A law of the iterated logarithm for iterated random walks, with application to random recursive trees

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Abstract

Consider a Crump-Mode-Jagers process generated by an increasing random walk whose increments have finite second moment. Let $Y_k(t)$ be the number of individuals in generation $k \in \mathbb{N}$ born in the time interval $[0, t]$. We prove a law of the iterated logarithm for $Y_k(t)$ with fixed k, as $t \to +\infty$. As a consequence, we derive a law of the iterated logarithm for the number of vertices at a fixed level k in a random recursive tree, as the number of vertices goes to ∞.

Key words: Crump-Mode-Jagers process; law of the iterated logarithm; profile; random recursive tree; standard random walk
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1 A law of the iterated logarithm for iterated random walks

Let $\xi_1, \xi_2, \ldots$ be independent copies of an almost surely (a.s.) positive random variable ξ. Denote by $S := (S_n)_{n\in\mathbb{N}}$ the *standard random walk* with increments ξ_n for $n \in \mathbb{N}$, that is, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$. The corresponding *renewal process* $(Y(t))_{t\geq 0}$ is defined by

$$Y(t) := \sum_{n\geq 1} \mathbb{1}_{\{S_n \leq t\}}, \quad t \geq 0.$$

Put $V(t) := \mathbb{E}Y(t)$ for $t \geq 0$. The function V is called *renewal function*.

Now we recall the construction of a general branching process (a.k.a. Crump-Mode-Jagers process) generated by S. There is a population of individuals initiated at time 0 by one individual, the ancestor. An individual born at time $t \geq 0$ produces offspring whose birth times have the same distribution as $(t + S_n)_{n\in\mathbb{N}}$. All individuals act independently

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of each other. For $k \in \mathbb{N}$, an individual resides in the kth generation if it has exactly k ancestors. For $k \in \mathbb{N}$ and $t \geq 0$, denote by $S^{(k)}$ the collection of the birth times in the kth generation and by $Y_k(t)$ the number of the kth generation individuals with birth times $\leq t$. Put $V_k(t) := \mathbb{E}Y_k(t)$. Plainly, $Y_1(t) = Y(t)$ and $V_1(t) = V(t)$ for $t \geq 0$. Following [2, 14] we call the sequence $(S^{(k)})_{k\geq 2}$ an *iterated standard random walk*.

Let (ξ_1, η_1), $(\xi_2, \eta_2), \dots$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ, η) with positive arbitrarily dependent components. Put

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

The sequence $T := (T_n)_{n\in\mathbb{N}}$ is called *(globally) perturbed random walk*. A counterpart of $(S^{(k)})_{k\geq 2}$, obtained by replacing in the aforementioned construction S with T, is called an *iterated perturbed random walk*.

Now we review some previous work on iterated random walks and more general processes. Denote by D the Skorokhod space of càdlàg functions on $[0, \infty)$.

- Iterated standard random walks.

  - Theorem 1.3 in [10] is a functional central limit theorem (FCLT) for $(Y_k)_{k\in\mathbb{N}}$, properly scaled, normalized and centered, on $D^{\mathbb{N}}$ equipped with the product J_1-topology, under the assumption $\mathbb{E}\xi^2 < \infty$.

  - Theorem 2.6 in [12] derives the asymptotics of $\operatorname{Var} Y_k(t)$ as $t \to \infty$ when $k \in \mathbb{N}$ is fixed under the assumptions that $\mathbb{E}\xi^2 < \infty$ and that the distribution of ξ is nonlattice (see Section 3 for the definition).

  - Theorems 2.1 and 2.2 in [11] prove weak convergence of finite-dimensional distributions $(Y_{\lfloor k(t)u\rfloor}(t))_{u>0}$, properly normalized and centered, under the assumption that the distribution of ξ is exponential. Here, k is a positive function satisfying $k(t) \to +\infty$ and $k(t) = o(t)$ as $t \to \infty$.

- Iterated perturbed random walks.

  -Theorem 2.8 in [14] is a FCLT for a counterpart of $(Y_k)_{k\in\mathbb{N}}$, properly scaled, normalized and centered, on $D^{\mathbb{N}}$ equipped with the product J_1-topology, under the assumptions $\mathbb{E}\xi^2 < \infty$ and $\mathbb{E}\eta^a < \infty$ for some $a > 0$. Also, this paper proves the elementary renewal theorem and its refinement, Blackwell's theorem and the key renewal theorem for a counterpart of V_k with $k \in \mathbb{N}$ fixed.

  - The paper [2] proves the elementary renewal theorem, Blackwell's theorem and the key renewal theorem for a counterpart of $V_{k(t)}(t)$ under various assumptions imposed on the distributions of ξ and η. Here, k is an integer-valued function satisfying $k(t) \to +\infty$ and $k(t) = o(t^{2/3})$ as $t \to \infty$. The most interesting observation of the cited paper is that, under the assumptions that $\mathbb{E}\xi^3 < \infty$, that $\mathbb{E}\eta^2 < \infty$ and that the distribution of ξ is spread out, the asymptotics of $V_{k(t)}(t)$ exhibits a phase transition at generations $k(t)$ of order $t^{1/2}$.

  - Proposition 3.1 and Theorems 3.2 and 3.3 in [3] prove weak convergence of the finite-dimensional distributions for a counterpart of $(Y_{\lfloor k(t)u\rfloor}(t))_{u>0}$, properly normalized and centered, under the assumptions $\mathbb{E}\xi^2 < \infty$, $\mathbb{E}\eta < \infty$ and that the distribution of ξ is nonlattice. Here, k is a positive function satisfying $k(t) \to +\infty$ and $k(t) = o(t^{1/3})$ as $t \to \infty$.

- Theorems 3.1 and 3.2 in [13] provide an improvement over the aforementioned result from [3], in which the distribution of ξ is not required to be nonlattice and, more importantly, the condition $k(t) = o(t^{1/3})$ as $t \to \infty$ is replaced with $k(t) = o(t^{1/2})$.

- Theorem 1 in [12] proves weak convergence of the finite-dimensional distributions for a counterpart of $(Y_{\lfloor k(t)u \rfloor}(t))_{u>0}$, properly normalized and centered, under the assumptions that $\mathbb{E}\xi^2 = \infty$, that the distribution of ξ belongs to the domain of attraction of an α-stable distribution, $\alpha \in (1, 2]$ and that $\mathbb{E}\min(\eta, t) = O(t^{2-\gamma})$ as $t \to \infty$ for some $\gamma \in (1, 2)$ specified in the paper. Here, k is a positive function satisfying $k(t) \to +\infty$ and $k(t) = o(t^{(\gamma-1)/2})$ as $t \to \infty$.

- More general iterated sequences.

  - Theorem 3.2 in [6] is a FCLT for a counterpart of $(Y_k)_{k\in\mathbb{N}}$, properly scaled, normalized and centered, on $D^{\mathbb{N}}$ equipped with the product J_1-topology, under the assumption that an appropriate FCLT holds for Y_1 and some further assumptions.

For a family (x_t) of real numbers we write $C((x_t))$ for the set of its limit points. Recall that $0! = 1$. Now we state a law of the iterated logarithm for Y_k for fixed $k \in \mathbb{N}$.

Theorem 1.1. *Assume that $\sigma^2 := \operatorname{Var}\xi \in (0, \infty)$. Then, for each fixed $k \in \mathbb{N}$,*

$$C\Bigg(\Bigg(\frac{a_k\big(Y_k(t) - t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} : t > \mathrm{e}\Bigg)\Bigg) = [-1, 1] \quad \text{a.s.},$$

where

$$a_k := \sigma^{-1}\mu^{k+1/2}(k-1)!(2k-1)^{1/2}$$

and $\mu := \mathbb{E}\xi < \infty$. In particular,

$$\limsup_{t\to\infty} \frac{a_k\big(Y_k(t) - t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} = 1 \quad \text{a.s.}$$

and

$$\liminf_{t\to\infty} \frac{a_k\big(Y_k(t) - t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} = -1 \quad \text{a.s.}$$

The centering $t^k/(k!\mu^k)$ can be replaced with $\mathbb{E}Y_k(t)$ everywhere.

2 Application to random recursive trees

In this section we state a law of the iterated logarithm for the profile of the random recursive tree (RRT) and prove it using Theorem 1.1 in the special case when the random variable ξ has an exponential distribution of unit mean. For our purposes, the following continuous-time construction of the RRT is convenient (see, e.g., Example 6.1 in [8]). At time 0, the RRT consists of 1 vertex, the root, located at level 0. This vertex generates offspring at arrival times of a unit intensity Poisson process. These offspring are located at level 1. More generally, each vertex of the tree, immediately after its birth, starts to generate offspring at rate 1, and all vertices act independently. If some vertex is located at level k, then its offspring appear at level $k+1$, so that the level of any vertex is its distance to the root. Clearly, one can identify the birth times of the vertices at level $k \in \mathbb{N}$ with the process $S^{(k)}$, as defined in Section 1, with the random variable ξ having an exponential distribution of unit mean. Let $\tau_1 < \tau_2 < \ldots$ be the birth times of the vertices of the RRT, excluding the root born at time $\tau_0 = 0$. For $n \in \mathbb{N}$, at time τ_n, the tree consists of $n+1$ vertices. For $k \in \mathbb{N}$, let $X_n(k) = Y_k(\tau_n)$ be the number of vertices in this tree having distance k to the root at time τ_n. The function $k \mapsto X_n(k)$ is called the profile of the RRT. Its asymptotic behavior as $n \to \infty$ has been much studied. For example, a central limit theorem for $X_n(k)$ with fixed k has been obtained in [5]; see also [10] for a functional version. As a corollary of Theorem 1.1 we shall prove the following law of the iterated logarithm for $X_n(k)$.

Theorem 2.1. *For each fixed $k \in \mathbb{N}$,*

$$C\bigg(\bigg(\frac{(k-1)!(2k-1)^{1/2}\big(X_n(k)-(\log n)^k/k!\big)}{(2(\log n)^{2k-1}\log\log\log n)^{1/2}} : n > \mathrm{e}^{\mathrm{e}}\bigg)\bigg) = [-1,1] \quad \text{a.s.}$$

For $k=1$, the claim is known (see Theorem 3' in [19]) since the sequence $(X_n(1))_{n\in\mathbb{N}}$ has the same joint distribution as $(B_1 + \ldots + B_n)_{n\in\mathbb{N}}$, where $B_1, B_2, \ldots$ are independent Bernoulli random variables with $\mathbb{P}\{B_k = 1\} = 1/k$.

Proof of Theorem 2.1. Let ξ be a random variable having an exponential distribution of unit mean. Then $a_k = (k-1)!(2k-1)^{1/2}$ since $\mu = \sigma^2 = 1$, and Theorem 1.1 takes the form

$$C\bigg(\bigg(\frac{a_k\big(Y_k(t)-t^k/k!\big)}{(2t^{2k-1}\log\log t)^{1/2}} : t > \mathrm{e}\bigg)\bigg) = [-1,1] \quad \text{a.s.} \tag{1}$$

For $t \geq 0$, let $n(t) \in \{0,1,\ldots\}$ be the unique index with $\tau_{n(t)} \leq t < \tau_{n(t)+1}$. Then, $(n(t)+1)_{t\geq 0}$ is the Yule process for which it is known (see Theorems 1 and 2 on pp. 111-112 in [1]) that $\lim_{t\to\infty} \mathrm{e}^{-t} n(t) = W$ a.s., where W is a random variable satisfying $W > 0$ a.s. It follows that $\lim_{t\to\infty}(\log n(t) - t) = \log W$ a.s. and $\lim_{t\to\infty} t^{-1}\log n(t) = 1$ a.s. Consequently,

$$\frac{t^k - (\log n(t))^k}{t^{k-1}} = (t - \log n(t)) \cdot \frac{\sum_{j=0}^{k-1} t^j (\log n(t))^{k-1-j}}{t^{k-1}} \underset{t\to\infty}{\longrightarrow} -k \cdot \log W \quad \text{a.s.}$$

Note that $Y_k(\tau_{n(t)}) = Y_k(t)$. The identity

$$\frac{Y_k(\tau_{n(t)}) - (\log n(t))^k/k!}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}} = \bigg(\frac{Y_k(t)-t^k/k!}{(2t^{2k-1}\log\log t)^{1/2}} + \frac{t^k - (\log n(t))^k}{k!(2t^{2k-1}\log\log t)^{1/2}}\bigg) \times \frac{(2t^{2k-1}\log\log t)^{1/2}}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}},$$

in which

$$\lim_{t\to\infty} \frac{(2t^{2k-1}\log\log t)^{1/2}}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^k - (\log n(t))^k}{k!(2t^{2k-1}\log\log t)^{1/2}} = 0 \quad \text{a.s.},$$

combined with (1) entails that

$$C\bigg(\bigg(\frac{a_k\big(Y_k(\tau_{n(t)})-(\log n(t))^k/k!\big)}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}}:t\geq \tau_{\lceil \mathrm{e}^{\mathrm{e}}\rceil}\bigg)\bigg)=[-1,1]\quad \text{a.s.}$$

It follows that

$$C\bigg(\bigg(\frac{a_k\big(Y_k(\tau_n)-(\log n)^k/k!\big)}{(2(\log n)^{2k-1}\log\log\log n)^{1/2}}:n> \mathrm{e}^{\mathrm{e}}\bigg)\bigg)=[-1,1]\quad \text{a.s.}$$

This completes the proof of Theorem 2.1 since $X_n(k)=Y_k(\tau_n)$. □

3 Auxiliary results

For the proof of Theorem 1.1 we shall need the following strong approximation result, which follows, for instance, from Theorem 12.13 on p. 227 in [15].

Lemma 3.1. *Assume that $\sigma^2=\mathrm{Var}\,\xi\in(0,\infty)$. Then there exists a standard Brownian motion W such that*

$$\lim_{t\to\infty}\frac{\sup_{0\leq s\leq t}\big|Y(s)-V(s)-\sigma\mu^{-3/2}W(s)\big|}{(t\log\log t)^{1/2}}=0\quad \text{a.s.},$$

where $\mu=\mathbb{E}\xi<\infty$.

Now we lay down the ground for the subsequent proofs. For $t\geq 0$, $k\geq 2$ and $r\in\mathbb{N}$, let $Y_{k-1}^{(r)}(t)$ be the number of successors in the kth generation with birth times within $[S_r,\,S_r+t]$ of the first generation individual with birth time S_r. Then

$$Y_k(t)=\sum_{r\geq 1}Y_{k-1}^{(r)}(t-S_r)\,\mathbb{1}_{\{S_r\leq t\}}\,. \tag{2}$$

By the branching property, $(Y_{k-1}^{(1)}(t))_{t\geq 0}$, $(Y_{k-1}^{(2)}(t))_{t\geq 0},\ldots$ are independent copies of $(Y_{k-1}(t))_{t\geq 0}$ which are also independent of S. Passing in (2) to expectations we obtain, for $k\geq 2$ and $t\geq 0$,

$$V_k(t)=\int_{[0,\,t]}V_{k-1}(t-y)\mathrm{d}V(y)=\int_{[0,\,t]}V(t-y)\mathrm{d}V_{k-1}(y). \tag{3}$$

Thus, V_k is the k-fold Lebesgue-Stieltjes convolution of V with itself.

For fixed $d>0$, the distribution of a positive random variable is called d-lattice if it is concentrated on the lattice $(nd)_{n\in\mathbb{N}_0}$ and not concentrated on $(nd_1)_{n\in\mathbb{N}_0}$ for any $d_1>d$. The number d is called span of the corresponding lattice distribution. The distribution of a positive random variable is called nonlattice if it is not d-lattice for any $d>0$. Lemma 3.2 collects some properties of $V_k=\mathbb{E}Y_k$.

Lemma 3.2. *Fix any $k\in\mathbb{N}$.*
(a) Assume that $\mu=\mathbb{E}\xi<\infty$. Then

$$\lim_{t\to\infty}\frac{V_k(t)}{t^k}=\frac{1}{k!\mu^k}.$$

(b) Assume that $\mu = \mathbb{E}\xi < \infty$. Then

$$\lim_{t\to\infty} \frac{V_k(t+h) - V_k(t)}{t^{k-1}} = \frac{h}{(k-1)!\mu^k}$$

for each $h > 0$ if the distribution of ξ is nonlattice and $h = id$, $i \in \mathbb{N}$ if the distribution of ξ is d-lattice.
(c) Assume that $\mathbb{E}\xi^2 < \infty$. Then

$$-\infty < \liminf_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} \le \limsup_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} < \infty.$$

(d) For all $x, h \ge 0$,

$$V_k(x+h) - V_k(x) \le (V(h)+1)(V(x+h))^{k-1}. \tag{4}$$

Proof. (a) See, for instance, Theorem 1.16 on p. 38 in [18].
(b) When the distribution of ξ is nonlattice, this is a particular case ($\eta = \xi$) of Theorem 2.4 in [14]. Assume now that, for some $d > 0$, the distribution of ξ is d-lattice. Then $\lim_{t\to\infty}(V_1(t+h) - V_1(t)) = \mu^{-1}h$ for $h = id$, $i \in \mathbb{N}$ by Blackwell's theorem, see Theorem 1.10 in [18]. With this at hand, the same proof by induction as in [14] also works in the lattice case.
(c) Assume that the distribution of ξ is nonlattice. Using Theorem 2.2 in [14], with $\eta = \xi$ in the notation of that paper, we conclude that, for each fixed $k \in \mathbb{N}$,

$$V_k(t) - \frac{t^k}{k!\mu^k} \sim \frac{bkt^{k-1}}{(k-1)!\mu^{k-1}}, \quad t \to \infty,$$

where $b = \mathbb{E}\xi^2/(2\mu^2) - 1 \in \mathbb{R}$.

Assume now that, for some $d > 0$, the distribution of ξ is d-lattice. Then for each $t \ge 0$ there exists $n \in \mathbb{N}_0$ such that $t \in [nd, (n+1)d)$. Hence, by monotonicity,

$$\limsup_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} \le \limsup_{n\to\infty} \frac{V_k((n+1)d) - (nd)^k/(k!\mu^k)}{(nd)^{k-1}}$$
$$= \limsup_{n\to\infty} \frac{V_k((n+1)d) - V_k(nd)}{(nd)^{k-1}} + \limsup_{n\to\infty} \frac{V_k(nd) - (nd)^k/(k!\mu^k)}{(nd)^{k-1}}.$$

The former limit superior (actually, the full limit) is finite according to Lemma 3.2(b), and the latter limit superior (the full limit) is finite according to Lemma 5.1 with $\eta = \xi$. The finiteness of the lower limit follows analogously.
(d) We use mathematical induction in k. If $k = 1$, then (4) expresses a known fact that the renewal function $V + 1$ is subadditive, see, for instance, Theorem 1.7 on p. 10 in [18]. Assume that (4) holds for $k = j$ and note that, by monotonicity and (3), $V_j(h) \le (V(h))^j \le (V(h)+1)(V(x+h))^{j-1}$ for $x, h \ge 0$. We obtain, by another application of (3),

$$V_{j+1}(x+h) - V_{j+1}(x) = \int_{[0,\,x]} (V_j(x+h-y) - V_j(x-y)){\rm d}V(y) + \int_{(x,\,x+h]} V_j(x+h-y){\rm d}V(y)$$
$$\le (V(h)+1) \int_{[0,\,x]} (V(x+h-y))^{j-1}{\rm d}V(y) + V_j(h)(V(x+h) - V(x))$$
$$\le (V(h)+1)(V(x+h))^{j-1}V(x) + (V(h)+1)(V(x+h))^{j-1}(V(x+h) - V(x))$$
$$= (V(h)+1)(V(x+h))^j.$$

□

Here is another important ingredient for our proof of Theorem 1.1.

Lemma 3.3. *Fix any* $k \in \mathbb{N}$. *Assume that* $\operatorname{Var} \xi \in (0, \infty)$. *Then*

$$\mathbb{E} \sup_{0 \le s \le t} (Y_k(s) - V_k(s))^2 = O(t^{2k-1}), \quad t \to \infty. \tag{5}$$

Proof. In the setting of iterated perturbed random walks a counterpart of (5) is a consequence of Lemmas 4.2(b) and 3.1(c) in [6] under the assumptions $\mathbb{E}\xi^2 < \infty$ and $\mathbb{E}\eta < \infty$. Relation (5) itself follows on putting $\eta = \xi$. □

4 Proof of Theorem 1.1

The possibility of replacing $t \mapsto t^k/(k!\mu^k)$ with V_k is justified by Lemma 3.2(c).

Since Y_1 is a renewal process, the case $k = 1$ of Theorem 1.1 was known, see Proposition 3.5 in [9]. Thus, in what follows it is tacitly assumed that $k \ge 2$.

Throughout the proof, for notational simplicity, we assume that if the distribution of ξ is lattice, its lattice span is 1. Using (2) we obtain a basic decomposition for the present proof: for $k \ge 2$ and $t \ge 0$:

$$Y_k(t) - V_k(t) = \sum_{r \ge 1} \big(Y_{k-1}^{(r)}(t - S_r) - V_{k-1}(t - S_r)\big) \mathbb{1}_{\{S_r \le t\}} + \Big(\sum_{r \ge 1} V_{k-1}(t - S_r) \mathbb{1}_{\{S_r \le t\}} - V_k(t)\Big) =: I_k(t) + J_k(t).$$

We shall prove that

$$C\bigg(\bigg(\frac{a_k J_k(t)}{(2t^{2k-1} \log\log t)^{1/2}} : t > \mathrm{e}\bigg)\bigg) = [-1, 1] \quad \text{a.s.} \tag{6}$$

and that

$$\lim_{t \to \infty} \frac{I_k(t)}{(t^{2k-1} \log\log t)^{1/2}} = 0 \quad \text{a.s.}, \tag{7}$$

that is, the term J_k gives the principal contribution, whereas the contribution of I_k is negligible.

First, we deal with (6). Recalling (3) write, with the help of integration by parts, for $k \ge 2$ and $t \ge 0$,

$$\begin{aligned} J_k(t) &= \int_{[0,\,t]} V_{k-1}(t - x) \mathrm{d}(Y(x) - V(x)) = \int_{[0,\,t)} (Y(t - x) - V(t - x)) \mathrm{d}V_{k-1}(x) \\ &= \int_{[0,\,t)} (Y(t - x) - V(t - x) - \sigma\mu^{-3/2} W(t - x)) \mathrm{d}V_{k-1}(x) \\ &+ \sigma\mu^{-3/2} \int_{[0,\,t)} W(t - x) \mathrm{d}V_{k-1}(x)\Big) =: A_k(t) + \sigma\mu^{-3/2} B_k(t), \end{aligned}$$

where W is a standard Brownian motion appearing in Lemma 3.1. By Lemmas 3.1 and 3.2(a),

$$|A_k(t)| \le \sup_{0\le u\le t} |Y(u) - V(u) - \sigma\mu^{-3/2}W(u)| V_{k-1}(t) = o\big((t^{2k-1}\log\log t)^{1/2}\big), \quad t\to\infty \quad \text{a.s.}$$

Further,

$$B_k(t) = \frac{1}{(k-1)!\mu^{k-1}}\int_{(0,\,t]}(t-x)^{k-1}\mathrm{d}W(x) + \int_{(0,\,t]}\Big(V_{k-1}(t-x) - \frac{(t-x)^{k-1}}{(k-1)!\mu^{k-1}}\Big)\mathrm{d}W(x) =: ((k-1)!\mu^{k-1})^{-1}B_{1,k}(t) + B_{2,k}(t).$$

We intend to prove that

$$\lim_{t\to\infty}\frac{B_{2,k}(t)}{t^{k-1/2}} = 0 \quad \text{a.s.}$$

To this end, it suffices to show that, for all $\varepsilon > 0$,

$$\sum_{n\ge 1}\mathbb{P}\{\sup_{t\in[n,\,n+1]}|B_{2,k}(t)| > \varepsilon n^{k-1/2}\} < \infty. \tag{8}$$

Indeed, if this is true, then, by the Borel–Cantelli lemma,

$$\sup_{t\in[n,\,n+1]}|B_{2,k}(t)| \le \varepsilon n^{k-1/2}$$

for n large enough a.s. Hence, for all large enough n and $t\in[n,\,n+1]$,

$$|B_{2,k}(t)| \le \sup_{t\in[n,\,n+1]}|B_{2,k}(t)| \le \varepsilon n^{k-1/2} \le \varepsilon t^{k-1/2} \quad \text{a.s.}$$

Thus, $\limsup_{t\to\infty}|B_{2,k}(t)|/t^{k-1/2} \le \varepsilon$ a.s. which entails the claim.

Let us prove (8). In what follows C_1, $C_2, \dots$ will denote positive constants, whose values are of no importance. Put $f_k(t) := V_{k-1}(t) - ((k-1)!\mu^{k-1})^{-1}t^{k-1}$ for $k\ge 2$ and $t\ge 0$. Write

$$\begin{aligned}\sup_{t\in[n,\,n+1]}|B_{2,k}(t) - B_{2,k}(n)| &= \sup_{t\in[0,\,1]}\Big|\int_{(0,\,n+t]} f_k(n+t-x)\mathrm{d}W(x) - \int_{(0,\,n]} f_k(n-x)\mathrm{d}W(x)\Big| \\ &= \sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]} f_k(n+t-x)\mathrm{d}W(x) + \int_{(0,\,n]}(f_k(n+t-x) - f_k(n-x))\mathrm{d}W(x)\Big| \\ &\le \sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]} f_k(n+t-x)\mathrm{d}W(x)\Big| + \sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(n+t-x) - f_k(n-x))\mathrm{d}W(x)\Big|.\end{aligned}$$

Note that the variable $B_{2,k}(n)$ has a normal distribution with zero mean and variance $\int_0^n (f_k(x))^2\mathrm{d}x$. By Lemma 3.2(c), for large enough n, $\int_0^n (f_k(x))^2\mathrm{d}x \le C_1 n^{2k-3}$. Hence, for all $\varepsilon > 0$ and large n,

$$\mathbb{P}\{|B_{2,k}(n)| > \varepsilon n^{k-1/2}\} \le \Big(\frac{2}{\pi}\Big)^{1/2}\int_{\varepsilon C_1^{-1/2} n}^{\infty} \mathrm{e}^{-x^2/2}\mathrm{d}x \le \Big(\frac{2C_1}{\varepsilon^2\pi}\Big)^{1/2}\frac{\mathrm{e}^{-\varepsilon^2 n^2/(2C_1)}}{n}.$$

The right-hand side is the nth term of a summable sequence.

Observe that the process $B_{2,k}$ is a.s. continuous. Indeed,

$$B_{2,k}(t)=\int_{[0,\,t)}W(t-x)\mathrm{d}V_{k-1}(x)-\frac{1}{(k-1)!\mu^{k-1}}\int_{[0,\,t)}W(t-x)\mathrm{d}x^{k-1},$$

and each of the summands is a.s. continuous as the Lebesgue-Stieltjes convolution of an a.s. continuous function and nondecreasing function. In view of the a.s. continuity, which entails the a.s. boundedness on $[0,1]$, we infer, for all $\varepsilon>0$,

$$-\log\mathbb{P}\{\sup_{t\in[0,\,1]}|B_{2,k}(t)|>\varepsilon n^{k-1/2}\}\ \sim\ \frac{\varepsilon^2 n^{2k-1}}{2\int_0^1 f_k^2(y)\mathrm{d}y},\quad n\to\infty \tag{9}$$

by a large deviation bound for a.s. bounded Gaussian processes, see formula (1.1) in [17]. Since the variable $\sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)\Big|$ has the same distribution as $\sup_{t\in[0,\,1]}|B_{2,k}(t)|$ we conclude that the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)\Big|>\varepsilon n^{k-1/2}\Big\}$$

is summable.

To proceed, we note that the variable $\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(n+t-x)-f_k(n-x))\mathrm{d}W(x)\Big|$ has the same distribution as $\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x)\Big|$. Whenever a Skorokhod integral is well-defined, it coincides with the result of (formal) integration by parts. In particular,

$$\begin{aligned}\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x)&=(f_k(n+t)-f_k(n))W(n)-\int_{(0,\,n]}W(x)\mathrm{d}_x(f_k(x+t)-f_k(x))\\ =(f_k(n+t)-f_k(n))W(n)&+\int_{(0,\,t]}W(x)\mathrm{d}f_k(x)+\int_{(0,\,n-t]}(W(x+t)-W(x))\mathrm{d}f_k(x+t)\\ &-\int_{(n-t,\,n]}W(x)\mathrm{d}f_k(x+t).\end{aligned}$$

Hence, since the function V_{k-1} is nondecreasing,

$$\begin{aligned}\sup_{t\in[0,1]}\Big|\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x)\Big|&\leq(V_{k-1}(n+1)-V_{k-1}(n))|W(n)|\\ +((k-1)!\mu^{k-1})^{-1}((n+1)^{k-1}-n^{k-1})|W(n)|&+\sup_{t\in[0,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|\\ +((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}x^{k-1}\Big|&+\sup_{t\in[0,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)\\ +((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,1]}&\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_x(x+t)^{k-1}\\ +\sup_{t\in[0,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_xV_{k-1}(x+t)&+((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_x(x+t)^{k-1}.\end{aligned} \tag{10}$$

We shall only treat the terms involving V_{k-1}, for the analysis of the terms involving $t \mapsto t^{k-1}$ is analogous but easier. We start with the penultimate term in (10). Observe that

$$\sup_{t\in[0,\,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_x V_{k-1}(x+t)\leq \sup_{t\in[0,\,1]}(V_{k-1}(n+t)-V_{k-1}(n))\sup_{n-t\leq z\leq n}|W(z)|$$
$$\leq (V_{k-1}(n+1)-V_{k-1}(n))\sup_{n-1\leq z\leq n}|W(z)|.$$

By Lemma 3.2(b), for large n, $V_{k-1}(n+1)-V_{k-1}(n)\leq C_2 n^{k-2}$. The random variable $\sup_{z\in[n-1,\,n]}|W(z)|$ has the same distribution as $\sup_{z\in[0,1]}|W(z)+W'(n-1)|$, where $W'(n-1)$ is a copy of $W(n-1)$ which is independent of $\sup_{z\in[0,1]}|W(z)|$. Hence, for all $\varepsilon>0$ and large n,

$$\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[n-1,\,n]}|W(z)|>\varepsilon n^{k-1/2}\}$$
$$\leq \mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[0,1]}|W(z)|>\varepsilon n^{k-1/2}/2\}$$
$$+\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))|W'(n-1)|>\varepsilon n^{k-1/2}/2\}=:R_{n,1}+R_{n,2}.$$

Further,

$$R_{n,2}\leq\Big(\frac{2}{\pi}\Big)^{1/2}\int_{2^{-1}C_2^{-1}\varepsilon n}^{\infty}\mathrm{e}^{-x^2/2}\mathrm{d}x\leq\Big(\frac{8C_2^2}{\varepsilon^2\pi}\Big)^{1/2}\frac{\mathrm{e}^{-\varepsilon^2n^2/(8C_2^2)}}{n}.$$

The right-hand side is the nth term of a summable sequence. Using the inequalities

$$\mathbb{P}\{\sup_{t\in[0,\,1]}|W(t)|>x\}\leq 2\mathbb{P}\{\sup_{t\in[0,\,1]}W(t)>x\}=2\mathbb{P}\{|W(1)|>x\},\quad x>0 \tag{11}$$

we also conclude that the sequence $(R_{n,1})_{n\in\mathbb{N}}$ is summable. Thus, for all $\varepsilon>0$, the sequence

$$n\mapsto\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[n-1,\,n]}|W(z)|>\varepsilon n^{k-1/2}\}$$

is summable, and so is

$$n\mapsto\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))|W(n)|>\varepsilon n^{k-1/2}\}$$

because $|W(n)|\leq\sup_{z\in[n-1,\,n]}|W(z)|$. For all $\varepsilon>0$, the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|>\varepsilon n^{k-1/2}\Big\}$$

is summable in view of the bound

$$\sup_{t\in[0,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|\leq\int_{(0,\,1]}|W(x)|\mathrm{d}V_{k-1}(x)\leq\sup_{t\in[0,\,1]}|W(t)|V_{k-1}(1)$$

and (11).

Finally,

$$\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)\le V_{k-1}(n)\sup_{t\in[0,\,1]}\ \sup_{z\in[0,\,n]}|W(z+t)-W(z)|.$$

By Lemma 3.2(a), $V_{k-1}(n)\le C_3n^{k-1}$ for large n. By Lemma 1.2.1 on p. 29 in [4], given $\delta>0$ there exists $C=C(\delta)>0$ such that, for all $\varepsilon>0$ and $n\ge 2$,

$$\mathbb{P}\{V_{k-1}(n)\sup_{t\in[0,\,1]}\ \sup_{z\in[0,\,n]}|W(z+t)-W(z)|>\varepsilon n^{k-1/2}\}\le C(n+1)\exp\Big(-\frac{\varepsilon^2 n}{C_3^2(2+\delta)}\Big).$$

This proves that the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)>\varepsilon n^{k-1/2}\Big\}$$

is summable. Combining fragments together we arrive at (8).

We are now in position to prove (6). By Theorem 1 in [16],

$$\limsup_{t\to\infty}\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}=1\quad\text{a.s.}$$

Since $-W$ is also a Brownian motion we infer

$$\liminf_{t\to\infty}\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}=-1\quad\text{a.s.}$$

and thereupon

$$C\bigg(\bigg(\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}:t>\mathrm{e}\bigg)\bigg)=[-1,1]\quad\text{a.s.}$$

because the random function $t\mapsto B_{1,k}(t)t^{1/2-k}(\log\log t)^{-1/2}$ is a.s. continuous on (e,∞). This completes the proof of (6).

Now we pass to a proof of (7). Recall that $k\ge 2$. Invoking Lemmas 3.2(a) and 3.3 yields

$$\begin{aligned}\mathbb{E}(I_k(t))^2&=\int_{[0,\,t]}\mathbb{E}\big(Y_{k-1}(t-x)-V_{k-1}(t-x)\big)^2\mathrm{d}V(x)\\&\le\mathbb{E}\big(\sup_{s\in[0,\,t]}(Y_{k-1}(s)-V_{k-1}(s))^2\big)\cdot V(t)=O(t^{2k-2}),\quad t\to\infty.\end{aligned}\tag{12}$$

By Markov's inequality and (12), for all $\varepsilon>0$,

$$\sum_{n\ge1}\mathbb{P}\Big\{\frac{I_k(n^{3/2})}{n^{(3/2)(k-1/2)}}>\varepsilon\Big\}\le\sum_{n\ge1}\frac{\mathbb{E}(I_k(n^{3/2}))^2}{\varepsilon^2n^{3(k-1/2)}}<\infty.$$

Hence,

$$\lim_{n\to\infty}\frac{I_k(n^{3/2})}{n^{(3/2)(k-1/2)}}=0\quad\text{a.s.}\tag{13}$$

It remains to pass from an integer argument to a continuous argument. For any $t \geq 0$, there exists $n \in \mathbb{N}_0$ such that $t \in [n^{3/2}, (n+1)^{3/2})$. By monotonicity,

$$\frac{I_k(t)}{t^{k-1/2}} \leq \frac{I_k((n+1)^{3/2})}{n^{(3/2)(k-1/2)}} + \frac{\int_{[0,\,(n+1)^{3/2}]} V_{k-1}((n+1)^{3/2}-x)\mathrm{d}Y(x) - \int_{[0,\,n^{3/2}]} V_{k-1}(n^{3/2}-x)\mathrm{d}Y(x)}{n^{(3/2)(k-1/2)}}.$$

Relation (13) implies that the first summand on the right-hand side converges to 0 a.s. as $n \to \infty$. The second summand is equal to

$$\int_{(n^{3/2},\,(n+1)^{3/2}]} V_{k-1}((n+1)^{3/2}-x)\mathrm{d}Y(x) + \int_{[0,\,n^{3/2}]} (V_{k-1}((n+1)^{3/2}-x) - V_{k-1}(n^{3/2}-x))\mathrm{d}Y(x) =: X_{k,1}(n) + X_{k,2}(n).$$

By monotonicity,

$$X_{k,1}(n) \leq V_{k-1}((n+1)^{3/2} - n^{3/2})\big(Y((n+1)^{3/2}) - Y(n^{3/2})\big) = o(n^{k/2+1}) = o(n^{(3/2)(k-1/2)}), \quad n \to \infty \quad \text{a.s.}$$

Here, the penultimate equality is justified by the strong law of large numbers for renewal processes $\lim_{n\to\infty} n^{-1}Y(n) = \mu^{-1}$ a.s. and $V_{k-1}((n+1)^{3/2} - n^{3/2}) = O(n^{(k-1)/2})$ as $n \to \infty$ which holds true by Lemma 3.2(a).

Using Lemma 3.2(d) we infer

$$X_{k,2}(n) \leq \big(V((n+1)^{3/2} - n^{3/2}) + 1\big)\big(V((n+1)^{3/2})\big)^{k-2} Y(n^{3/2}) = O(n^{(3/2)(k-2/3)}) = o(n^{(3/2)(k-1/2)}), \quad n \to \infty \quad \text{a.s.}$$

The penultimate equality is secured by Lemma 3.2(a) and the strong law of large numbers for renewal processes.

We have shown that

$$\limsup_{t\to\infty} t^{-(k-1/2)} I_k(t) \leq 0 \quad \text{a.s.}$$

An analogous argument proves the converse inequality for the lower limit. The proof of Theorem 1.1 is complete.

5 Appendix

Lemma 5.1 is a lattice analogue of Theorem 2.2 in [14] dealing with iterated perturbed random walks. In the proof of Lemma 3.2(c) we only need a version of Lemma 5.1 for iterated standard random walks.

Lemma 5.1. *Let $d > 0$. Assume that the distributions of ξ and η are d-lattice and that $\mathbb{E}\xi^2 < \infty$ and $\mathbb{E}\eta < \infty$. Then, for each fixed $k \in \mathbb{N}$,*

$$V_k^*(nd) - \frac{(nd)^k}{k!\mu^k} \sim \frac{(nd)^{k-1}\Big(\frac{d}{2\mu}\big(2k-1\big) + k\big(\frac{\mathbb{E}\xi^2}{2\mu^2} - \frac{\mathbb{E}\eta}{\mu}\big)\Big)}{\mu^{k-1}(k-1)!}, \quad n \to \infty, \qquad (14)$$

where $\mu = \mathbb{E}\xi < \infty$ and V_k^* is a counterpart of V_k for iterated perturbed random walks.

Proof. We use mathematical induction in k. Let $k=1$. Put $U(t):=V(t)+1$ for $t\geq 0$, so that U is the renewal function. Since $V_1^*(t)=\sum_{j\geq 1}\mathbb{P}\{S_{j-1}+\eta_j\leq t\}=:V^*(t)$ for $t\geq 0$, we infer, for $n\in\mathbb{N}$,

$$V^*(nd)-\frac{nd}{\mu}=\int_{[0,\,nd]}\Big(U(nd-x)-\frac{nd-x}{\mu}\Big)\mathrm{d}\mathbb{P}\{\eta\leq x\}-\frac{1}{\mu}\int_0^{nd}\mathbb{P}\{\eta>x\}\mathrm{d}x$$
$$=\sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\}-\frac{1}{\mu}\int_0^{nd}\mathbb{P}\{\eta>x\}\mathrm{d}x.$$

According to formula (5.14) on p. 59 in [7],

$$\lim_{n\to\infty}\Big(U(nd)-\frac{nd}{\mu}\Big)=\frac{d}{2\mu}+\frac{\mathbb{E}\xi^2}{2\mu^2}=:D.$$

Hence, given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $U(nd)-\mu^{-1}nd\leq D+\varepsilon$ for all $n\geq n_0$. With this at hand, for $n\geq n_0$,

$$\sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\}=\sum_{r=1}^{n-n_0}\ldots+\sum_{r=n-n_0+1}^{n}\ldots$$
$$\leq D+\varepsilon+\sup_{1\leq r\leq n_0-1}\Big(U(rd)-\frac{rd}{\mu}\Big)\mathbb{P}\{\eta\geq(n-n_0+1)d\}.$$

Letting $n\to\infty$ and then $\varepsilon\to 0+$ we conclude that

$$\limsup_{n\to\infty}\sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\}\leq D.$$

The converse inequality for the lower limit follows analogously. Noting that $\lim_{n\to\infty}\int_0^{nd}\mathbb{P}\{\eta>x\}\mathrm{d}x=\mathbb{E}\eta$ completes the proof of (14) with $k=1$.

Assume now that (14) holds for $k\leq j$. In particular, given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$V_k^*(nd)-\frac{(nd)^k}{k!\mu^k}\leq(C_k+\varepsilon)\frac{(nd)^{k-1}}{(k-1)!\mu^{k-1}},\quad 1\leq k\leq j,$$

where

$$C_k:=\frac{d}{2\mu}\big(2k-1\big)+k\Big(\frac{\mathbb{E}\xi^2}{2\mu^2}-\frac{\mathbb{E}\eta}{\mu}\Big),\quad k\in\mathbb{N}.$$

Recalling (3) we obtain

$$V_{j+1}^*(nd)-\frac{(nd)^{j+1}}{(j+1)!\mu^{j+1}}=\sum_{r=1}^{n}\Big(V^*((n-r)d)-\frac{(n-r)d}{\mu}\Big)\Big(V_j^*(rd)-V_j^*((r-1)d)\Big)$$
$$+\frac{d}{\mu}\sum_{r=1}^{n-1}\Big(V_j^*(rd)-\frac{(rd)^j}{j!\mu^j}\Big)+\frac{d^{j+1}}{j!\mu^{j+1}}\Big(\sum_{r=1}^{n-1}r^j-\frac{n^{j+1}}{j+1}\Big).$$

Hence, for $n \geq n_0$,

$$A_j(n) := \sum_{r=1}^{n} \Big(V^*((n-r)d) - \frac{(n-r)d}{\mu} \Big) \Big(V_j^*(rd) - V_j^*((r-1)d) \Big) = \sum_{r=1}^{n-n_0} \ldots + \sum_{r=n-n_0+1}^{n} \ldots$$
$$\leq (C_1 + \varepsilon) V_j^*((n-n_0)d) + \sup_{1 \leq r \leq n_0 - 1} \Big(V^*(rd) - \frac{rd}{\mu} \Big) \big(V_j^*(nd) - V_j^*((n-n_0)d) \big).$$

Invoking parts (a) and (b) of Lemma 3.2 and letting $n \to \infty$ and then $\varepsilon \to 0+$ yields

$$\limsup_{n\to\infty} \frac{A_j(n)}{(nd)^j} \leq \frac{C_1}{j!\mu^j}.$$

Further,

$$B_j(n) := \sum_{r=1}^{n-1} \Big(V_j^*(rd) - \frac{(rd)^j}{j!\mu^j} \Big) = \sum_{r=1}^{n_0-1} \ldots + \sum_{r=n_0}^{n} \ldots \leq n_0 \sup_{1 \leq r \leq n_0-1} \Big(V_j^*(rd) - \frac{(rd)^j}{j!\mu^j} \Big)$$
$$+ (C_j + \varepsilon) \sum_{r=n_0}^{n} \frac{(rd)^{j-1}}{(j-1)!\mu^{j-1}} = (C_j + \varepsilon) \frac{d^{j-1}}{(j-1)!\mu^{j-1}} \frac{n^j}{j} + o(n^j), \quad n \to \infty$$

having utilized Faulhaber's formula for the last equality. Thus,

$$\limsup_{n\to\infty} \frac{d}{\mu} \frac{B_j(n)}{(nd)^j} \leq \frac{C_j}{j!\mu^j}.$$

Analogous arguments prove the converse inequalities for the lower limits involving both $A_j(n)$ and $B_j(n)$. Finally,

$$\frac{d^{j+1}}{j!\mu^{j+1}} \Big(\sum_{r=1}^{n-1} r^j - \frac{n^{j+1}}{j+1} \Big) \sim \frac{d}{2\mu} \frac{(nd)^j}{j!\mu^j}, \quad n \to \infty$$

by another application of Faulhaber's formula.

Combining all the fragments together we conclude that

$$V_{j+1}^*(nd) - \frac{(nd)^{j+1}}{(j+1)!\mu^{j+1}} \sim \Big(C_1 + C_j + \frac{d}{2\mu} \Big) \frac{(nd)^j}{j!\mu^j} = C_{j+1} \frac{(nd)^j}{j!\mu^j}, \quad n \to \infty.$$

□

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