# A functional limit theorem for general shot noise processes

Alexander Iksanov * and Bohdan Rashytov †

October 27, 2019

**Abstract**

By a general shot noise process we mean a shot noise process in which the counting process of shots is arbitrary locally finite. Assuming that the counting process of shots satisfies a functional limit theorem in the Skorokhod space with a locally Hölder continuous Gaussian limit process and that the response function is regularly varying at infinity we prove that the corresponding general shot noise process satisfies a similar functional limit theorem with a different limit process and different normalization and centering functions. For instance, if the limit process for the counting process of shots is a Brownian motion, then the limit process for the general shot noise process is a Riemann-Liouville process. We specialize our result for five particular counting processes. Also, we investigate Hölder continuity of the limit processes for general shot noise processes.

Keywords: Hölder continuity; shot noise process; weak convergence in the Skorokhod space
2000 Mathematics Subject Classification: Primary: 60F17
Secondary: 60G55

## 1 Introduction and main result

Let $(S_k)_{k\in\mathbb{N}_0}$ be a not necessarily monotone sequence of nonnegative random variables. Define the counting process $(N(t))_{t\geq 0}$ by

$$N(t) := \sum_{k\geq 0} \mathbb{1}_{\{S_k\leq t\}}, \quad t\geq 0,$$

where $\mathbb{1}_A = 1$ if the event $A$ holds and $= 0$, otherwise. Throughout the paper we always assume that $N(t) < \infty$ almost surely (a.s.) for $t \geq 0$.

Denote by $D := D[0,\infty)$ the Skorokhod space of right-continuous real-valued functions which are defined on $[0,\infty)$ and have finite limits from the left at each positive point. For a function $h \in D$, the random process $X := (X(t))_{t\geq 0}$ which is the main object of our investigation is given by

$$X(t) := \sum_{k\geq 0} h(t-S_k)\, \mathbb{1}_{\{S_k\leq t\}} = \int_{[0,\,t]} h(t-y)\mathrm{d}N(y), \quad t\geq 0.$$

We call $X$ *general shot noise process*, for no assumptions are imposed apart from $N(t) < \infty$ a.s. Plainly, $X \in D$ a.s.

*Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine; e-mail: iksan@univ.kiev.ua

†Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine; e-mail: mr.rashytov@gmail.com

Denote by $Y_1 := (Y_1(t))_{t\geq 0}$, $Y_2 := (Y_2(t))_{t\geq 0}, \ldots$ independent and identically distributed (i.i.d.) random processes with paths in $D$. Assume that, for $k \in \mathbb{N}_0$, $Y_{k+1}$ is independent of $(S_0, \ldots, S_k)$. In particular, the case of complete independence of $(Y_j)_{j\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}_0}$ is not excluded. Set

$$Y(t) := \sum_{k\geq 0} Y_{k+1}(t - S_k)\, \mathbb{1}_{\{S_k \leq t\}}, \quad t \geq 0$$

and call $Y := (Y(t))_{t\geq 0}$ *random process with immigration at random times*. The interpretation is that associated with the $k$th immigrant arriving at the system at time $S_{k-1}$ is the process $Y_k$ which defines a model-dependent 'characteristic' of the $k$th immigrant. For instance, $Y_k(t-S_{k-1})$ may be the fitness of the $k$th immigrant at time $t$. The value of $Y(t)$ is then given by the sum of 'characteristics' of all immigrants arriving at the system up to and including time $t$. Assume that the function $g(t) := \mathbb{E}Y_1(t)$ is finite for all $t \geq 0$, not identically 0 and that $g \in D$. To investigate weak convergence of the process $Y$, properly normalized and centered, it is natural to use decomposition

$$Y(t) = \sum_{k\geq 0}(Y_{k+1}(t - S_k) - g(t - S_k))\, \mathbb{1}_{\{S_k\leq t\}} + \sum_{k\geq 0} g(t - S_k)\, \mathbb{1}_{\{S_k\leq t\}}, \quad t \geq 0. \tag{1}$$

For fixed $t > 0$, while the first summand is the terminal value of a martingale, the second is the value at time $t$ of a general shot noise process. The summands should be treated separately, for each of these requires a specific approach. Weak convergence of the first summand in (1) will be investigated in [11].

In the present paper we are aimed at proving a functional limit theorem for a general shot noise process $X$ under natural assumptions. Besides being of independent interest our findings pave the way towards controlling the asymptotic behavior of the second summand in (1). These taken together with prospective results from [11] should eventually lead to understanding of the asymptotics of processes $Y$.

A rich source of random processes $Y$ with immigration at random times are queueing systems and various branching processes with or without immigration. For example, particular instances of random variables $Y(t)$ are given by the number of the $k$th generation individuals ($k \geq 2$) with positions $\leq t$ in a branching random walk; the number of customers served up to and including time $t$ or the number of busy servers at time $t$ in a $GEN/G/\infty$ queuing system, where GEN means that the arrival of customers is regulated by a general point process. Nowadays rather popular objects of research are queueing systems in which an input process is more complicated than the renewal process, for instance, a Cox process (also known as a doubly stochastic Poisson process) [8] or a Hawkes process [10, 28] and branching processes with immigration governed by a process which is more general than the renewal process, for instance, an inhomogeneous Poisson process [34] or a Cox process [7]. Note that some authors investigated the processes $X$, $Y$ or the like from purely mathematical viewpoint. An incomplete list of relevant publications includes the works [9, 30, 31, 32] and the recent article [29]. On the other hand, we stress that the results obtained in the aforementioned papers do not overlap with ours.

The present work was preceded by several articles [16, 19, 20, 21, 27] in which weak convergence of renewal shot noise processes has been investigated. The latter processes is a particular case of processes $X$ in which the input sequence $(S_k)_{k\in\mathbb{N}_0}$ is a standard random walk. Development of elements of the weak convergence theory for renewal shot noise processes was motivated by and effectively used for the asymptotic analysis of various characteristics of several random regenerative structures: the order of random permutations [12], the number of zero and nonzero blocks of weak random compositions [2, 24], the number of collisions in coalescents with multiple collisions [13], the number of busy servers in a $G/G/\infty$ queuing system [18], random process with immigration at the epochs of a renewal process [22, 23]. Chapter 3 of the monograph [17] provides a survey of results obtained in the aforementioned articles, pointers to relevant literature and a detailed discussion of possible applications.

To formulate our main result we need additional notation. Denote by $W_\alpha := (W_\alpha(u))_{u\geq 0}$ a centered Gaussian process which is a.s. locally Hölder continuous with exponent $\alpha > 0$ and satisfies $W_\alpha(0) = 0$ a.s. In particular, for all $T > 0$, all $0 \leq x, y \leq T$ and some a.s. finite random variable $M_T$

$$|W_\alpha(x) - W_\alpha(y)| \leq M_T |x-y|^\alpha. \tag{2}$$

Define the random process $Y_{\alpha,\rho} := (Y_{\alpha,\rho}(u))_{u\geq 0}$ by

$$Y_{\alpha,\rho}(u) := \rho \int_0^u (u-y)^{\rho-1} W_\alpha(y) \mathrm{d}y, \quad u > 0, \quad Y_{\alpha,\rho}(0) := \lim_{u\to+0} Y_{\alpha,\rho}(u), \tag{3}$$

when $\rho > 0$ and by

$$Y_{\alpha,\rho}(u) := u^\rho W_\alpha(u) + |\rho| \int_0^u (W_\alpha(u) - W_\alpha(u-y)) y^{\rho-1} \mathrm{d}y, \quad u > 0, \quad Y_{\alpha,\rho}(0) := \lim_{u\to+0} Y_{\alpha,\rho}(u), \tag{4}$$

when $-\alpha < \rho < 0$. Also, put $Y_{\alpha,0} = W_\alpha$. Using (2) we conclude that $Y_{\alpha,\rho}(0) = 0$ a.s. whenever $\rho > -\alpha$.

Convergence of the integrals in (3) and (4) and a.s. continuity of the processes $Y_{\alpha,\rho}$ will be proved in Lemma 2.1 below. When $W_\alpha$ is a Brownian motion (so that $\alpha = 1/2 - \varepsilon$ for any $\varepsilon \in (0, 1/2)$), the process $Y_{\alpha,\rho}$ can be represented as a Skorokhod integral

$$Y_{\alpha,\rho}(u) := \int_{[0,\,u]} (u-y)^\rho \mathrm{d}W_\alpha(y), \quad u \geq 0. \tag{5}$$

The so defined process is called *Riemann–Liouville process* or *fractionally integrated Brownian motion* with exponent $\rho$ for $\rho > -1/2$. Since these processes appear for several times in our presentation we reserve a special notation for them, $R_\rho$. When $W_\alpha$ is a more general Gaussian process satisfying the standing assumptions, the process $Y_{\alpha,\rho}$ may be called *fractionally integrated Gaussian process*. Note that, for positive integer $\rho$, the process $Y_{\alpha,\rho}$ is up to a multiplicative constant an $r$-times integrated process $W_\alpha$. This can be easily checked with the help of integration by parts.

Throughout the paper we assume that the spaces $D$ and $D \times D$ are endowed with the $J_1$-topology and denote weak convergence in these spaces by $\overset{J_1}{\Longrightarrow}$ and $\overset{J_1(D\times D)}{\Longrightarrow}$, respectively. Comprehensive information concerning the $J_1$-topology can be found in the books [4, 26]. In what follows we use the notation $\mathbb{R}^+ := [0, \infty)$.

**Theorem 1.1.** *Let $h \in D$ be an eventually nondecreasing function of bounded variation which is regularly varying*[1] *at $\infty$ of index $\beta \geq 0$. Assume that $\lim_{t\to\infty} h(t) = \infty$ when $\beta = 0$ and that*

$$\frac{N(t\cdot) - b(t\cdot)}{a(t)} \overset{J_1}{\Longrightarrow} W_\alpha(\cdot), \quad t \to \infty, \tag{6}$$

*where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is regularly varying at $\infty$ of positive index, and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function. Then*

$$\frac{X(t\cdot) - \int_{[0,\,t\cdot]} h(t\cdot - y) \mathrm{d}b(y)}{a(t)h(t)} \overset{J_1}{\Longrightarrow} Y_{\alpha,\beta}(\cdot), \quad t \to \infty.$$

[1] This means that $\lim_{t\to\infty}(h(tx)/h(t)) = x^\beta$ for all $x > 0$.

**Remark 1.2.** *A perusal of the proof given below reveals that the assumption* $\lim_{t\to\infty} h(t) = \infty$ *when* $\beta = 0$ *is not needed if* $h$ *is nondecreasing on* $\mathbb{R}^+$ *rather than eventually nondecreasing.*

Since $b$ is nondecreasing and $h$ is locally bounded and almost everywhere continuous function (in view of $h \in D$), the integral $\int_{[0,\,t]} h(t-y)\mathrm{d}b(y)$ exists as a Riemann-Stieltjes integral.

The remainder of the article is organized as follows. In Section 2 we investigate local Hölder continuity of the limit processes in Theorem 1.1. In Section 3 we give five specializations of Theorem 1.1 for particular sequences $(S_k)_{k\in\mathbb{N}_0}$. Finally, we prove Theorem 1.1 in Section 4.

## 2 Hölder continuity of the limit processes

For the subsequent presentation, it is convenient to define the process $W_\alpha$ on the whole line. To this end, put $W_\alpha(x) = 0$ for $x < 0$. The right-hand side of (4) can then be given in an equivalent form

$$Y_{\alpha,\rho}(u) = |\rho| \int_0^\infty (W_\alpha(u) - W_\alpha(u-y))y^{\rho-1}\mathrm{d}y, \quad u > 0. \tag{7}$$

It is important for us that formula (2) still holds true for negative $x, y$. More precisely, we claim that, for all $T > 0$, all $-\infty < x, y \le T$ and the same random variable $M_T$ as in (2),

$$|W_\alpha(x) - W_\alpha(y)| \le M_T|x-y|^\alpha. \tag{8}$$

This inequality is trivially satisfied in the case $x \vee y \le 0$ and follows from (2) in the case $x \wedge y \ge 0$. Assume now that $x \wedge y \le 0 < x \vee y$. Then $|W_\alpha(x) - W_\alpha(y)| = |W_\alpha(x \vee y)| \le M_T(x \vee y)^\alpha \le M_T|x-y|^\alpha$. Here, the first inequality is a consequence of (2) with $y = 0$.

**Lemma 2.1.** *Let* $\rho > -\alpha$. *The following assertions hold:*
*1)* $|Y_{\alpha,\rho}(u)| < \infty$ *a.s. for each fixed* $u > 0$*;*
*2) the process* $Y_{\alpha,\rho}$ *is a.s. locally Hölder continuous with exponent* $\min(1,\, \alpha+\rho)$ *if* $\alpha + \rho \ne 1$ *and with arbitrary positive exponent less than* 1 *if* $\alpha + \rho = 1$*; more precisely, in the latter situation we have, for any* $T^* > T$,

$$\sup_{0 \le u \ne v \le T} \frac{|Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)|}{|u-v|\log(T^*|u-v|^{-1})} < \infty \quad \text{a.s.}$$

*Proof.* The case $\rho = 0$ is trivial. Fix $T > 0$.
Proof of 1). Using (2) we obtain, for all $u \in [0, T]$,

$$|Y_{\alpha,\rho}(u)| \le \rho \int_0^u (u-y)^{\rho-1}|W_\alpha(y)|\mathrm{d}y \le M_T\rho \int_0^u (u-y)^{\rho-1}y^\alpha \mathrm{d}y = M_T\rho\mathrm{B}(\rho, \alpha+1)u^{\rho+\alpha} < \infty \quad \text{a.s.}$$

in the case $\rho > 0$ and

$$\begin{aligned} |Y_{\alpha,\rho}(u)| &\le u^\rho|W_\alpha(u)| + |\rho| \int_0^u |W_\alpha(u) - W_\alpha(u-y))y^{\rho-1}\mathrm{d}y \\ &\le M_Tu^{\rho+\alpha} + |\rho|M_T(\rho+\alpha)^{-1}u^{\rho+\alpha} = M_T\alpha(\rho+\alpha)^{-1}u^{\rho+\alpha} < \infty \quad \text{a.s.} \end{aligned}$$

in the case $-\alpha < \rho < 0$.
Proof of 2). By virtue of symmetry it is enough to investigate the case $0 \le v < u \le T$, and this is tacitly assumed throughout the proof.

Assume first that $-\alpha < \rho < 0$. Appealing to (8) and (2) we conclude that, for $v > 0$,

$$\begin{aligned}
|\rho|^{-1}\left|Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)\right| &= \left|\int_0^\infty (W_\alpha(u) - W_\alpha(u-y) - W_\alpha(v) + W_\alpha(v-y))y^{\rho-1}\mathrm{d}y\right| \\
&\le \int_0^{u-v} \left|W_\alpha(u) - W_\alpha(u-y)\right| y^{\rho-1}\mathrm{d}y \\
&+ \int_0^{u-v} \left|W_\alpha(v) - W_\alpha(v-y)\right| y^{\rho-1}\mathrm{d}y \\
&+ \int_{u-v}^\infty \left|W_\alpha(u) - W_\alpha(v)\right| y^{\rho-1}\mathrm{d}y \\
&+ \int_{u-v}^\infty \left|W_\alpha(u-y) - W_\alpha(v-y)\right| y^{\rho-1}\mathrm{d}y \\
&\le 2M_T\left(\int_0^{u-v} y^{\rho-1+\alpha}\mathrm{d}y + (u-v)^\alpha \int_{u-v}^\infty y^{\rho-1}\mathrm{d}y\right) \\
&= 2M_T\alpha(|\rho|(\rho+\alpha))^{-1}(u-v)^{\rho+\alpha} \quad \text{a.s.}
\end{aligned}$$

We already know from the proof of part 1) that a similar inequality holds when $v = 0$. Thus, the claim of part 2) has been proved in the case $-\alpha < \rho < 0$.

Assume now that $\rho \ge 1$. We infer with the help of (2) that

$$\begin{aligned}
\left|Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)\right| &= \left|\rho\int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1})W_\alpha(y)\mathrm{d}y + \rho\int_v^u (u-y)^{\rho-1}W_\alpha(y)\mathrm{d}y\right| \\
&\le \rho\int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1})\left|W_\alpha(y)\right|\mathrm{d}y + \rho\int_v^u (u-y)^{\rho-1}\left|W_\alpha(y)\right|\mathrm{d}y \\
&\le \rho M_T\int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1})y^\alpha\mathrm{d}y + \rho M_T\int_v^u (u-y)^{\rho-1}y^\alpha\mathrm{d}y \\
&= \rho M_T\int_0^u (u-y)^{\rho-1}y^\alpha\mathrm{d}y - \rho M_T\int_0^v (v-y)^{\rho-1}y^\alpha\mathrm{d}y \\
&= \rho M_T \mathrm{B}(\rho,\alpha+1)(u^{\rho+\alpha} - v^{\rho+\alpha}) \\
&\le \rho(\rho+\alpha)M_T\mathrm{B}(\rho,\alpha+1)T^{\rho+\alpha-1}(u-v) \quad \text{a.s.,}
\end{aligned}$$

where the last inequality follows from the mean value theorem for differentiable functions.

It remains to investigate the case $0 < \rho < 1$. We shall use the following decomposition

$$\begin{aligned}
\left|Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)\right| &= \left|\rho\int_0^u (u-y)^{\rho-1}W_\alpha(y)\mathrm{d}y - \rho\int_0^v (v-y)^{\rho-1}W_\alpha(y)\mathrm{d}y\right| \\
&= \left|\rho\int_0^v (W_\alpha(v) - W_\alpha(v-y))(y^{\rho-1} - (y+u-v)^{\rho-1})\mathrm{d}y\right. \\
&- \left.\rho\int_0^{u-v} (W_\alpha(v) - W_\alpha(u-y))y^{\rho-1}\mathrm{d}y + W_\alpha(v)(u^\rho - v^\rho)\right| \\
&\le I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 := \rho\int_0^v |W_\alpha(v) - W_\alpha(v-y)|(y^{\rho-1} - (y+u-v)^{\rho-1})\mathrm{d}y,$$

$$I_2 := \rho\int_0^{u-v} |W_\alpha(v) - W_\alpha(u-y)|y^{\rho-1}\mathrm{d}y \quad \text{та} \quad I_3 := |W_\alpha(v)|(u^\rho - v^\rho).$$

The summand $I_1$ can be estimated as follows

$$\begin{aligned} I_1 &\leq \rho M_T \int_0^v y^\alpha (y^{\rho-1} - (y+u-v)^{\rho-1}) \mathrm{d}y \\ &= \rho M_T (u-v)^{\alpha+\rho} \int_0^{v/(u-v)} t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) \mathrm{d}t. \end{aligned}$$

Using the inequality $x^{\rho-1} - (x+1)^{\rho-1} \leq (1-\rho)x^{\rho-2}$ for $x > 0$ gives

$$\begin{aligned} I_1 &\leq \rho(1-\rho) M_T (u-v)^{\alpha+\rho} \int_0^{v/(u-v)} t^{\alpha+\rho-2} \mathrm{d}t \\ &= \rho(1-\rho)(\alpha+\rho-1)^{-1} M_T v^{\alpha+\rho-1}(u-v) \\ &\leq \rho(1-\rho)(\alpha+\rho-1)^{-1} M_T T^{\alpha+\rho-1}(u-v) in \end{aligned}$$

in the case $\alpha + \rho > 1$ and

$$\begin{aligned} I_1 &\leq \rho M_T (u-v)^{\alpha+\rho} \left( \int_0^1 t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) \mathrm{d}t + \int_1^\infty t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) \mathrm{d}t \right) \\ &\leq \rho M_T (u-v)^{\alpha+\rho} \left( \int_0^1 t^{\alpha+\rho-1} \mathrm{d}t + (1-\rho) \int_1^\infty t^{\alpha+\rho-2} \mathrm{d}t \right) \\ &= \rho M_T \left( \frac{1}{\alpha+\rho} + \frac{1-\rho}{1-\alpha-\rho} \right) (u-v)^{\alpha+\rho} \end{aligned}$$

in the case $0 < \alpha + \rho < 1$. Also,

$$\begin{aligned} I_1 &\leq \rho M_T (u-v) \int_0^{v/(u-v)} \left( 1 - \left( \frac{t}{t+1} \right)^\alpha \right) \mathrm{d}t \\ &\leq \rho M_T (u-v) \int_0^{v/(u-v)} \frac{\mathrm{d}t}{t+1} \\ &= \rho M_T (u-v) \log \frac{u}{u-v} \\ &\leq \rho M_T (u-v) \log \frac{T}{u-v} \end{aligned}$$

in the case $\alpha + \rho = 1$.

Further,

$$I_2 \leq \rho M_T \int_0^{u-v} (u-v-y)^\alpha y^{\rho-1} \mathrm{d}y = \rho M_T \mathrm{B}(\rho, \alpha+1)(u-v)^{\alpha+\rho}.$$

In the case $\alpha + \rho > 1$ the inequality $(u-v)^{\alpha+\rho} \leq T^{\alpha+\rho-1}(u-v)$ has to be additionally used.

Finally,

$$I_3 \leq M_T v^\alpha (u^\rho - v^\rho) \leq M_T (u^{\alpha+\rho} - v^{\alpha+\rho}).$$

The right-hand side does not exceed $M_T(u-v)^{\alpha+\rho}$ in the case $0 < \alpha + \rho \leq 1$ in view of subadditivity of $x \mapsto x^{\alpha+\rho}$ on $[0, \infty)$ and $M_T(\alpha+\rho)T^{\alpha+\rho-1}(u-v)$ in the case $\alpha + \rho > 1$.

The proof of Lemma 2.1 is complete. □

## 3 Applications of Theorem 1.1

In this section we give five examples of particular sequences $(S_k)_{k\in\mathbb{N}_0}$ which satisfy limit relation (6) with four different Gaussian processes $W_\alpha$. Throughout the section we always assume, without further notice, that $h$ satisfies the assumptions of Theorem 1.1.

In the case where the sequence $(S_k)_{k\in\mathbb{N}_0}$ is a.s. nondecreasing, the counting process $(N(t))_{t\geq 0}$ is nothing else but a generalized inverse function for $(S_k)$, that is,

$$N(t) = \inf\{k \in \mathbb{N} : S_k > t\} \quad \text{a.s.}, \quad t \geq 0. \tag{9}$$

In view of this, if a functional limit theorem for $S_{[ut]}$ in the $J_1$-topology on $D$ holds, and the limit process is a.s. continuous, then the corresponding functional limit theorem for $N(ut)$ in the $J_1$-topology on $D$ is a simple consequence. A detailed discussion of this fact can be found, for instance, in [15]. If the sequence $(S_k)_{k\in\mathbb{N}_0}$ is not monotone (as, for instance, at point 2 below), then equality (9) is no longer true, and the proof of a functional limit theorem for $N(ut)$ requires an additional specific argument in every particular case.

1. Delayed standard random walk. Let $\xi_1, \xi_2, \ldots$ be i.i.d. nonnegative random variables which are independent of a nonnegative random variable $S_0$. The random sequence $(S_k)_{k\in\mathbb{N}_0}$ defined by $S_k := S_0 + \xi_1 + \ldots + \xi_k$ for $k \in \mathbb{N}_0$ is called *delayed standard random walk* provided that $\mathbb{P}\{S_0 = 0\} < 1$. In the case $S_0 = 0$ a.s. the term *zero-delayed standard random walk* is used. It is well-known (see, for instance, Theorem 1b(i) in [5]) that

a) if $\sigma^2 := \operatorname{Var}\xi_1 \in (0, \infty)$, then

$$\frac{N(t\cdot) - \mu^{-1}t(\cdot)}{(\sigma^2\mu^{-3}t)^{1/2}} \overset{J_1}{\Longrightarrow} B(\cdot), \quad t \to \infty, \tag{10}$$

where $\mu := \mathbb{E}\xi_1 < \infty$, and $(B(u))_{u\geq 0}$ is a standard Brownian motion (so that relation (6) holds with $b(t) = \mu^{-1}t$ and $a(t) = (\sigma^2\mu^{-3}t)^{1/2}$);

b) if

$$\sigma^2 = \infty \quad \text{and} \quad \int_{[0,\,x]} y^2 \mathbb{P}\{\xi_1 \in \mathrm{d}y\} \sim L(x), \quad x \to \infty \tag{11}$$

for some $L$ slowly varying at $\infty$, then

$$\frac{N(t\cdot) - \mu^{-1}t(\cdot)}{\mu^{-3/2}c(t)} \overset{J_1}{\Longrightarrow} B(\cdot), \quad t \to \infty, \tag{12}$$

where $c$ is a positive measurable function satisfying $\lim_{t\to\infty} c(t)^{-2}tL(c(t)) = 1$ (so that relation (6) holds with $b(t) = \mu^{-1}t$ and $a(t) = \mu^{-3/2}c(t)$; since $c$ is asymptotically inverse for $t \mapsto t^2/L(t)$, an application of Proposition 1.5.15 in [6] enables us to conclude that $c$, hence also $a$ are regularly varying at $\infty$ of index 1/2).

Thus, according to Theorem 1.1, we have

$$\frac{X(t\cdot) - \mu^{-1}\int_0^{t\cdot} h(y)\mathrm{d}y}{(\sigma^2\mu^{-3}t)^{1/2}h(t)} \overset{J_1}{\Longrightarrow} \beta\int_0^{(\cdot)} (\cdot - y)^{\beta-1}B(y)\mathrm{d}y = R_\beta(\cdot), \quad t \to \infty,$$

provided that $\sigma^2 \in (0, \infty)$ (in the case $\beta = 0$, the limit process is $R_0 = B$), and

$$\frac{X(t\cdot) - \mu^{-1}\int_0^{t\cdot} h(y)\mathrm{d}y}{\mu^{-3/2}c(t)h(t)} \overset{J_1}{\Longrightarrow} R_\beta(\cdot), \quad t \to \infty$$

provided that conditions (11) hold. In particular, irrespective of whether the variance is finite or not the limit process is a fractionally integrated Brownian motion with parameter $\beta$. As far as zero-delayed standard random walks are concerned, the aforementioned results can be found in Theorem 1.1 (A1, A2) of [16].

2. Perturbed random walks. Let $(\xi_1, \eta_1)$, $(\xi_2, \eta_2)\dots$ be i.i.d. random vectors with nonnegative coordinates. Put

$$S_1 := \eta_1, \quad S_n := \xi_1 + \ldots + \xi_{n-1} + \eta_n, \quad n \geq 2.$$

The so defined sequence $(S_n)_{n\in\mathbb{N}}$ is called *perturbed random walk*. Various properties of perturbed random walks are discussed in the monograph [17].

Assume that $\sigma^2 = \operatorname{Var}\xi_1 \in (0,\infty)$ and $\mathbb{E}\eta^a < \infty$ for some $a > 0$. Put $F(x) := \mathbb{P}\{\eta_1 \leq x\}$ for $x \in \mathbb{R}$. According to Theorem 3.2 in [2],

$$\frac{N(t\cdot) - \mu^{-1}\int_0^{t\cdot} F(y)\mathrm{d}y}{\sqrt{\sigma^2\mu^{-3}t}} \overset{J_1}{\Longrightarrow} B(\cdot), \quad t \to \infty,$$

where $\mu = \mathbb{E}\xi_1 < \infty$. Therefore, by Theorem 1.1,

$$\frac{X(t\cdot) - \mu^{-1}\int_0^{t\cdot} h(y)F(y)\mathrm{d}y}{(\sigma^2\mu^{-3}t)^{1/2}h(t)} \overset{J_1}{\Longrightarrow} R_\beta(\cdot), \quad t \to \infty.$$

3. Random walks with long memory. Let $\xi_1$, $\xi_2, \dots$ be i.i.d. positive random variables with finite mean. Assume that these are independent of random variables $\theta_1$, $\theta_2, \dots$ which form a centered stationary Gaussian sequence with $\mathbb{E}\theta_1\theta_{k+1} \sim k^{2d-1}\ell(k)$ as $k \to \infty$ for some $d \in (0, 1/2)$. Put $S_0 := 0$ and

$$S_n - S_{n-1} = \xi_n e^{\theta_n}, \quad n \in \mathbb{N}.$$

Recall that a fractional Brownian motion with Hurst index $H \in (0,1)$ is a centered Gaussian process $B_H := (B_H(u))_{u\geq 0}$ with covariance $\mathbb{E}B_H(u)B_H(v) = 2^{-1}(u^{2H} + v^{2H} - (u-v)^{2H})$ for $u, v \geq 0$. This process has stationary increments and is self-similar of index $H$. Therefore, for any $p > 0$,

$$\mathbb{E}|B_H(u) - B_H(v)|^p = (u-v)^{Hp}\mathbb{E}|B_H(1)|^p, \quad u, v \geq 0.$$

According to the Kolmogorov-Chentsov sufficient conditions, there exists a version of $B_H$ (which we also denote by $B_H$) which is a.s. Hölder continuous with exponent smaller than $H - 1/p$ for any $p > 0$, hence also, smaller than $H$.

According to Example 4.25 on p. 357 in [3],

$$\frac{N(t\cdot) - m_1^{-1}t(\cdot)}{(d(2d+1))^{-1/2}m_1^{-3/2-d}m_2t^{d+1/2}(\ell(t))^{1/2}} \overset{J_1}{\Longrightarrow} B_{d+1/2}(\cdot), \quad t \to \infty,$$

where $m_1 := \mathbb{E}S_1 = \mathbb{E}\xi_1\mathbb{E}e^{\theta_1}$ and $m_2 := \mathbb{E}\xi_1\mathbb{E}\theta_1e^{\theta_1}$. An application of Theorem 1.1 yields

$$\frac{X(t\cdot) - m_1^{-1}\int_0^{t\cdot} h(y)\mathrm{d}y}{(d(2d+1))^{-1/2}m_1^{-3/2-d}m_2t^{d+1/2}(\ell(t))^{1/2}h(t)} \overset{J_1}{\Longrightarrow} \beta\int_0^{(\cdot)}(\cdot - y)^{\beta-1}B_{d+1/2}(y)\mathrm{d}y, \quad t \to \infty$$

if $\beta > 0$. If $\beta = 0$, the limit process is $B_{d+1/2}$.

4. Counting process in a branching random walk. Assume that the random variables $\xi_k$ defined at point 1 are a.s. positive. For some integer $k \geq 2$, we take in the role of $N(t)$

the number of the $k$th generation individuals with positions $\le t$ in a branching random walk in which the first generation individuals are located at the points $S_1 = \xi_1$, $S_2 = \xi_1 + \xi_2, \dots$ (a more precise definition can be found in Section 1.2 of [25]).

Assume that $\sigma^2 = \operatorname{Var} \xi_1 \in (0, \infty)$. Theorem 1.3 in [25] implies that

$$\frac{N(t\cdot) - (t\cdot)^k/(k!\mu^k)}{((k-1)!)^{-1}\sqrt{\sigma^2 \mu^{-2k-1} t^{2k-1}}} \overset{J_1}{\Longrightarrow} R_{k-1}(\cdot), \quad t \to \infty,$$

where $\mu = \mathbb{E}\xi_1 < \infty$. Of course, for $k = 1$ this limit relation is also valid and amounts to (10) as it must be. By Lemma 2.1, the process $R_{k-1}$ is a.s. locally Hölder continuous with any positive exponent smaller than $k - 1/2$. Thus, Theorem 1.1 applies and gives

$$\begin{aligned} \frac{X(t\cdot) - ((k-1)!\mu^k)^{-1} \int_0^{t\cdot} h(t \cdot -y) y^{k-1} \mathrm{d}y}{((k-1)!)^{-1}\sqrt{\sigma^2 \mu^{-2k-1} t^{2k-1}}\, h(t)} &\overset{J_1}{\Longrightarrow} \beta \int_0^{(\cdot)} (\cdot - y)^{\beta-1} R_{k-1}(y) \mathrm{d}y \\ &= (k-1)\mathrm{B}(k-1, \beta+1) R_{\beta+k-1}(\cdot), \quad t \to \infty, \end{aligned}$$

where $\mathrm{B}(\cdot, \cdot)$ is the beta function. The latter equality can be checked as follows: for $u > 0$

$$\begin{aligned} \beta \int_0^u (u-y)^{\beta-1} R_{k-1}(y) \mathrm{d}y &= \beta \int_0^u (u-y)^{\beta-1} (k-1) \int_0^y (y-x)^{k-2} B(x) \mathrm{d}x \mathrm{d}y \\ &= \beta(k-1) \int_0^u B(x) \int_x^u (u-y)^{\beta-1} (y-x)^{k-2} \mathrm{d}y \mathrm{d}x \\ &= \beta(k-1) \int_0^u B(x) \int_0^{u-x} (u-x-y)^{\beta-1} y^{k-2} \mathrm{d}y \mathrm{d}x \\ &= \beta(k-1)\mathrm{B}(k-1, \beta) \int_0^u (u-x)^{\beta+k-2} B(x) \mathrm{d}x \\ &= \frac{\beta(k-1)\mathrm{B}(k-1,\beta)}{\beta+k-1} R_{\beta+k-1}(u) \\ &= (k-1)\mathrm{B}(k-1, \beta+1) R_{\beta+k-1}(u). \end{aligned}$$

5. Inhomogeneous Poisson process. Let $(N(t))_{t\ge 0}$ be an inhomogeneous Poisson process with $\mathbb{E}N(t) = m(t)$ for a nondecreasing function $m(t)$ satisfying

$$m(t) \sim ct^w, \quad t \to \infty, \tag{13}$$

where $c, w > 0$. Without loss of generality we can identify $(N(t))_{t\ge0}$ with the process $(N^*(m(t)))_{t\ge0}$, where $(N^*(t))_{t\ge0}$ is a homogeneous Poisson process with $\mathbb{E}N^*(t) = t$, $t \ge 0$. According to (10),

$$\frac{N^*(t\cdot) - (t\cdot)}{t^{1/2}} \overset{J_1}{\Longrightarrow} B(\cdot), \quad t \to \infty. \tag{14}$$

Dini's theorem in combination with (13) ensures that

$$\lim_{t\to\infty} \sup_{u\in[0,T]} \Big| \frac{m(tu)}{ct^w} - u^w \Big| = 0 \tag{15}$$

for all $T > 0$. It is known (see, for instance, Lemma 2.3 on p. 159 in [14]) that the composition mapping $(x, \varphi) \mapsto (x \circ \varphi)$ is continuous on continuous functions $x : \mathbb{R}_+ \to \mathbb{R}$ and continuous nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Using this fact in conjunction with (14), (15) and continuous mapping theorem we infer

$$\frac{N(t\cdot) - m(t\cdot)}{(ct^w)^{1/2}} \overset{J_1}{\Longrightarrow} B((\cdot)^w), \quad t \to \infty. \tag{16}$$

Thus, the limit process is a time-changed Brownian motion. An application of Theorem 1.1 yields

$$\frac{X(t\cdot) - \mu^{-1}\int_0^{t\cdot} h(y)\mathrm{d}m(y)}{(ct^w)^{1/2}h(t)} \overset{J_1}{\Longrightarrow} \beta\int_0^{(\cdot)} (\cdot - y)^{\beta-1} B(y^w)\mathrm{d}y, \quad t\to\infty$$

if $\beta > 0$. If $\beta = 0$, the limit process is $B((\cdot)^w)$.

# 4 Proof of Theorem 1.1

To prove weak convergence of finite-dimensional distributions we need an auxiliary lemma.

**Lemma 4.1.** *Let* $f : \mathbb{R}^+ \times \mathbb{R}^+ \to (0,\infty)$ *be a function which is nondecreasing in the second coordinate and satisfies* $\lim_{t\to\infty} f(t,x) = x^\beta$ *for all* $x>0$ *and some* $\beta\geq 0$*. For* $t,x\geq 0$*, set*

$$Z(t,x) := \int_{[0,\,x]} W_\alpha(x-y)\mathrm{d}_y f(t,y),$$

$$Z(x) := \int_{[0,\,x]} W_\alpha(x-y)\mathrm{d}y^\beta \ \textit{for}\ \beta>0\ \textit{and}\ Z(x) := W_\alpha(x)\ \textit{for}\ \beta=0.$$

*Then, for any* $u,v>0$*,*

$$\lim_{t\to\infty} \mathbb{E}Z(t,u)Z(t,v) = \mathbb{E}Z(u)Z(v).$$

*Proof.* Fix any $u,v>0$. For each $t>0$, denote by $Q_t^{(u)}$ and $Q_t^{(v)}$ independent random variables with the distribution functions

$$\mathbb{P}\{Q_t^{(u)}\leq y\} = \begin{cases} 0, & \text{if } y<0,\\ \frac{f(t,y)}{f(t,u)}, & \text{if } y\in[0,u],\\ 1, & \text{if } y>u \end{cases} \quad\text{and}\quad \mathbb{P}\{Q_t^{(v)}\leq y\} = \begin{cases} 0, & \text{if } y<0,\\ \frac{f(t,y)}{f(t,v)}, & \text{if } y\in[0,v],\\ 1, & \text{if } y>v. \end{cases}$$

Also, denote by $Q^{(u)}$ and $Q^{(v)}$ independent random variables with the distribution functions

$$\mathbb{P}\{Q^{(u)}\leq y\} = \begin{cases} 0, & \text{if } y<0,\\ (\frac{y}{u})^\beta, & \text{if } y\in[0,u],\\ 1, & \text{if } y>u \end{cases} \quad\text{and}\quad \mathbb{P}\{Q^{(v)}\leq y\} = \begin{cases} 0, & \text{if } y<0,\\ (\frac{y}{v})^\beta, & \text{if } y\in[0,v],\\ 1, & \text{if } y>v. \end{cases}$$

By assumption,

$$(Q_t^{(u)}, Q_t^{(v)}) \overset{\mathrm{d}}{\to} (Q^{(u)}, Q^{(v)}), \quad t\to\infty.$$

Define the function $r(x,y) := \mathbb{E}W_\alpha(x)W_\alpha(y)$ on $\mathbb{R}^+\times\mathbb{R}^+$. Using a.s. continuity of $W_\alpha$, Lebesgue's dominated convergence theorem and the fact that, according to Theorem 3.2 on p. 63 in [1], $\mathbb{E}(\sup_{z\in[0,T]} W_\alpha(z))^2<\infty$ we conclude that $r$ is continuous, hence also bounded on $[0,T]\times[0,T]$ for all $T>0$. This entails

$$r(u-Q_t^{(u)}, v-Q_t^{(v)}) \overset{\mathrm{d}}{\longrightarrow} r(u-Q^{(u)}, v-Q^{(v)}), \quad t\to\infty$$

and thereupon

$$\lim_{t\to\infty} \mathbb{E}r(u-Q_t^{(u)}, v-Q_t^{(v)}) = \mathbb{E}r(u-Q^{(u)}, v-Q^{(v)})$$

by Lebesgue's dominated convergence theorem. Further,

$$\begin{aligned}\mathbb{E}Z(t,u)Z(t,v) &= f(t,u)f(t,v)\int_{[0,\,u]}\int_{[0,\,v]}\mathbb{E}W_\alpha(u-y)W_\alpha(v-z)\mathrm{d}_y\left(\frac{f(t,y)}{f(t,u)}\right)\mathrm{d}_z\left(\frac{f(t,z)}{f(t,v)}\right)\\ &= f(t,u)f(t,v)\mathbb{E}r\left(u-Q_t^{(u)},v-Q_t^{(v)}\right)\underset{t\to\infty}{\longrightarrow}(uv)^\beta\mathbb{E}r\left(u-Q^{(u)},v-Q^{(v)}\right).\end{aligned}$$

It remains to note that while in the case $\beta>0$ we have

$$\begin{aligned}&(uv)^\beta\mathbb{E}r(u-Q^{(u)},v-Q^{(v)})\\ =&\int_{[0,\,u]}\int_{[0,\,v]}r(u-y,v-z)\mathrm{d}_y\left(u^\beta\mathbb{P}\{\mathrm{Q^{(u)}\le y}\}\right)\mathrm{d}_z\left(v^\beta\mathbb{P}\{\mathrm{Q^{(v)}\le z}\}\right)\\ =&\int_{[0,\,u]}\int_{[0,\,v]}r(u-y,v-z)\mathrm{d}y^\beta\mathrm{d}z^\beta=\mathbb{E}\int_{[0,\,u]}W_\alpha(u-y)\mathrm{d}y^\beta\int_{[0,\,v]}W_\alpha(v-z)\mathrm{d}z^\beta\\ =&\ \mathbb{E}Z(u)Z(v),\end{aligned}$$

in the case $\beta=0$ we have

$$(uv)^\beta\mathbb{E}r(u-Q^{(u)},v-Q^{(v)})=r(u,v)=\mathbb{E}W_\alpha(u)W_\alpha(v)=\mathbb{E}Z(u)Z(v).$$

The proof of Lemma 4.1 is complete. □

*Proof of Theorem 1.1.* Since $h$ is eventually nondecreasing, there exists $t_0>0$ such that $h(t)$ is nondecreasing for $t>t_0$. Being a regularly varying function of nonnegative index, $h$ is eventually positive. Hence, increasing $t_0$ if needed we can ensure that $h(t)>0$ for $t>t_0$. We first show that the behavior of the function of bounded variation $h$ on $[0,t_0]$ does not affect weak convergence of the general shot noise process. Once this is done, we can assume, without loss of generality, that $h(0)=0$ and that $h$ is nondecreasing on $\mathbb{R}^+$.

Integrating by parts yields

$$\begin{aligned}X(tu)-\int_{[0,\,tu]}(h(tu-y)\mathrm{d}b(y) &= \int_{[0,\,u]}(h(t(u-y))\mathrm{d}_y(N(ty)-b(ty))\\ &= (h(tu)-h((tu)-))(N(0)-b(0))\\ &+ \int_{(0,\,u]}(N(ty)-b(ty)))\mathrm{d}_y(-h(t(u-y)))\end{aligned}\tag{17}$$

For all $T>0$,

$$\frac{\sup_{u\in[0,\,T]}|h(tu)-h((tu)-)||N(0)-b(0)|}{a(t)h(t)}\le\frac{h(tT)}{h(t)}\frac{|N(0)-b(0)|}{a(t)}\overset{\mathbb{P}}{\to}0,\quad t\to\infty$$

because $h$ is regularly varying at $\infty$. Denote by $\mathbb{V}_0^{t_0}(h)$ the total variation of $h$ on $[0,t_0]$. By assumption, $\mathbb{V}_0^{t_0}(h)<\infty$. For all $T>0$,

$$\frac{\sup_{u\in[0,\,T]}\int_{[u-t_0/t,\,u)}(N(ty)-b(ty))\mathrm{d}_y(-h(t(u-y)))}{a(t)h(t)}\le\frac{\sup_{u\in[0,\,T]}|N(tu)-b(tu)|}{a(t)}\frac{\mathbb{V}_0^{t_0}(h)}{h(t)}\overset{\mathbb{P}}{\to}0$$

as $t\to\infty$. The convergence to 0 is justified by the facts that, according to (6), the first factor converges in distribution to $\sup_{u\in[0,\,T]}|W_\alpha(u)|$, whereas the second trivially converges to 0. Recall that, when $\beta=0$, $\lim_{t\to\infty}h(t)=\infty$ holds by assumption, whereas, when $\beta>0$, it holds automatically. Thus, as was claimed, while investigating the asymptotic behavior of the second summand on the right-hand side of (17) we can and do assume that $h(0)=0$ and that $h$ is nondecreasing on $\mathbb{R}^+$.

Skorokhod's representation theorem ensures that there exist versions $(\hat{N}(t))_{t\geq 0}$ and $(\hat{W}_\alpha(t))_{t\geq 0}$ of the processes $(N(t))_{t\geq 0}$ and $(W_\alpha(t))_{t\geq 0}$ such that, for all $T>0$,

$$\lim_{t\to\infty}\sup_{u\in[0,\,T]}|\hat{W}_\alpha^{(t)}(u)-\hat{W}_\alpha(u)|=0\quad\text{a.s.},\tag{18}$$

where $\hat{W}_\alpha^{(t)}(u):=\frac{\hat{N}(tu)-b(tu)}{a(t)}$ for $t>0$ and $u\geq 0$. For each $t>0$, set $h_t(x):=h(tx)/h(t)$, $x\geq 0$,

$$X_t(u):=(a(t))^{-1}\int_{(0,\,u]}(N(ty)-b(ty))\mathrm{d}_y(-h_t(u-y)),\quad u\geq 0$$

and

$$\hat{X}_t^*(u):=\int_{(0,\,u]}\hat{W}_\alpha^{(t)}(y)\mathrm{d}_y(-h_t(u-y)),\quad u\geq 0.$$

The distributions of the processes $(X_t(u))_{u\geq 0}$ and $(\hat{X}_t^*(u))_{u\geq 0}$ are the same. Hence, it remains to check that

$$\lim_{t\to\infty}\int_{(0,\,u]}(\hat{W}_\alpha^{(t)}(y)-\hat{W}_\alpha(y))\mathrm{d}_y(-h_t(u-y))=0\quad\text{a.s.}\tag{19}$$

in the $J_1$-topology on $D$ and

$$\int_{(0,\,\cdot]}W_\alpha(y)\mathrm{d}(-h_t(\cdot-y))\overset{J_1}{\Longrightarrow}Y_{\alpha,\beta}(\cdot),\quad t\to\infty.\tag{20}$$

In view of (18) and monotonicity of $h_t$, we have, for all $T>0$ as $t\to\infty$,

$$\begin{aligned}&\sup_{u\in[0,\,T]}\left|\int_{(0,\,u]}\left(\hat{W}_\alpha^{(t)}(u)-\hat{W}_\alpha(u)\right)\mathrm{d}_y(-h_t(u-y))\right|\\ \leq\;&\sup_{u\in[0,\,T]}\left|\hat{W}_\alpha^{(t)}(u)-\hat{W}_\alpha(u)\right|h_t(T)\longrightarrow 0\quad\text{a.s.}\end{aligned}$$

which proves (19).

Since $W_\alpha$ is a Gaussian process, the convergence of the finite-dimensional distributions in (20) which is equivalent to the convergence of covariances follows from Lemma 4.1. While applying the lemma we use the equalities

$$Y_{\alpha,\,\beta}(u)=\int_{(0,\,u]}W_\alpha(y)\mathrm{d}_y(-(u-y)^\beta)$$

when $\beta>0$ and $Y_{\alpha,\,\beta}(u)=W_\alpha(u)$ when $\beta=0$. Our next step is to prove tightness on $D[0,T]$, for all $T>0$, of

$$\hat{X}_t(u):=\int_{(0,\,u]}W_\alpha(y)\mathrm{d}_y(-h_t(u-y))=\int_{[0,\,u)}W_\alpha(u-y)\mathrm{d}_yh_t(y),\quad u\geq 0.$$

By Theorem 15.5 in [4], it is enough to show that, for any $r_1,r_2>0$, there exist $t_0,\delta>0$ such that, for all $t\geq t_0$,

$$\mathbb{P}\{\sup_{0\leq u,v\leq T,|u-v|\leq\delta}|\hat{X}_t(u)-\hat{X}_t(v)|>r_1\}\leq r_2.\tag{21}$$

Put $l := \max(u, v)$. Recalling that $W_\alpha(s) = 0$ for $s < 0$ (see the beginning of Section 2) we have, for $0 \le u, v \le T$ and $|u - v| \le \delta$,

$$\begin{aligned} |\hat{X}_t(u) - \hat{X}_t(v)| &= \left| \int_{[0,\,l)} (W_\alpha(u-y) - W_\alpha(v-y)) \mathrm{d}_y h_t(y) \right| \\ &\le M_T |u-v|^\alpha h_t(T) \le M_T |u-v|^\alpha \lambda \end{aligned}$$

for large enough $t$ and a positive constant $\lambda$. The existence of $\lambda$ is justified by the relation $\lim_{t\to\infty} h_t(T) = T^\beta < \infty$. Decreasing $\delta$ if needed, we ensure that inequality (21) holds for any positive $r_1$ and $r_2$. The proof of Theorem 1.1 is complete. □

**Acknowledgements.** The authors thank the anonymous referee for a useful comment which enabled them to improve the presentation of part 1 (delayed standard random walk) in Section 3.

# References

[1] R. J. Adler, *An introduction to continuity, extrema, and related topics for general Gaussian processes.* Institute of Mathematical Statistics. 1990.

[2] G. Alsmeyer, A. Iksanov and A. Marynych, *Functional limit theorems for the number of occupied boxes in the Bernoulli sieve.* Stoch. Proc. Appl. **127** (2017), 995–1017.

[3] J. Beran, Y. Feng, S. Ghosh and R. Kulik, *Long-memory processes. Probabilistic properties and statistical methods.* Springer, 2013.

[4] P. Billingsley, *Convergence of probability measures.* Wiley, 1968.

[5] N. H. Bingham, *Maxima of sums of random variables and suprema of stable processes.* Z.Wahrscheinlichkeitstheor. Verwandte Geb. **26** (1973), 273–296.

[6] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation.* Cambridge University Press, 1989.

[7] O. A. Butkovsky, *Limit behavior of a critical branching process with immigration.* Mathemat. Notes. **92** (2012), 612–618.

[8] E. A. Chernavskaya, *Limit theorems for an infinite-server queuing system.* Mathemat. Notes. **98** (2015), 653–666.

[9] D. J. Daley, *Asymptotic properties of stationary point processes with generalized clusters.* Z. Wahrscheinlichkeitstheorie Verw. Geb. **21** (1972), 65–76.

[10] A. Daw and J. Pender, *Queues driven by Hawkes processes.* Stoch. Systems. **8** (2018), 192–229.

[11] C. Dong and A. Iksanov, *Weak convergence of random processes with immigration at random times.* Preprint available at `https://arxiv.org/abs/1906.11605`

[12] A. Gnedin, A. Iksanov and A. Marynych, *A generalization of the Erdos-Turan law for the order of random permutation.* Combinat. Probab. Comput. **21** (2012), 715–733.

[13] A. Gnedin, A. Iksanov, A. Marynych and M. Möhle, *The collision spectrum of* $\Lambda$*-coalescents.* Ann. Appl. Probab. **28** (2018), 3857–3883.

[14] A. Gut, *Stopped random walks. Limit theorems and applications.* 2nd Edition, Springer, 2009.

[15] D. L. Iglehart and W. Whitt, *The equivalence of functional limit theorems for counting processes and associated partial sums.* Ann. Math. Statist. **42** (1971), 1372–1378.

[16] A.Iksanov, *Functional limit theorems for renewal shot noise processes with increasing response functions.* Stoch. Proc. Appl. **123** (2013), 1987–2010.

[17] A. Iksanov, *Renewal theory for perturbed random walks and similar processes.* Birkhäuser, 2016.

[18] A. Iksanov, W. Jedidi and F. Bouzzefour, *Functional limit theorems for the number of busy servers in a* $G/G/\infty$ *queue.* J. Appl. Probab. **55** (2018), 15–29.

[19] A. Iksanov, Z. Kabluchko and A. Marynych, *Weak convergence of renewal shot noise processes in the case of slowly varying normalization*, Stat. Probab. Letters. **114** (2016), 67–77.

[20] A. Iksanov, Z. Kabluchko, A. Marynych and G. Shevchenko, *Fractionally integrated inverse stable subordinators.* Stoch. Proc. Appl. **127** (2016), 80–106.

[21] A. Iksanov, A. Marynych and M. Meiners, *Limit theorems for renewal shot noise processes with eventually decreasing response functions.* Stoch. Proc. Appl. **124** (2014), 2132–2170.

[22] A. Iksanov, A. Marynych and M. Meiners, *Asymptotics of random processes with immigration I: scaling limits.* Bernoulli. **23** (2017), 1233–1278.

[23] A. Iksanov, A. Marynych and M. Meiners, *Asymptotics of random processes with immigration II: convergence to stationarity.* Bernoulli. **23** (2017), 1279–1298.

[24] A. Iksanov, A. Marynych and V. Vatutin, *Weak convergence of finite-dimensional distributions of the number of empty boxes in the Bernoulli sieve.* Theor. Probab. Appl. **59** (2015), 87–113.

[25] A. Iksanov and Z. Kabluchko, *A functional limit theorem for the profile of random recursive trees.* Electron. Commun. Probab. **23** (2018), paper no. 87, 13 pp.

[26] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes*, 2nd Edition, Springer, 2003.

[27] Z. Kabluchko and A. Marynych, *Renewal shot noise processes in the case of slowly varying tails.* Theor. Stoch. Proc. **21(37)** (2016), 14–21.

[28] D. T. Koops, M. Saxena, O. J. Boxma and M. Mandjes, *Infinite-server queues with Hawkes input.* J. Appl. Probab. **55** (2018), 920–943.

[29] G. Pang and Y. Zhou, *Functional limit theorems for a new class of non-stationary shot noise processes.* Stoch. Proc. Appl. **128** (2018), 505–544.

[30] J. Rice, *On generalized shot noise.* Adv. Appl. Probab. **9** (1977), 553–565.

[31] V. Schmidt, *On finiteness and continuity of shot noise processes*. Optimization. **16** (1985), 921–933.

[32] M. Westcott, *On the existence of a generalized shot-noise process*. Studies in probability and statistics (papers in honour of Edwin J. G. Pitman), 73–88. North-Holland, 1976.

[33] W. Whitt, *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues*. Springer, 2002.

[34] A. Yakovlev and N. Yanev, *Age and residual lifetime distributions for branching processes*. Statist. Probab. Letters. **77** (2007), 503–513.