Functional limit theorems for discounted exponential functional of random walk and discounted convergent perpetuity

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Abstract

Let (ξ_1, η_1), $(\xi_2, \eta_2), \ldots$ be independent and identically distributed $\mathbb{R}^2$-valued random vectors. Put $S_0 := 0$ and $S_k := \xi_1 + \ldots + \xi_k$ for $k \in \mathbb{N}$. We prove a functional central limit theorem for a discounted exponential functional of the random walk $\sum_{k\geq 0} e^{-S_k/t}$, properly normalized and centered, as $t \to \infty$. In combination with a theorem obtained recently in Iksanov et al (2021) this leads to an ultimate functional central limit theorem for a discounted convergent perpetuity $\sum_{k\geq 0} e^{-S_k/t}\eta_{k+1}$, again properly normalized and centered, as $t \to \infty$. The latter result complements Vervaat's (1979) one-dimensional central limit theorem. Our argument is different from that used by Vervaat. The functional limit theorem is not informative in the case where $\xi_k = \eta_k$. As a remedy, we show that $\sum_{k\geq 0} e^{-S_k/t}\xi_{k+1}$ concentrates tightly around the point t in a deterministic manner.

Key words: exponential functions of random walk; functional central limit theorem; perpetuity; standard random walk
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1 Introduction

Let ξ_1, $\xi_2, \ldots$ be independent copies of a real-valued random variable ξ. Denote by $(S_k)_{k\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the zero-delayed standard random walk with jumps ξ_k, that is, $S_0 := 0$ and $S_k := \xi_1 + \ldots + \xi_k$ for $k \in \mathbb{N}$. For each $t > 0$, put $X(t) := \sum_{k\geq 0} e^{-S_k/t}$. Plainly, $X(t) < \infty$ a.s. for all/some $t > 0$ if, and only if, $\lim_{k\to\infty} S_k = +\infty$ a.s. Under this condition we call $(X(t))_{t>0}$ *discounted exponential functional of random walk.* Our terminology is inherited from a continuous-time counterpart $\int_0^\infty \exp(-S(y))\mathrm{d}y$, where $(S(y))_{y\geq 0}$ is a Lévy process diverging to $+\infty$, known in the literature as 'exponential functional of Lévy process', see, for instance, [2, 4]. Distributional properties of $X(t)$, with $t > 0$ fixed, were investigated in [11]. The latter authors call $X(t)$ *exponential functional of random walk.* There are many works investigating the asymptotics of $\sum_{k=0}^n f(S_k)$ as $n \to \infty$ for various functions f (for instance, exponential) in the situation that the corresponding series is divergent. This circle of problems which is very different from that related to discounted exponential functional is discussed in the books [1] and [10].

As usual, we write $\overset{\mathbb{P}}{\to}$ to denote convergence in probability, and $\Rightarrow$ and $\overset{\mathrm{d}}{\longrightarrow}$ to denote weak convergence in a function space and weak convergence of one-dimensional distributions,

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respectively. We prefer to use the notation $Y_t(u) \Rightarrow Y(u)$ as $t \to \infty$ in place of $Y_t(\cdot) \Rightarrow Y(\cdot)$. Also, we denote by $D(0,\infty)$ ($D[0,\infty)$) the Skorokhod space of right-continuous functions defined on $(0,\infty)$ (on $[0,\infty)$) with finite limits from the left at positive points.

Here is our first result.

Theorem 1.1. *Assume that $\mu := \mathbb{E}\xi \in (0,\infty)$ and $\sigma^2 := \mathrm{Var}\,\xi \in (0,\infty)$. Then, as $t \to \infty$,*

$$\frac{1}{t^{1/2}}\Big(\sum_{k\geq 0} e^{-uS_k/t} - \frac{t}{\mu u}\Big) \ \Rightarrow\ \Big(\frac{\sigma^2}{\mu^3}\Big)^{1/2} \int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y) \tag{1}$$

in the J_1-topology on $D(0,\infty)$, where $B := (B(y))_{y\geq 0}$ is a standard Brownian motion.

Remark 1.2. Under the assumptions $\xi \geq 0$ a.s. and $\mathbb{E}\xi^p < \infty$ for all $p > 0$ a one-dimensional central limit theorem for $\sum_{k\geq 0} e^{-S_k/t}$ as $t \to \infty$ was proved in Theorem 2 of [5] with the help of the method of moments.

Remark 1.3. The limit process in Theorem 1.1 is an a.s. continuous centered Gaussian process on $(0,\infty)$ with

$$\mathbb{E}\int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y)\int_{[0,\,\infty)} e^{-vy}\mathrm{d}B(y) = \frac{1}{u+v}, \quad u,v > 0.$$

Integration by parts yields an alternative representation

$$\int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y) = u\int_0^\infty e^{-uy}B(y)\mathrm{d}y, \quad u > 0.$$

Let (ξ_1,η_1), $(\xi_2,\eta_2),\ldots$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ,η) with arbitrarily dependent components. Whenever the random series $\sum_{k\geq 0} e^{-S_k/t}\eta_{k+1}$ is a.s. convergent for each $t > 0$, following [8], we call its sum *discounted convergent perpetuity.* We refer to the cited article for the justification of the term and standard limit theorems: a strong law of large numbers, a functional central limit theorem and a law of the iterated logarithm. In particular, Proposition 1.4 given next is Theorem 1.2 in [8], up to the change of variable.

Proposition 1.4. *Assume that $\mu = \mathbb{E}\xi \in (0,\infty)$, $\mathbb{E}\eta = 0$ and $\mathtt{s}^2 := \mathrm{Var}\,\eta \in (0,\infty)$. Then, as $t \to \infty$,*

$$\frac{1}{t^{1/2}}\sum_{k\geq 0} e^{-uS_k/t}\eta_{k+1} \ \Rightarrow\ \Big(\frac{\mathtt{s}^2}{\mu}\Big)^{1/2}\int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y) \tag{2}$$

in the J_1-topology on $D(0,\infty)$, where $(B(y))_{y\geq 0}$ is a standard Brownian motion.

A combination of Theorem 1.1 and Proposition 1.4 plus some additional work enable us to treat the case $\mathbb{E}\eta \neq 0$, thereby leading to an ultimate version of the functional central limit theorem for discounted convergent perpetuities.

Corollary 1.5. *Assume that $\mu = \mathbb{E}\xi \in (0,\infty)$, $\sigma^2 = \mathrm{Var}\,\xi \in [0,\infty)$, $\mathtt{m} := \mathbb{E}\eta \in \mathbb{R}$, $\mathtt{s}^2 = \mathrm{Var}\,\eta \in [0,\infty)$, $\sigma^2 + \mathtt{s}^2 > 0$. Then*

$$\frac{1}{t^{1/2}}\Big(\sum_{k\geq 0} e^{-uS_k/t}\eta_{k+1} - \frac{\mathtt{m}t}{\mu u}\Big) \ \Rightarrow\ v\int_{[0,\,\infty)} e^{-uy}\mathrm{d}B(y) \tag{3}$$

in the J_1-topology on $D(0,\infty)$. Here, $(B(y))_{y\geq 0}$ is a standard Brownian motion,

$$v^2 := \mu^{-1}\mathrm{Var}\,(\mu^{-1}\mathtt{m}\xi - \eta) = \sigma^2\mu^{-3}\mathtt{m}^2 - 2\gamma\mathtt{m}\mu^{-2} + \mathtt{s}^2\mu^{-1} \in [0,\infty), \quad \gamma := \mathrm{Cov}\,(\xi,\eta) = \mathbb{E}\xi\eta - \mu\mathtt{m} \in \mathbb{R}.$$

The constant v^2 is positive unless $\xi = c\eta$ for some $c \in \mathbb{R}$.

Remark 1.6. Putting in Corollary 1.5 $u = 1$ we obtain the one-dimensional central limit theorem

$$\frac{1}{t^{1/2}}\Big(\sum_{k\geq 0} e^{-S_k/t}\eta_{k+1} - \frac{\mathtt{m}t}{\mu}\Big) \overset{\mathrm{d}}{\longrightarrow} 2^{-1/2}v\,\mathrm{Normal}\,(0,1), \quad t\to\infty,$$

where Normal $(0, 1)$ is a random variable with the standard normal distribution. This result was proved in Theorem 6.1 of [12]. Our argument is different from that exploited in [12].

A perusal of the proof of Corollary 1.5 reveals that the distributional asymptotic behavior of the process on the left-hand side of (3) is driven by the competitive distributional fluctuations of the functional versions of the random walks (S_k) and $(\eta_1 + \ldots + \eta_k)$. The latter are represented in the limit by the same integral functional of (generally) dependent Brownian motions $-B_1$ and B_2. Because of the minus sign and the dependence the contributions of the two random walks compensate each other. When ξ and η are linearly dependent, the Brownian motion B_1 is a multiple of B_2. As a result, the contributions of the random walks get mutually neutralized, so that the limit variance v^2 is equal to 0. This means that $t^{-1/2}$ is not a proper normalization in this case. According to Theorem 1.7, when $\eta = c\xi$ for nonzero real c, no normalization is needed at all, and $\sum_{k\geq 0} e^{-S_k/t}\eta_{k+1}$ concentrates tightly around the point ct in a deterministic manner for t large enough.

Theorem 1.7. *Assume that $\mu = \mathbb{E}\xi \in (0,\infty)$ and $\sigma^2 = \mathrm{Var}\,\xi \in [0,\infty)$. Then*

$$\lim_{t\to\infty}\Big(\sum_{k\geq 0} e^{-S_k/t}\xi_{k+1} - t\Big) = \frac{\mathbb{E}\xi^2}{2\mu} \quad \text{a.s.}$$

The rest of the paper is structured as follows. We prove Theorem 1.1, Corollary 1.5 and Theorem 1.7 in Sections 2, 3 and 4 respectively. The appendix collects some auxiliary facts.

2 Proof of Theorem 1.1

For $t \in \mathbb{R}$, put $N(t) = \#\{k \geq 0 : S_k \leq t\}$. Since $\lim_{k\to\infty} S_k = +\infty$ a.s., we have $N(t) < \infty$ a.s. Write, for $u > 0$,

$$\sum_{k\geq 0} e^{-uS_k/t} - (\mu u)^{-1}t = \sum_{k\geq 0} e^{-uS_k/t}\,\mathbb{1}_{\{S_k\leq 0\}} + \int_{(0,\,\infty)} e^{-ux/t}\mathrm{d}(N(x) - \mu^{-1}x).$$

The first summand does not contribute to the limit. Indeed, for any $0 < c < d < \infty$,

$$\sup_{u\in[c,\,d]}\sum_{k\geq 0} e^{-uS_k/t}\,\mathbb{1}_{\{S_k\leq 0\}} \leq \sum_{k\geq 0} e^{-dS_k/t}\,\mathbb{1}_{\{S_k\leq 0\}} \to N(0), \quad t\to\infty \;\text{ a.s.}$$

Further, integration by parts followed by change of variable yields, for any $T > 0$,

$$\int_{(0,\,\infty)} e^{-ux/t}\mathrm{d}(N(x) - \mu^{-1}x) + N(0) = u\Big(\int_0^T e^{-ux}(N(xt) - \mu^{-1}xt)\mathrm{d}x + \int_T^\infty e^{-ux}(N(xt) - \mu^{-1}xt)\mathrm{d}x\Big).$$

By Propositon 5.1, as $t \to \infty$,

$$t^{-1/2}\big(N(ut) - \mu^{-1}ut\big) \;\Rightarrow\; (\sigma^2\mu^{-3})^{1/2}B(u) \tag{4}$$

in the J_1-topology on $D[0,\infty)$. Let $(t_k)_{k\in\mathbb{N}}$ be any sequence of positive numbers satisfying $\lim_{k\to\infty} t_k = \infty$. By Skorokhod's representation theorem there exists versions $\tilde{N}_k$ of the process on the left-hand side of (4) with t_k replacing t which converge a.s. to $\tilde{B}$, a version of $(\sigma^2\mu^{-3})^{1/2}B$, in the J_1-topology on $D[0,\infty)$ as $k\to\infty$. Since the limit B is a.s. continuous, the convergence is even locally uniform on $[0,\infty)$ with probability one, that is, for any $T>0$,

$$\lim_{k\to\infty} \sup_{x\in[0,\,T]} |\tilde{N}_k(x) - \tilde{B}(x)| = 0 \quad \text{a.s.}$$

Fix any $0<c<d<\infty$ and $T>0$. The process

$$t_k^{-1/2}\Big(\int_0^T e^{-ux}(N(xt_k) - \mu^{-1}xt_k)\mathrm{d}x\Big)_{u\in[c,\,d]}$$

has the same distribution as $(\int_0^T e^{-ux}\tilde{N}_k(x)\mathrm{d}x)_{u\in[c,\,d]}$. Furthermore,

$$\sup_{u\geq 0}\Big|\int_0^T e^{-ux}\tilde{N}_k(x)\mathrm{d}x - \int_0^T e^{-ux}\tilde{B}(x)\mathrm{d}x\Big| \leq T \sup_{x\in[0,\,T]} |\tilde{N}_k(x) - \tilde{B}(x)| \to 0, \quad k\to\infty \ \text{ a.s.}$$

Since the diverging sequence $(t_k)_{k\in\mathbb{N}}$ is arbitrary, this entails

$$t^{-1/2}\int_0^T e^{-ux}(N(xt) - \mu^{-1}xt)\mathrm{d}x \ \Rightarrow\ (\sigma^2\mu^{-3})^{1/2}\int_0^T e^{-ux}B(x)\mathrm{d}x, \quad t\to\infty$$

in the J_1-topology on $D[0,\infty)$. An appeal to Theorem 3.1 in [3] reveals that it suffices to prove the following asymptotic relations:

$$\sup_{u\in[c,\,d]}\Big|\int_0^T e^{-ux}B(x)\mathrm{d}x - \int_0^\infty e^{-ux}B(x)\mathrm{d}x\Big| \overset{\mathbb{P}}{\to} 0, \quad T\to\infty \quad \text{a.s.} \tag{5}$$

and

$$\lim_{T\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{t^{-1/2}\sup_{u\in[c,\,d]}\Big|\int_T^\infty e^{-ux}(N(xt) - \mu^{-1}xt)\mathrm{d}x\Big| > \varepsilon\Big\} = 0 \tag{6}$$

for every $\varepsilon>0$.

The proof of (5) is trivial. Further, by Proposition 5.2(b), for some constant $A>0$ and large t,

$$\mathbb{E}|N(t) - \mu^{-1}t| \leq At^{1/2}. \tag{7}$$

For large $t, T>0$, the probability in (6) is bounded from above by

$$\mathbb{P}\Big\{t^{-1/2}\int_T^\infty e^{-cx}|N(xt) - \mu^{-1}xt|\mathrm{d}x > \varepsilon\Big\}$$
$$\leq \varepsilon^{-1}t^{-1/2}\int_T^\infty e^{-cx}\mathbb{E}|N(xt) - \mu^{-1}xt|\mathrm{d}x \leq A\varepsilon^{-1}\int_T^\infty e^{-cx}x^{1/2}\mathrm{d}x.$$

We have used Markov's inequality for the next to the last step and (7) for the last step. The integral in the upper bound does not depend on t and tends to 0 as $T\to\infty$. This proves (6).

3 Proof of Corollary 1.5

In view of Theorem 1.1 and Proposition 1.4 the most important thing that remains to be done is determining the dependence between the Brownian motions appearing in (1) and (2).

Let $(\mathbf{B}(u))_{u\geq 0} := ((B_1(u), B_2(u)))_{u\geq 0}$ denote a two-dimensional Wiener process such that $\mathbf{B}(1)$ has the covariance matrix

$$\Gamma = \begin{pmatrix} \sigma^2 & \gamma \\ \gamma & \mathtt{s}^2 \end{pmatrix}$$

($\mathbf{B}$ is a two-dimensional standard Brownian motion whenever ξ and η are uncorrelated). Assume that we can prove that

$$t^{-1/2}\Big(\sum_{k\geq 0} e^{-uS_k/t} - (\mu u)^{-1}t,\ \sum_{k\geq 0} e^{-uS_k/t}(\eta_{k+1} - \mathtt{m})\Big) \Rightarrow \Big(\int_{[0,\infty)} e^{-uy}\mathrm{d}(-\mu^{-1}B_1(\mu^{-1}y)),\ \int_{[0,\infty)} e^{-uy}\mathrm{d}B_2(\mu^{-1}y)\Big),\quad t\to\infty \tag{8}$$

in the product J_1-topology on $D(0,\infty)\times D(0,\infty)$. Writing

$$\sum_{k\geq 0} e^{-uS_k/t}\eta_{k+1} - (\mu u)^{-1}\mathtt{m}t = \mathtt{m}\Big(\sum_{k\geq 0} e^{-uS_k/t} - (\mu u)^{-1}t\Big) + \sum_{k\geq 0} e^{-uS_k/t}(\eta_{k+1} - \mathtt{m}),\quad u>0$$

and using the fact that the summation functional is continuous at continuous functions we arrive at (3) after noting that the process $(-\mathtt{m}\mu^{-1}B_1(\mu^{-1}y) + B_2(\mu^{-1}y))_{y\geq 0}$ has the same distribution as $(vB(y))_{y\geq 0}$.

Put $T_0 := 0$ and $T_k := \eta_1 + \ldots + \eta_k$ for $k\in\mathbb{N}$. It is a standard fact that

$$t^{-1/2}(S_{\lfloor ut\rfloor} - \mu ut,\ T_{\lfloor ut\rfloor} - \mathtt{m}ut)\ \Rightarrow\ \mathbf{B}(u),\quad t\to\infty \tag{9}$$

in the J_1-topology on $D[0,\infty)\times D[0,\infty)$. Here and hereafter, $\lfloor x\rfloor$ denotes the integer part of real x. We claim that

$$t^{-1/2}\Big(S_{\lfloor ut\rfloor} - \mu\lfloor ut\rfloor,\ T_{\lfloor ut\rfloor} - \mathtt{m}\lfloor ut\rfloor,\ \sum_{k\geq 0} e^{-uS_k/t}(\eta_{k+1} - \mathtt{m})\Big) \Rightarrow \Big(B_1(u), B_2(u), \int_{[0,\infty)} e^{-\mu uy}\mathrm{d}B_2(y)\Big),\quad t\to\infty \tag{10}$$

in the product J_1-topology on $D[0,\infty)\times D[0,\infty)\times D(0,\infty)$.

Proof of (10). While tightness of the distributions of the first two coordinates on the left-hand side of (10) follows from (9), tightness of the distributions of the third coordinate is ensured by (2). This justifies tightness of the distributions of the vectors on the left-hand side of (10). Passing to weak convergence of the finite-dimensional distributions we shall prove that, for any $\ell\in\mathbb{N}$, any real $\alpha_{1,j}$, $\alpha_{2,j}$ and $\alpha_{3,j}$, $j\in\mathbb{N}$, $j\leq\ell$ and any $u_{1,j}\geq 0$, $u_{2,j}\geq 0$ and $u_{3,j}>0$, $j\in\mathbb{N}$, $j\leq\ell$,

$$t^{-1/2}\sum_{j=1}^{\ell}\Big(\alpha_{1,j}(S_{\lfloor u_{1,j}t\rfloor} - \mu\lfloor u_{1,j}t\rfloor) + \alpha_{2,j}(T_{\lfloor u_{2,j}t\rfloor} - \mathtt{m}\lfloor u_{2,j}t\rfloor) + \alpha_{3,j}\sum_{k\geq 0} e^{-u_{3,j}S_k/t}(\eta_{k+1} - \mathtt{m})\Big) \xrightarrow{\mathrm{d}} \sum_{j=1}^{\ell}\Big(\alpha_{1,j}B_1(u_{1,j}) + \alpha_{2,j}B_2(u_{2,j}) + \alpha_{3,j}\int_{[0,\infty)} e^{-\mu u_{3,j}y}\mathrm{d}B_2(y)\Big),\quad t\to\infty. \tag{11}$$

For $k\in\mathbb{N}$, denote by $\mathcal{F}_k$ the σ-algebra generated by $(\xi_j,\eta_j)_{1\le j\le k}$. The left-hand side in (11) can be represented as follows:

$$t^{-1/2}\sum_{k\ge 1}\Big(\Big(\sum_{j=1}^{\ell}\alpha_{1,j}\,\mathbb{1}_{\{k\le\lfloor u_{1,j}t\rfloor\}}\Big)(\xi_k-\mu)+\Big(\sum_{j=1}^{\ell}\Big(\alpha_{2,j}\,\mathbb{1}_{\{k\le\lfloor u_{2,j}t\rfloor\}}+\alpha_{3,j}e^{-u_{3,j}S_{k-1}/t}\Big)\Big)(\eta_k-\mathtt{m})\Big). \tag{12}$$

For each $t>0$, this is a terminal value of the martingale

$$\Big(t^{-1/2}\sum_{k=1}^{n}\Big(\Big(\sum_{j=1}^{\ell}\alpha_{1,j}\,\mathbb{1}_{\{k\le\lfloor u_{1,j}t\rfloor\}}\Big)(\xi_k-\mu)\\ +\Big(\sum_{j=1}^{\ell}\Big(\alpha_{2,j}\,\mathbb{1}_{\{k\le\lfloor u_{2,j}t\rfloor\}}+\alpha_{3,j}e^{-u_{3,j}S_{k-1}/t}\Big)\Big)(\eta_k-\mathtt{m})\Big),\mathcal{F}_n\Big)_{n\in\mathbb{N}}.$$

Thus, by the martingale central limit theorem (Theorem 2.5(a) in [7]), (11) follows if we can show that

$$t^{-1}\sum_{k\ge1}\mathbb{E}\Big(\Big(\sum_{j=1}^{\ell}\alpha_{1,j}\,\mathbb{1}_{\{k\le\lfloor u_{1,j}t\rfloor\}}\Big)(\xi_k-\mu)\\ +\Big(\sum_{j=1}^{\ell}\Big(\alpha_{2,j}\,\mathbb{1}_{\{k\le\lfloor u_{2,j}t\rfloor\}}+\alpha_{3,j}e^{-u_{3,j}S_{k-1}/t}\Big)\Big)(\eta_k-\mathtt{m})\Big)^2\Big|\mathcal{F}_{k-1}\Big)\\ \overset{\mathbb{P}}{\to}\ \mathbb{E}\Big(\sum_{j=1}^{\ell}\Big(\alpha_{1,j}B_1(u_{1,j})+\alpha_{2,j}B_2(u_{2,j})+\alpha_{3,j}\int_{[0,\,\infty)}e^{-\mu u_{3,j}y}\mathrm{d}B_2(y)\Big)\Big)^2,\quad t\to\infty \tag{13}$$

and that each of the 3ℓ summands in (12) satisfies the conditional Lindeberg-Feller condition. The latter is trivial for the summands with the coefficients $\alpha_{1,j}$ and $\alpha_{2,j}$ and was checked in the proof of Theorem 1.2 in [8] (stated here as Proposition 1.4) for the summands with the coefficients $\alpha_{3,j}$.

We intend to prove (13). Recall that

$$\mathbb{E}B_1(u_1)B_1(u_2)=\sigma^2(u_1\wedge u_2)\quad\text{and}\quad\mathbb{E}B_2(u_1)B_2(u_2)=\mathtt{s}^2(u_1\wedge u_2),\quad u_1,u_2\ge0,$$

where $x\wedge y=\min(x,y)$. Also, we note that

$$\mathbb{E}\int_{[0,\,\infty)}e^{-\mu u_1 y}\mathrm{d}B_2(y)\int_{[0,\,\infty)}e^{-\mu u_2 y}\mathrm{d}B_2(y)=\mathtt{s}^2\int_0^\infty e^{-\mu(u_1+u_2)y}\mathrm{d}y=\mathtt{s}^2\frac{1}{\mu(u_1+u_2)},\ u_1,u_2>0.$$

Since the increments of $\mathbf{B}$ are independent,

$$\mathbb{E}B_1(u_1)B_2(u_2)=\mathbb{E}B_1(u_1\wedge u_2)B_2(u_1\wedge u_2)=\gamma(u_1\wedge u_2),\quad u_1,u_2\ge0. \tag{14}$$

Further,

$$\mathbb{E}B_2(u_1)\int_{[0,\,\infty)}e^{-\mu u_2 y}\mathrm{d}B_2(y)=\mathbb{E}\int_{[0,\,\infty)}\mathbb{1}_{\{z\in[0,\,u_1]\}}\,\mathrm{d}B_2(z)\int_{[0,\,\infty)}e^{-\mu u_2 y}\mathrm{d}B_2(y)\\ =\mathtt{s}^2\int_0^{u_1}e^{-\mu u_2 y}\mathrm{d}y=\mathtt{s}^2\frac{1-e^{-\mu u_1 u_2}}{\mu u_2},\quad u_1\ge0,u_2>0.$$

Using (14) and arguing similarly, we obtain

$$\mathbb{E}B_1(u_1)\int_{[0,\infty)} e^{-\mu u_2 y}\mathrm{d}B_2(y)=\gamma\frac{1-e^{-\mu u_1 u_2}}{\mu u_2},\quad u_1\geq 0, u_2>0.$$

The left-hand side of (13) is equal to

$$\begin{aligned}t^{-1}\Big(\sigma^2\Big(\sum_{j=1}^{\ell}\alpha_{1,j}^2\lfloor u_{1,j}t\rfloor+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{1,j_1}\alpha_{1,j_2}(\lfloor u_{1,j_1}t\rfloor\wedge\lfloor u_{1,j_2}t\rfloor)\Big)\\ +\mathtt{s}^2\Big(\sum_{j=1}^{\ell}\alpha_{2,j}^2\lfloor u_{2,j}t\rfloor+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{2,j_1}\alpha_{2,j_2}(\lfloor u_{2,j_1}t\rfloor\wedge\lfloor u_{2,j_2}t\rfloor)\Big)\\ +\mathtt{s}^2\Big(\sum_{j=1}^{\ell}\alpha_{3,j}^2\sum_{k\geq 1}e^{-2u_{3,j}S_{k-1}/t}+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{3,j_1}\alpha_{3,j_2}\sum_{k\geq 1}e^{-(u_{3,j_1}+u_{3,j_2})S_{k-1}/t}\Big)\\ +2\gamma\sum_{j_1,j_2=1}^{\ell}\alpha_{1,j_1}\alpha_{2,j_2}(\lfloor u_{1,j_1}t\rfloor\wedge\lfloor u_{2,j_2}t\rfloor)+2\mathtt{s}^2\sum_{j_1,j_2=1}^{\ell}\alpha_{2,j_1}\alpha_{3,j_2}\sum_{k=1}^{\lfloor u_{2,j_1}t\rfloor}e^{-u_{3,j_2}S_{k-1}/t}\\ +2\gamma\sum_{j_1,j_2=1}^{\ell}\alpha_{1,j_1}\alpha_{3,j_2}\sum_{k=1}^{\lfloor u_{1,j_1}t\rfloor}e^{-u_{3,j_2}S_{k-1}/t}\Big).\end{aligned}$$

We are going to apply Lemma 5.3 with particular $(R_k)_{k\in\mathbb{N}}$ and $(\tau_k)_{k\in\mathbb{N}}$ that we now specify. By the strong law of large numbers for random walks, for each ω from some set of probability measure one, the sequences $(R_k)_{k\in\mathbb{N}}=(S_k(\omega))_{k\in\mathbb{N}_0}$ and $(\tau_k)_{k\in\mathbb{N}}$ given by $\tau_k=1$ for all $k\in\mathbb{N}$ satisfy (22) with $\theta=\mu$ and $\rho=1$. Hence, by elementary calculations and Lemma 5.3, the expression in the last centered formula converges a.s. as $t\to\infty$ to

$$\begin{aligned}\sigma^2\Big(\sum_{j=1}^{\ell}\alpha_{1,j}^2u_{1,j}+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{1,j_1}\alpha_{1,j_2}(u_{1,j_1}\wedge u_{1,j_2})\Big)\\ +\mathtt{s}^2\Big(\sum_{j=1}^{\ell}\alpha_{2,j}^2u_{2,j}+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{2,j_1}\alpha_{2,j_2}(u_{2,j_1}\wedge u_{2,j_2})\Big)\\ +\mathtt{s}^2\Big(\sum_{j=1}^{\ell}\alpha_{3,j}^2\frac{1}{2\mu u_{3,j}}+2\sum_{1\leq j_1<j_2\leq\ell}\alpha_{3,j_1}\alpha_{3,j_2}\frac{1}{\mu(u_{3,j_1}+u_{3,j_2})}\Big)+2\gamma\sum_{j_1,j_2=1}^{\ell}\alpha_{1,j_1}\alpha_{2,j_2}(u_{1,j_1}\wedge u_{2,j_2})\\ +2\mathtt{s}^2\sum_{j_1,j_2=1}^{\ell}\alpha_{2,j_1}\alpha_{3,j_2}\frac{1-e^{-\mu u_{2,j_1}u_{3,j_2}}}{\mu u_{3,j_2}}+2\gamma\sum_{j_1,j_2=1}^{\ell}\alpha_{1,j_1}\alpha_{3,j_2}\frac{1-e^{-\mu u_{1,j_1}u_{3,j_2}}}{\mu u_{3,j_2}}.\end{aligned}$$

According to the formulae in the previous paragraph the latter is equal to the right-hand side of (13). This completes the proof of (11), hence of (10).

From (10) and the proofs of Theorem 3.1 on p. 162 in [6] and our Proposition 5.1 it follows that

$$\begin{aligned}t^{-1/2}\Big(N(ut)-\mu^{-1}ut,\sum_{k\geq 0}e^{-uS_k/t}(\eta_{k+1}-\mathtt{m})\Big)\\ \Rightarrow\Big(-\mu^{-1}B_1(\mu^{-1}u),\int_{[0,\infty)}e^{-uy}\mathrm{d}B_2(\mu^{-1}y)\Big),\quad t\to\infty\quad(15)\end{aligned}$$

in the product J_1-topology on $D[0,\infty)\times D(0,\infty)$. Repeating now the proof of Theorem 1.1 but keeping in mind that the first coordinate on the left-hand side of (15) converges jointly with the second we arrive at (8).

To prove the last claim of the corollary, observe that

$$\mu v^2=\mathbb{E}((\mathtt{m}/\mu)(\xi-\mu)-(\eta-\mathtt{m}))^2=\mathbb{E}((\mathtt{m}/\mu)\xi-\eta)^2.$$

Thus, $v^2=0$ if, and only if, $\eta=(\mathtt{m}/\mu)\xi$ a.s. which is equivalent to $\eta=c\xi$ a.s. for some $c\in\mathbb{R}$, $c\neq 0$.

4 Proof of Theorem 1.7

Using $\sum_{k\geq 0}e^{-S_k/t}(1-e^{-\xi_{k+1}/t})=1$, we write

$$\sum_{k\geq 0}e^{-S_k/t}\xi_{k+1}-t=\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}-(1-e^{-\xi_{k+1}/t})t\Big).$$

With the help of

$$y-1+e^{-y}\leq y^2/2\quad\text{and}\quad e^y-1-y\leq y^2e^y/2,\quad y\geq 0$$

we infer

$$\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}-(1-e^{-\xi_{k+1}/t})t\Big)\,\mathbb{1}_{\{\xi_{k+1}\geq 0\}}\leq\frac{1}{2t}\sum_{k\geq 0}e^{-S_k/t}(\xi_{k+1}^+)^2$$

and

$$\begin{aligned}\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}-\big(1-e^{-\xi_{k+1}/t}\big)t\Big)\,\mathbb{1}_{\{\xi_{k+1}<0\}}&=\sum_{k\geq 0}e^{-S_k/t}\Big(\big(e^{\xi_{k+1}^-/t}-1\big)t-\xi_{k+1}^-\Big)\\&\leq\frac{1}{2t}\sum_{k\geq 0}e^{-(S_k-\xi_{k+1}^-)/t}(\xi_{k+1}^-)^2,\end{aligned}$$

respectively. We intend to use Lemma 5.3(a) with particular $(R_k)_{k\in\mathbb{N}}$ and $(\tau_k)_{k\in\mathbb{N}}$. By the strong law of large numbers for random walks, for each ω from some set of probability measure one, the sequences $(R_k)_{k\in\mathbb{N}}=(S_k(\omega))_{k\in\mathbb{N}_0}$ and $(\tau_k)_{k\in\mathbb{N}}=((\xi_k^+(\omega))^2)_{k\in\mathbb{N}}$ satisfy (22) with $\theta=\mu$ and $\rho=\mathbb{E}(\xi^+)^2$, and $(R_k)_{k\in\mathbb{N}}=(S_k(\omega)-\xi_{k+1}^-(\omega))_{k\in\mathbb{N}_0}$ and $(\tau_k)_{k\in\mathbb{N}}=((\xi_k^-(\omega))^2)_{k\in\mathbb{N}}$ satisfy (22) with $\theta=\mu$ and $\rho=\mathbb{E}(\xi^-)^2$. Hence, by Lemma 5.3(a),

$$\limsup_{t\to\infty}\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}-(1-e^{-\xi_{k+1}/t})t\Big)\leq\frac{\mathbb{E}(\xi^+)^2+\mathbb{E}(\xi^-)^2}{2\mu}=\frac{\mathbb{E}\xi^2}{2\mu}\quad\text{a.s.}$$

The function $y\mapsto y-1+e^{-y}$ is increasing on $[0,\infty)$. This in combination with the inequality

$$y-1+e^{-y}\geq y^2/2-y^3/6,\quad y\geq 0$$

enables us to conclude that, for any $r>0$,

$$\begin{aligned}\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}-\big(1-e^{-\xi_{k+1}/t}\big)t\Big)\,\mathbb{1}_{\{\xi_{k+1}\geq 0\}}&\geq\sum_{k\geq 0}e^{-S_k/t}\Big(\xi_{k+1}^+\wedge r-\big(1-e^{-(\xi_{k+1}^+\wedge r)/t}\big)t\Big)\\&\geq\frac{1}{2t}\sum_{k\geq 0}e^{-S_k/t}(\xi_{k+1}^+\wedge r)^2-\frac{1}{6t^2}\sum_{k\geq 0}e^{-S_k/t}(\xi_{k+1}^+\wedge r)^3.\end{aligned}$$

Further, in view of

$$e^y - 1 - y \geq y^2/2, \quad y \geq 0,$$

$$\sum_{k\geq 0} e^{-S_k/t}\Big(\xi_{k+1} - \big(1 - e^{-\xi_{k+1}/t}\big)t\Big)\,\mathbb{1}_{\{\xi_{k+1}<0\}} = \sum_{k\geq 0} e^{-S_k/t}\Big(\big(e^{\xi^-_{k+1}/t} - 1\big)t - \xi^-_{k+1}\Big)$$
$$\geq \frac{1}{2t}\sum_{k\geq 0} e^{-S_k/t}(\xi^-_{k+1})^2.$$

Thus, another application of Lemma 5.3(a) yields

$$\liminf_{t\to\infty} \sum_{k\geq 0} e^{-S_k/t}\Big(\xi_{k+1} - \big(1 - e^{-\xi_{k+1}/t}\big)t\Big) \geq \frac{\mathbb{E}(\xi^+ \wedge r)^2 + \mathbb{E}(\xi^-)^2}{2\mu} \quad \text{a.s.}$$

Since $r > 0$ is arbitrary and $\lim_{r\to\infty} \mathbb{E}(\xi^+ \wedge r)^2 = \mathbb{E}(\xi^+)^2$ by Levi's monotone convergence theorem, we obtain

$$\liminf_{t\to\infty} \sum_{k\geq 0} e^{-S_k/t}\Big(\xi_{k+1} - \big(1 - e^{-\xi_{k+1}/t}\big)t\Big) \geq \frac{\mathbb{E}\xi^2}{2\mu} \quad \text{a.s.}$$

5 Appendix

Propositions 5.1 and 5.2 given below are important ingredients in the proof of Theorem 1.1. We think that these results should be known. However, we have been unable to locate them in the literature. Hence, full proofs will be given.

In addition to $N(t)$, define $\nu(t) := \inf\{k \in \mathbb{N} : S_k > t\}$ and $\rho(t) := \sup\{k \in \mathbb{N}_0 : S_k \leq t\}$ for $t \geq 0$. Note that these random variables are a.s. finite whenever $\lim_{n\to\infty} S_n = +\infty$ a.s. and particularly when $\mathbb{E}S_1 \in (0, \infty)$.

Proposition 5.1. *Assume that* $\mu = \mathbb{E}\xi \in (0,\infty)$ *and* $\sigma^2 = \operatorname{Var}\xi \in (0,\infty)$. *Then*

$$t^{-1/2}\big(M(ut) - \mu^{-1}ut\big) \;\Rightarrow\; (\sigma^2\mu^{-3})^{1/2}B(u), \quad t\to\infty \tag{16}$$

in the J_1*-topology on* $D[0,\infty)$, *where* $M(t)$ *is either* $\nu(t)$ *or* $N(t)$, *or* $\rho(t)$.

Proof. In the case $M(t) = \nu(t)$, formula (16) was obtained in Theorem 3.1 on p. 162 in [6]. Since

$$\nu(t) \leq N(t) \leq \rho(t) + 1, \quad t \geq 0 \quad \text{a.s.}$$

it suffices to prove (16) with $M(t) = \rho(t)$. In view of the inequalities

$$S_{\rho(t)} \leq t \quad \text{and} \quad S_{\rho(t)+1} > t, \quad t \geq 0 \quad \text{a.s.}, \tag{17}$$

this can be done along the lines of the proof of the aforementioned Theorem 3.1 from [6].

The relation

$$\lim_{t\to\infty} t^{-1}\rho(t) = \mu^{-1} \quad \text{a.s.} \tag{18}$$

can be found in formula (4.8) on p. 80 in [6]. With (18) at hand an appeal to Theorem 2.1 on p. 158 in [6] yields

$$t^{-1/2}\big(S_{\rho(ut)} - \mu\rho(ut)\big) \;\Rightarrow\; (\sigma^2\mu^{-1})^{1/2}B(u), \quad t\to\infty \tag{19}$$

in the J_1-topology on $D[0,\infty)$. According to (17)

$$t^{-1/2}(ut-\mu\rho(ut))-t^{-1/2}\xi_{\rho(ut)+1} \le t^{-1/2}(S_{\rho(ut)}-\mu\rho(ut)) \le t^{-1/2}(ut-\mu\rho(ut)), \quad u\ge 0, t>0 \quad \text{a.s.}$$

This in combination with (19) tells us that it is enough to show that, for all $T>0$,

$$t^{-1/2}\sup_{u\in[0,T]}\xi_{\rho(ut)+1} \overset{\mathbb{P}}{\to} 0, \quad t\to\infty. \tag{20}$$

It can be checked that (20) is secured by $\mathbb{E}\xi^2<\infty$ and (18), see, for instance, the proof of formula (3.8) on p. 162 of [6] for a similar argument. □

Proposition 5.2. *Assume that $\mu=\mathbb{E}\xi\in(0,\infty)$ and $\sigma^2=\operatorname{Var}\xi\in(0,\infty)$. Then*

$$\mathbb{E}|N(t)-\mu^{-1}t| \sim (\sigma^2\mu^{-3})^{1/2}\mathbb{E}|B(1)|t^{1/2}, \quad t\to\infty.$$

Proof. Putting $u=1$ and taking the absolute values in (16) with $M(t)=N(t)$ we infer

$$t^{-1/2}|N(t)-\mu^{-1}t| \overset{\mathrm{d}}{\longrightarrow} (\sigma^2\mu^{-3})^{1/2}|B(1)|, \quad t\to\infty.$$

This in combination with Fatou's lemma yields

$$\liminf_{t\to\infty}t^{-1/2}\mathbb{E}|N(t)-\mu^{-1}t| \ge (\sigma^2\mu^{-3})^{1/2}\mathbb{E}|B(1)|.$$

To obtain the converse inequality for the limit superior, write

$$N(t)=\sum_{k=0}^{\nu(t)-1}\mathbb{1}_{\{S_k\le t\}}+\sum_{k\ge\nu(t)}\mathbb{1}_{\{S_k\le t\}}=\nu(t)+\sum_{k\ge 1}\mathbb{1}_{\{S_{\nu(t)+k}-S_{\nu(t)}\le t-S_{\nu(t)}\}}.$$

Observe that

$$\sum_{k\ge 1}\mathbb{1}_{\{S_{\nu(t)+k}-S_{\nu(t)}\le t-S_{\nu(t)}\}} \le \sum_{k\ge 0}\mathbb{1}_{\{S_{\nu(t)+k}-S_{\nu(t)}\le 0\}} \overset{\mathrm{d}}{=} N(0),$$

where $\overset{\mathrm{d}}{=}$ denotes equality of distributions. Further, note that, according to Theorem 1 in [9], $\mathbb{E}\xi_-^2<\infty$ ensures $\mathbb{E}N(0)<\infty$. Thus, $\mathbb{E}|N(t)-\mu^{-1}t|\le\mathbb{E}|\nu(t)-\mu^{-1}t|+\mathbb{E}N(0)$. By Theorem 8.4 on p. 98 in [6], $\mathbb{E}|\nu(t)-\mu^{-1}t|\sim(\sigma^2\mu^{-3})^{1/2}\mathbb{E}|B(1)|t^{1/2}$ as $t\to\infty$, whence

$$\limsup_{t\to\infty}t^{-1/2}\mathbb{E}|N(t)-\mu^{-1}t| \le (\sigma^2\mu^{-3})^{1/2}\mathbb{E}|B(1)|.$$

□

Lemma 5.3 is used in the proofs of Corollary 1.5 and Theorem 1.7.

Lemma 5.3. *Let $(R_k)_{k\in\mathbb{N}}$ and $(\tau_k)_{k\in\mathbb{N}}$ be sequences of real-valued and nonnegative numbers, respectively, satisfying*

$$\lim_{k\to\infty}\frac{R_k}{k}=\theta \quad \textit{and} \quad \lim_{k\to\infty}\frac{\tau_1+\ldots+\tau_k}{k}=\rho \tag{21}$$

for some finite positive constants θ and ρ.
(a) Then, for each $t>0$, the series $\sum_{k\ge1}e^{-R_k/t}\tau_k$ is absolutely convergent and

$$\lim_{t\to\infty}t^{-1}\sum_{k\ge 1}e^{-R_k/t}\tau_k=\frac{\rho}{\theta}; \tag{22}$$

(b) for fixed positive u and v,

$$\lim_{t\to\infty}t^{-1}\sum_{k=1}^{\lfloor vt\rfloor}e^{-uR_{k-1}/t}=\frac{1-e^{-\theta uv}}{\theta u}. \tag{23}$$

Proof. (a) In view of (21), while $\lim_{k\to\infty} e^{-R_k/t} = 0$ exponentially fast, $\lim_{k\to\infty} k^{-1}|\tau_k| = 0$. This justifies the absolute convergence of the series $\sum_{k\geq 1} e^{-R_k/t}\tau_k$. The proof given below was kindly suggested by the referee, in a slightly modified form.

Given $\varepsilon > 0$ there exist real constants c_+ and c_- such that

$$c_- + (\theta - \varepsilon)k \leq R_k \leq c_+ + (\theta + \varepsilon)k, \quad k \in \mathbb{N}. \tag{24}$$

As a preparation for what follows, note that

$$t^{-1}\sum_{k\geq 1}\big(e^{-(\theta+\varepsilon)k/t} - e^{-(\theta+\varepsilon)(k+1)/t}\big)k = t^{-1}\sum_{k\geq 1} e^{-(\theta+\varepsilon)k/t} \to \frac{1}{\theta+\varepsilon}, \quad t\to\infty,$$

where the equality is obtained with the help of summation by parts. Using this, (24) and the second part of (21) in combination with summation by parts and the Toeplitz theorem we infer

$$\begin{aligned}\liminf_{t\to\infty} t^{-1}\sum_{k\geq 1} e^{-R_k/t}\tau_k &\geq \liminf_{t\to\infty} t^{-1}\sum_{k\geq 1} e^{-(c_+ + (\theta+\varepsilon)k)/t}\tau_k \\ &= \lim_{t\to\infty} t^{-1}e^{-c_+/t}\sum_{k\geq 1}\frac{\tau_1 + \ldots + \tau_k}{k}\big(e^{-(\theta+\varepsilon)k/t} - e^{-(\theta+\varepsilon)(k+1)/t}\big)k = \frac{\rho}{\theta+\varepsilon}.\end{aligned}$$

The inequality

$$\limsup_{t\to\infty} t^{-1}\sum_{k\geq 1} e^{-R_k/t}\tau_k \leq \frac{\rho}{\theta-\varepsilon}$$

can be proved similarly.

(b) The proof of (23) is analogous but simpler, hence omitted. We only mention that in order to identify the limit it suffices to replace R_{k-1} with $\theta(k-1)$. This gives

$$t^{-1}\sum_{k=1}^{\lfloor vt\rfloor} e^{-\theta u(k-1)/t} = t^{-1}\frac{1 - e^{-\theta u\lfloor vt\rfloor/t}}{1 - e^{-\theta u/t}} \to \frac{1 - e^{-\theta uv}}{\theta u} \quad t\to\infty.$$

□

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