,k) y^{k+l} e^{-\lambda y}\,\mu(\mathrm{d}y)\\
&= -\frac{l!(k-1)!}{(l+k)!}(\mathcal{L}\mu)^{(k+l)}(\lambda),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(M^\lambda)_{i,j} &= \left(\int \exp(\lambda,-x,k)\phi(x)\mathrm{d}x\right)_{j,i}\\
&= -\mathbb{1}_{\{j\ge i\}}\mathcal{L}\mu^{(k+j-i)}(\lambda)\cdot\binom{j-1}{i-1}\frac{(j-i)!(k-1)!}{(j-i+k)!}\\
&= -\mathbb{1}_{\{j\ge i\}}\frac{(j-1)!(k-1)!}{(i-1)!(j-i+k)!}(\mathcal{L}\mu)^{(k+j-i)}(\lambda).
\end{aligned}
$$

In the non-lattice case, invoking Faulhaber's formula, we have

$$
\begin{aligned}
\sum_{x\in\mathbb{Z}}(-x)^l e^{-\lambda x}\phi(x) &= (-1)^l\mathbb{E}\left[\sum_{j=1}^N \sum_{0\le x<X_j} x^l (x-X_j)^{k-1} e^{-\lambda X_j}\right]\\
&= (-1)^l \int \sum_{0\le x<y} x^l (x-y)^{k-1} e^{-\lambda y}\,\mu(\mathrm{d}y)\\
&= (-1)^l \int \sum_{m=1}^k \binom{k-1}{m-1}(-y)^{k-m}\sum_{0\le x<y} x^{l+m-1} e^{-\lambda y}\,\mu(\mathrm{d}y)\\
&= (-1)^l \int \sum_{m=1}^k \binom{k-1}{m-1}(-y)^{k-m}\frac{B_{l+m}(y)-B_{l+m}(0)}{l+m} e^{-\lambda y}\,\mu(\mathrm{d}y)\\
&= P_{k,l}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\mathcal{L}\mu(z)\Big|_{z=\lambda},
\end{aligned}
$$

which gives

$$
\begin{aligned}
(M^\lambda)_{i,j} &= \left(\int \exp(\lambda,-x,k)\phi(x)\mathrm{d}x\right)_{j,i}\\
&= \mathbb{1}_{\{j\ge i\}}\binom{j-1}{i-1}P_{k,j-i}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\mathcal{L}\mu(z)\Big|_{z=\lambda}.
\end{aligned}
$$

□

8. Discussion and open problems. In this section we formulate several open problems which are closely related to the present framework.

OPEN PROBLEM 1. Prove a corresponding limit theorem for the multitype CMJ process.

OPEN PROBLEM 2. Provide functional versions of the theorems proved in this paper.

A drawback of our method in the non-lattice case is that, in order to find the asymptotic of the mean m_t^φ we need to assume that the measure μ is absolutely continuous with respect to Lebesgue measure or that at least (7.7) holds.

OPEN PROBLEM 3. In the non-lattice case, work out a proof that does not require (7.7).

The Gaussian fluctuations appearing in our theorems are caused by the finiteness of the second moment (2.9). In the case where the condition is not satisfied one still may ask for a generalization.

OPEN PROBLEM 4. Prove a version of the limit theorems with a stable limit.

One of the basic ingredients of the CMJ process is the underlying branching random walk $(S(u))_{u\in\mathcal{T}}$ with positive increments. However, the process $\mathcal{Z}^\varphi$ can also be defined for a branching random walk with two-sided increments and suitable φ.

OPEN PROBLEM 5. Investigate the behavior of $\mathcal{Z}_t^\varphi$ for a branching random walk $(S(u))_{u\in\mathcal{T}}$ with two-sided increments.

A central limit theorem is usually complemented by a law of the iterated logarithm (see, for instance, [24] for the central limit theorem and the law of the iterated logarithm for Nerman's martingale). This motivates the following problem.

OPEN PROBLEM 6. Prove a corresponding law of the iterated logarithm for $\mathcal{Z}_t^\varphi$.

The martingale limits $W^{(j)}(\lambda)$ play an important role in the asymptotic behavior of the general branching process. It is important to obtain more information about their distributions. In particular, the following problem seems to be quite relevant.

OPEN PROBLEM 7. Derive the first-order asymptotic behavior of the tail probabilities $\mathbb{P}(|W^{(j)}(\lambda)|>t)$ as $t\to\infty$ for $j=0,\dots,k(\lambda)-1$.

We also believe that the approach developed in the present paper might be useful for settling the following.

OPEN PROBLEM 8. Find large deviation estimates for $\mathcal{Z}_t^\varphi$.

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APPENDIX: AN AUXILIARY RESULT

LEMMA A.1. *Let $r\in\mathbb{N}$, $k_j\in\mathbb{N}_0$, $j=1,\dots,r$, $z_1,\dots,z_r$ be distinct complex numbers with $|z_j|\geq\rho$ and $b_{j,l}$, $j=1,\dots,r$, $l=0,\dots,k_j$ complex numbers. Then*

$$\sum_{j=1}^{r}\sum_{l=0}^{k_j} b_{j,l}n^l z_j^n = o(\rho^n) \quad \text{as } n\to\infty,\ n\in\mathbb{N} \tag{A.1}$$

implies $b_{j,l}=0$ for all $j=1,\dots,r$, $l=0,\dots,k_j$.

PROOF. We use induction on $K := k_1 + \cdots + k_r$. Suppose that $K = 0$ and denote by $V(z_1, \ldots, z_r) := (z_j^{m-1})_{m,j=1,\ldots,r}$ the Vandermonde matrix associated with $z_1, \ldots, z_r$. Then, putting $b_j := b_{j,0}$, each component of the vector

$$V(z_1, \ldots, z_r) \begin{pmatrix} b_1 z_1^n \\ \vdots \\ b_r z_r^n \end{pmatrix} = \left(\sum_{j=1}^{r} b_j z_j^{m-1} z_j^n \right)_{m=1,\ldots,r}$$

is $o(\rho^n)$ as $n \to \infty$. Since $z_1, \ldots, z_r$ are distinct, $\det V(z_1, \ldots, z_r) \neq 0$, hence we may multiply the last displayed equation by the inverse of $V(z_1, \ldots, z_r)$ from the left and conclude that $b_j z_j^n = o(\rho^n)$ as $n \to \infty$, which, in turn, gives $b_j = 0$ for $j = 1, \ldots, r$. For the induction step, we assume that the induction hypothesis holds whenever $k_1 + \ldots + k_r \leq K$. If now $k_1 + \ldots + k_r = K + 1$, then there is exists some $j_0 \in \{1, \ldots, r\}$ with $k_{j_0} > 0$. We define a linear operator L by

$$Lf(n) := f(n) - z_{j_0} f(n-1)$$

for any $f : \mathbb{Z} \mapsto \mathbb{C}$. If $f(n) = o(\rho^n)$, then so is $Lf(n)$. Moreover, if $f(n) = p(n) z^n$ for some polynomial p, then $Lf(n) = \tilde{p}(n) z^n$ for another polynomial $\tilde{p}$ with $\deg \tilde{p} \leq \deg p$ and if $f(n) = n^l z_{j_0}^n$, then $Lf(n) = (l n^{l-1} + p(n)) z_{j_0}^n$ for some polynomial p with $\deg p \leq l - 2$. Applying L to both sides of the relation (A.1) we infer

$$\sum_{j=1}^{r} \sum_{l=0}^{\tilde{k}_j} \tilde{b}_{j,l} n^l z_j^n = o(\rho^n) \quad \text{as } n \to \infty,\ n \in \mathbb{N}$$

for some $\tilde{k}_j \leq k_j$, $\tilde{k}_{j_0} = k_{j_0} - 1$ and $\tilde{b}_{j_0, k_{j_0}-1} = k_{j_0} b_{j_0, k_{j_0}}$. The induction hypothesis gives that $\tilde{b}_{j_0, k_{j_0}-1} = 0$ which implies that $b_{j_0, k_{j_0}} = 0$ as well. This allows us to replace k_{j_0} by $k_{j_0} - 1$ in (A.1). The claim now follows by induction. □

REFERENCES

[1] ALDOUS, D. J. and EAGLESON, G. K. (1978). On mixing and stability of limit theorems. *Ann. Probability* **6** 325–331. MR517416

[2] ALSMEYER, G. (1991). *Erneuerungstheorie. Teubner Skripten zur Mathematischen Stochastik. [Teubner Texts on Mathematical Stochastics]*. B. G.Teubner, Stuttgart Analyse stochastischer Regenerationsschemata. [Analysis of stochastic regeneration schemes]. MR1119301

[3] ASMUSSEN, S. and HERING, H. (1983). *Branching processes. Progress in Probability and Statistics* **3**. Birkhäuser Boston, Inc., Boston, MA. MR701538

[4] ATHREYA, K. B. (1969). Limit theorems for multitype continuous time Markov branching processes. I. The case of an eigenvector linear functional. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **12** 320–332. MR254927

[5] ATHREYA, K. B. (1969). Limit theorems for multitype continuous time Markov branching processes. II. The case of an arbitrary linear functional. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **13** 204–214. MR254928

[6] ATHREYA, K. B., GHOSH, A. P. and SETHURAMAN, S. (2008). Growth of preferential attachment random graphs via continuous-time branching processes. *Proc. Indian Acad. Sci. Math. Sci.* **118** 473–494. MR2450248

[7] ATHREYA, K. B., MCDONALD, D. and NEY, P. (1978). Limit theorems for semi-Markov processes and renewal theory for Markov chains. *Ann. Probab.* **6** 788–797. MR503952

[8] BALL, F., GONZÁLEZ, M., MARTÍNEZ, R. and SLAVTCHOVA-BOJKOVA, M. (2014). Stochastic monotonicity and continuity properties of functions defined on Crump-Mode-Jagers branching processes, with application to vaccination in epidemic modelling. *Bernoulli* **20** 2076–2101. MR3263099

[9] BHAMIDI, S., STEELE, J. M. and ZAMAN, T. (2015). Twitter event networks and the superstar model. *Ann. Appl. Probab.* **25** 2462–2502. MR3375881

[10] BHAMIDI, S., VAN DER HOFSTAD, R. and KOMJÁTHY, J. (2014). The front of the epidemic spread and first passage percolation. *J. Appl. Probab.* **51A** 101–121. MR3317353

[11] BRITTON, T., LEUNG, K. Y. and TRAPMAN, P. (2019). Who is the infector? General multi-type epidemics and real-time susceptibility processes. *Adv. in Appl. Probab.* **51** 606–631. MR3989528

[12] BRITTON, T. and TOMBA, G. S. (2019). Estimation in emerging epidemics: biases and remedies. *Journal of The Royal Society Interface* **16** 20180670.

[13] CHARMOY, P. H. A., CROYDON, D. A. and HAMBLY, B. M. (2017). Central limit theorems for the spectra of classes of random fractals. *Trans. Amer. Math. Soc.* **369** 8967–9013. MR3710650

[14] DEVROYE, L. (1987). Branching processes in the analysis of the heights of trees. *Acta Inform.* **24** 277–298. MR894557

[15] DURRETT, R. (2015). *Branching process models of cancer*. *Mathematical Biosciences Institute Lecture Series. Stochastics in Biological Systems* **1**. Springer, Cham; MBI Mathematical Biosciences Institute, Ohio State University, Columbus, OH. MR3363681

[16] GATZOURAS, D. (2000). On the lattice case of an almost-sure renewal theorem for branching random walks. *Adv. in Appl. Probab.* **32** 720–737. MR1788092

[17] HACCOU, P., JAGERS, P. and VATUTIN, V. A. (2007). *Branching processes: variation, growth, and extinction of populations*. *Cambridge Studies in Adaptive Dynamics* **5**. Cambridge University Press, Cambridge; IIASA, Laxenburg. MR2429372

[18] HALL, P. and HEYDE, C. C. (1980). *Martingale limit theory and its application*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London Probability and Mathematical Statistics. MR624435

[19] HENRY, B. (2017). Central limit theorem for supercritical binary homogeneous Crump-Mode-Jagers processes. *ESAIM Probab. Stat.* **21** 113–137. MR3716122

[20] HEYDE, C. C. (1970). A rate of convergence result for the super-critical Galton-Watson process. *J. Appl. Probability* **7** 451–454. MR288859

[21] HILDEBRANDT, T. H. (1940). On unconditional convergence in normed vector spaces. *Bulletin of the American Mathematical Society* **46** 959 – 962. MR0003448

[22] HOLMGREN, C. and JANSON, S. (2017). Fringe trees, Crump-Mode-Jagers branching processes and m-ary search trees. *Probab. Surv.* **14** 53–154. MR3626585

[23] IKSANOV, A. and KABLUCHKO, Z. (2018). A functional limit theorem for the profile of random recursive trees. *Electron. Commun. Probab.* **23** Paper No. 87, 13. MR3882228

[24] IKSANOV, A., KOLESKO, K. and MEINERS, M. (2021). Gaussian fluctuations and a law of the iterated logarithm for Nerman's martingale in the supercritical general branching process. *Electron. J. Probab.* **26** Paper No. 160, 22. MR4355681

[25] JAGERS, P. (1975). *Branching processes with biological applications*. Wiley-Interscience [John Wiley & Sons], London-New York-Sydney Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics. MR0488341

[26] JAGERS, P. (1989). General branching processes as Markov fields. *Stochastic Process. Appl.* **32** 183–212. MR1014449

[27] JAGERS, P. and NERMAN, O. (1984). Limit theorems for sums determined by branching and other exponentially growing processes. *Stochastic Process. Appl.* **17** 47–71. MR738768

[28] JAGERS, P. and NERMAN, O. (1984). The growth and composition of branching populations. *Adv. in Appl. Probab.* **16** 221–259. MR742953

[29] JANSON, S. (2004). Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Process. Appl.* **110** 177–245. MR2040966

[30] JANSON, S. (2018). Asymptotics of fluctuations in Crump-Mode-Jagers processes: the lattice case. *Adv. in Appl. Probab.* **50** 141–171. MR3905097

[31] JANSON, S. and NEININGER, R. (2008). The size of random fragmentation trees. *Probab. Theory Related Fields* **142** 399–442. MR2438697

[32] JOG, V. and LOH, P.-L. (2017). Analysis of centrality in sublinear preferential attachment trees via the Crump-Mode-Jagers branching process. *IEEE Trans. Network Sci. Eng.* **4** 1–12. MR3625951

[33] KALLENBERG, O. (2002). *Foundations of modern probability*, second ed. *Probability and its Applications (New York)*. Springer-Verlag, New York. MR1876169

[34] KIMMEL, M. and AXELROD, D. E. (2015). *Branching processes in biology*, second ed. *Interdisciplinary Applied Mathematics* **19**. Springer, New York. MR3310028

[35] LECKEY, K., MITSCHE, D. and WORMALD, N. (2020). The height of depth-weighted random recursive trees. *Random Structures Algorithms* **56** 851–866. MR4084192

[36] MEINERS, M. (2010). An almost-sure renewal theorem for branching random walks on the line. *J. Appl. Probab.* **47** 811–825. MR2731350

[37] MÓRI, T. F. and ROKOB, S. (2019). Moments of general time dependent branching processes with applications. *Acta Math. Hungar.* **159** 131–149. MR4003699

[38] NERMAN, O. (1979). On the Convergence of Supercritical General Branching Processes, PhD thesis, Chalmers University of Technology and the University of Göteborg.

[39] NERMAN, O. (1981). On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57** 365–395. MR629532

[40] OLOFSSON, P. (1998). The $x \log x$ condition for general branching processes. *J. Appl. Probab.* **35** 537–544. MR1659492

[41] OLOFSSON, P. and SINDI, S. S. (2014). A Crump-Mode-Jagers branching process model of prion loss in yeast. *J. Appl. Probab.* **51** 453–465. MR3217778

[42] PITTEL, B. (1994). Note on the heights of random recursive trees and random m-ary search trees. *Random Structures Algorithms* **5** 337–347. MR1262983

[43] RESNICK, S. (1992). *Adventures in stochastic processes*. Birkhäuser Boston, Inc., Boston, MA. MR1181423

[44] RESNICK, S. I. (2014). *A probability path*. *Modern Birkhäuser Classics*. Birkhäuser/Springer, New York Reprint of the fifth (2005) printing of the 1999 original [MR1664717]. MR3135152

[45] RUDAS, A. and TÓTH, B. (2009). Random tree growth with branching processes—a survey. In *Handbook of large-scale random networks*. *Bolyai Soc. Math. Stud.* **18** 171–202. Springer, Berlin. MR2582389

[46] SHI, Z. (2015). *Branching random walks*. *Lecture Notes in Mathematics* **2151**. Springer, Cham Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. MR3444654

[47] TRAPMAN, P., BALL, F., DHERSIN, J.-S., TRAN, V. C., WALLINGA, J. and BRITTON, T. (2016). Inferring R_0 in emerging epidemics—the effect of common population structure is small. *Journal of The Royal Society Interface* **13** 20160288.

[48] WIDDER, D. V. (1941). *The Laplace Transform*. *Princeton Mathematical Series, vol. 6*. Princeton University Press, Princeton, N. J. MR0005923